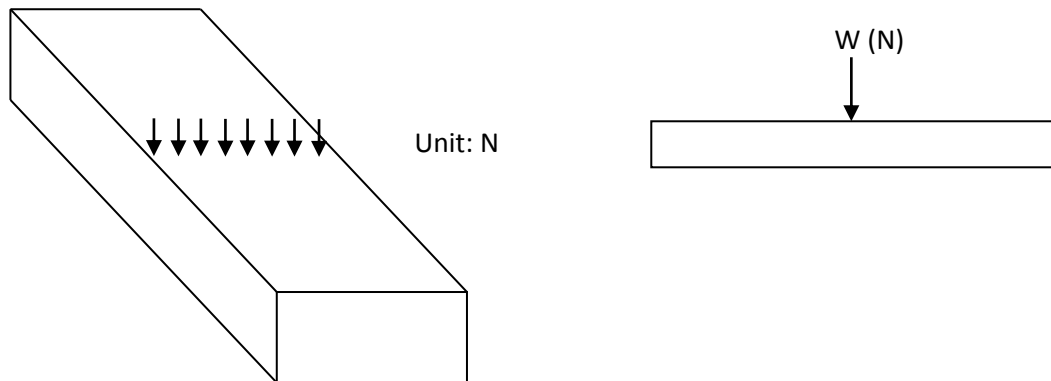


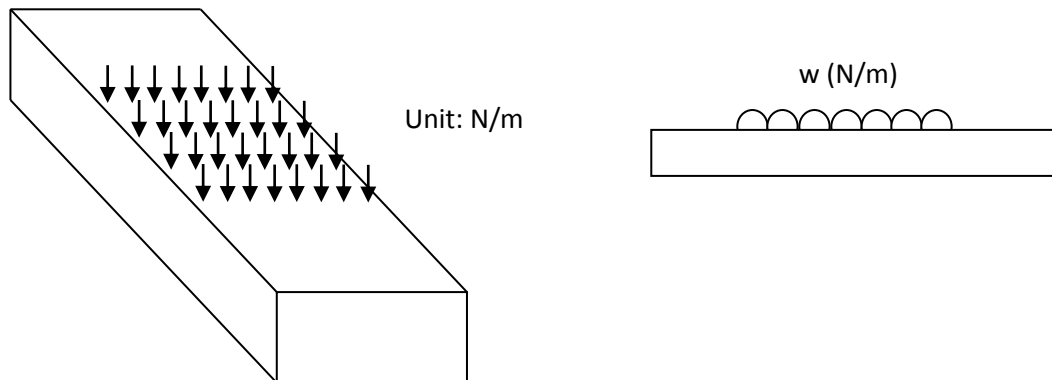
I. BEAMS

A beam is a structural member which is usually supported horizontally and that is subjected to external transverse forces (i.e. forces which are perpendicular to the length of the beam). The external forces considered may be due to concentrated loads or distributed loads, beam mass and supporting reactions.

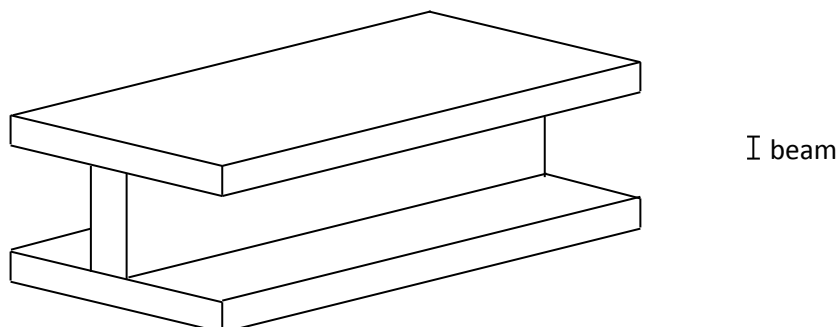
- A concentrated load is a load that acts on the surface of the beam along a line perpendicular to the longitudinal axis:



- A distributed load is a load that occupies a length of beam surface:



Beams should not just be regarded as rectangular cross-section bars of wood or metal. A beam can be of any cross-sectional shape. A very common form is the I beam:



II. EQUILIBRIUM OF BEAMS

For a horizontal beam to be in equilibrium when acted upon by transverse forces, the following condition must be satisfied:

- i. The sum of forces acting upwards must equal the sum of forces acting downwards:

$$\sum \text{upward forces} = \sum \text{downward forces}$$

- ii. The sum of the clockwise moments about any point must equal the sum of the anticlockwise moments about the same point:

$$\sum \text{clockwise moments about a point} = \sum \text{anticlockwise moments about the same point}$$

III. BEAM SUPPORTS and REACTIONS

Beams may be supported by a number of Joints or Constraints:

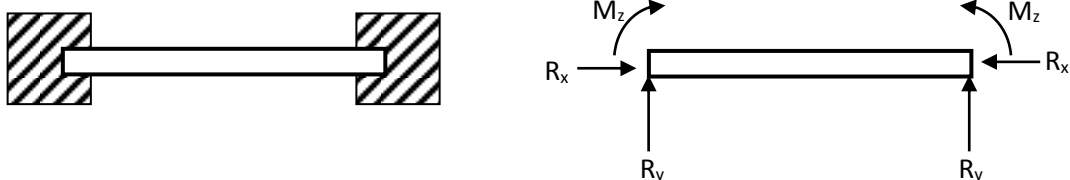
1. **Simply supported** (frictionless surface):



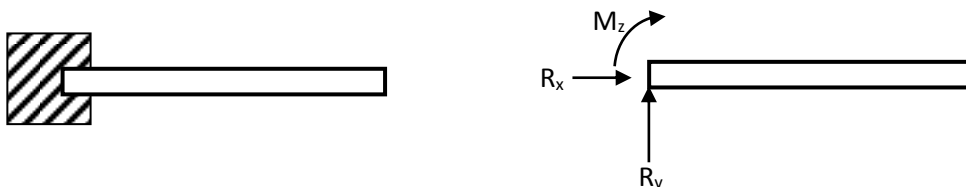
At the points of support, the downward forces acting on the beam are resisted by forces acting upwards. These upward forces are known as the beam support reactions R.

Supports are represented by denoting the motions they resist. In other words, supports are characterised by the support reactions.

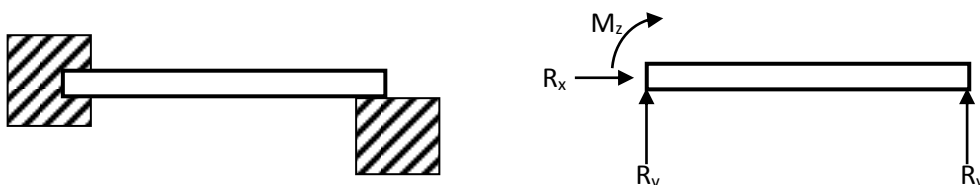
2. **Fixed** at both ends, this type of beam is called “**encastré**” beam:



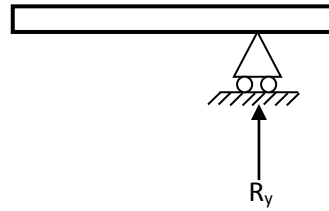
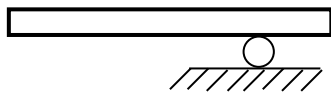
3. **Fixed** at one end only, this type is called **cantilever**:



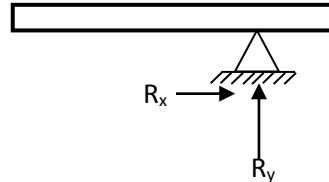
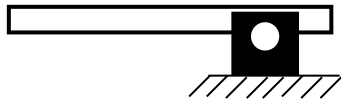
4. **Fixed** at one end and **simply supported** at the other, this type of beam is called **proper cantilever**:



5. Roller supports that resist vertical motion:

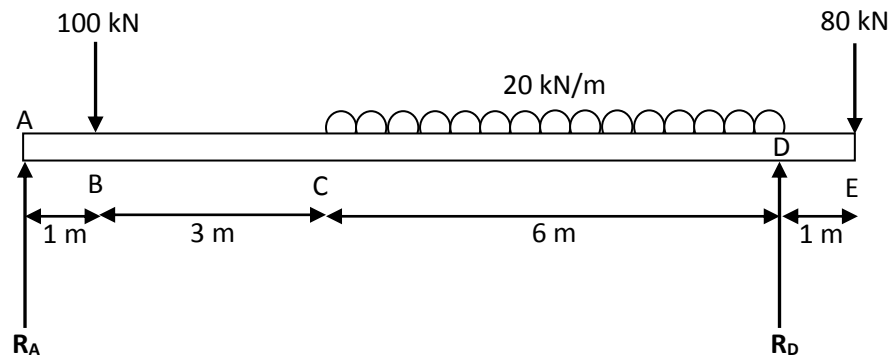


6. Pin supports that resist horizontal and vertical motion:

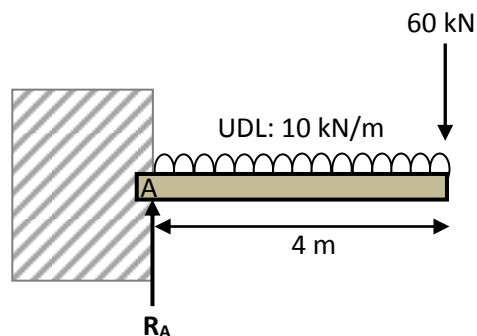


7. Examples:

- *Example 1:* A beam ABCDE is 11 m long and is simply supported at A and D. The beam carries point loads of magnitude 100 kN and 80 kN at B and E respectively, together with a uniform distributed load of 20 kN/m between C and D. AB = DE = 1 m and CD = 6 m. Calculate the reactions at A and D.

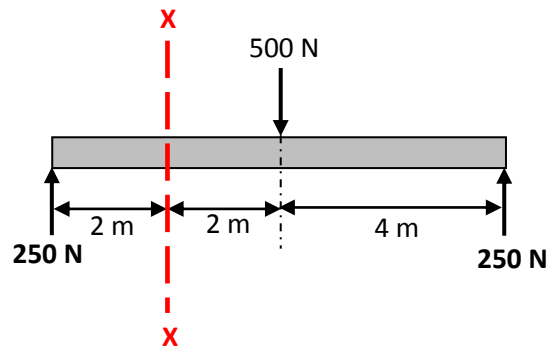


- *Example 2:* For the loaded cantilever shown in the following figure, calculate the reaction at point A.



IV. BENDING MOMENT, SHEAR FORCE

Let us consider the following Free Body Diagram (FBD) (force diagram) and the transverse section X-X as follows:



The moment about X due to the forces to the right of X and the moment about X due to the forces to the left of X are both tending to cause the beam at X to bend. Both the moment for the right side of X and the left side of X are 500 N.m. Indeed:

- left side: $M_x = 250 \times 2 = 500 \text{ N.m}$
- right side: $M_x = -250 \times 6 + 500 \times 2 = 500 \text{ N.m}$

This moment M_x is called the **bending moment** about the section X.

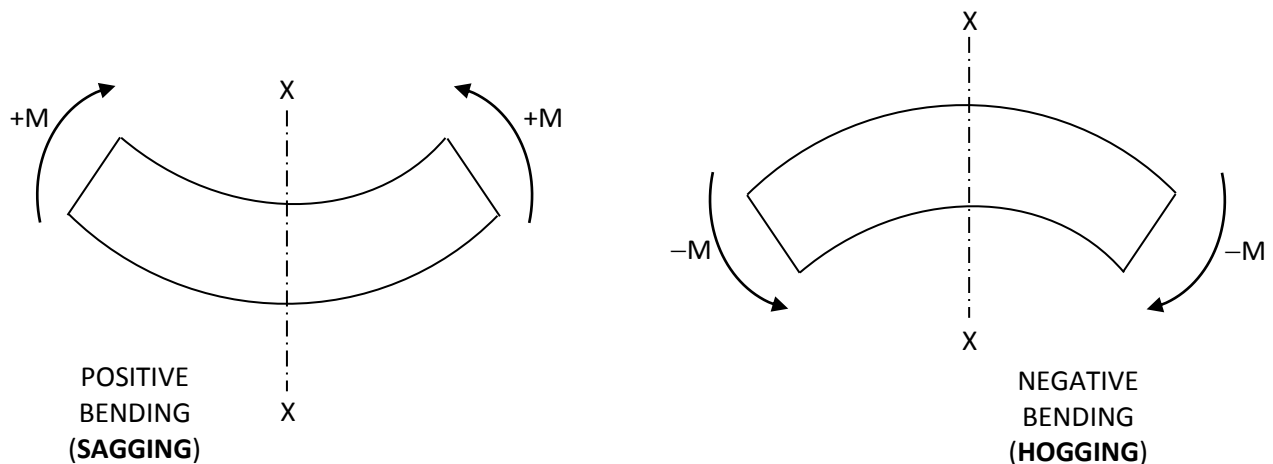
The bending moment at a transverse section of a beam is the algebraic sum of all the moment about the section of all the forces acting on one side of the section concerned, i.e.

$$\text{bending moment at any section} = \sum \text{moments of all the external forces acting either to the left or to the right of the section}$$

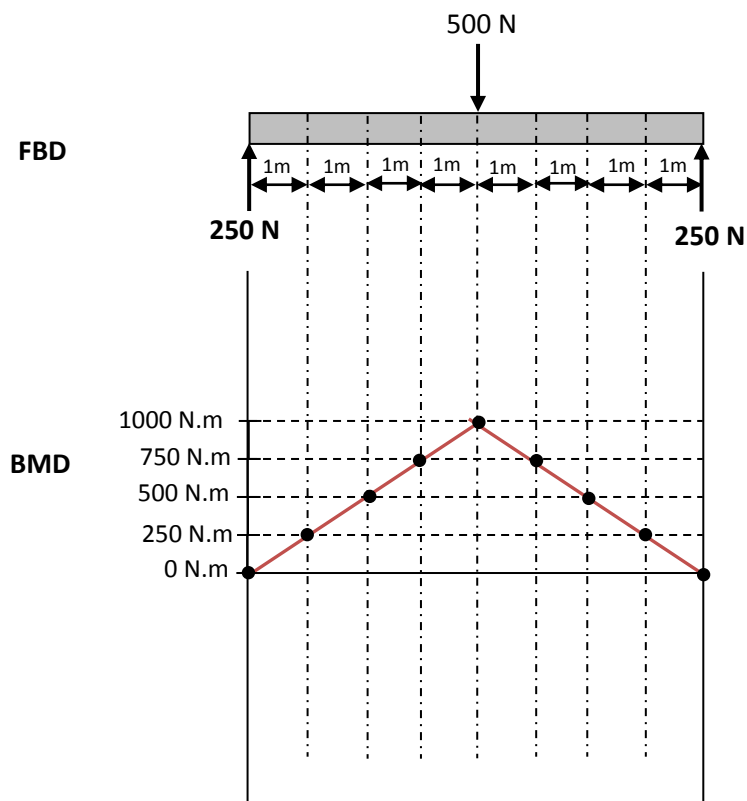
The symbol used for bending moment is M.

A bending moment is considered positive when the moment on the left of the section is clockwise and that on the right anticlockwise.

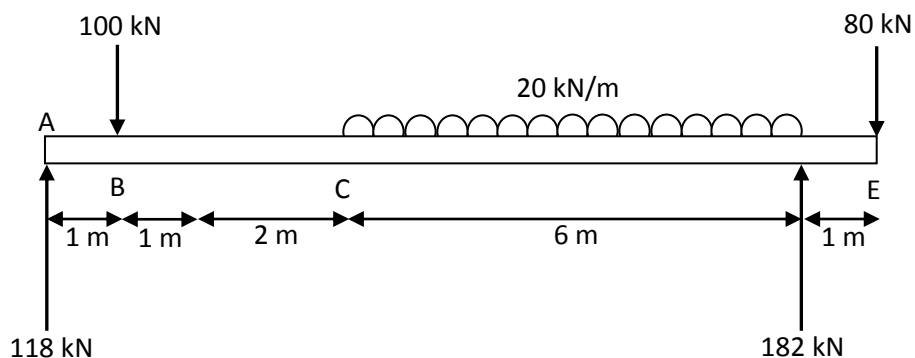
A bending moment is considered negative when the moment on the left of the section is anticlockwise and that on the right clockwise.



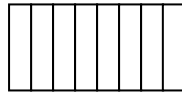
The value and location of the maximum bending moment $M_{\max}$ can be determined by considering the Free Body Diagram at different locations throughout the beam and plotting the Bending Moment Diagram (BMD) at these locations.



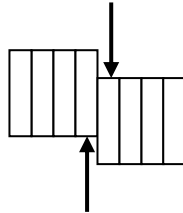
It would not be so easy to plot the BMD to find $M_{\max}$ in a more complex situation as in the following example,



The technique here to define $M_{\max}$ is to use the Shear Force Diagram (SFD) method. A shear force is the slapping effect inside a beam. Imagine a beam made of a bunch of magnets,



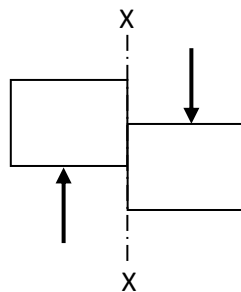
If one implements a load on that beam as follow,



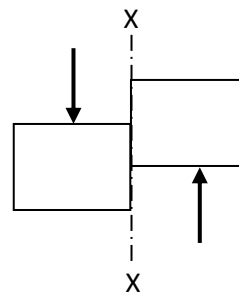
The beam is likely to “slide” rather than bend due to the effect of the shear forces. Any beam under the action of load undergo shear efforts throughout its structure.

A shear force is considered positive when the load on the left of the section is upward and that on the right is downward.

A shear force is considered negative when the load on the left of the section is downward and that on the right is upward.



POSITIVE
SHEAR FORCE



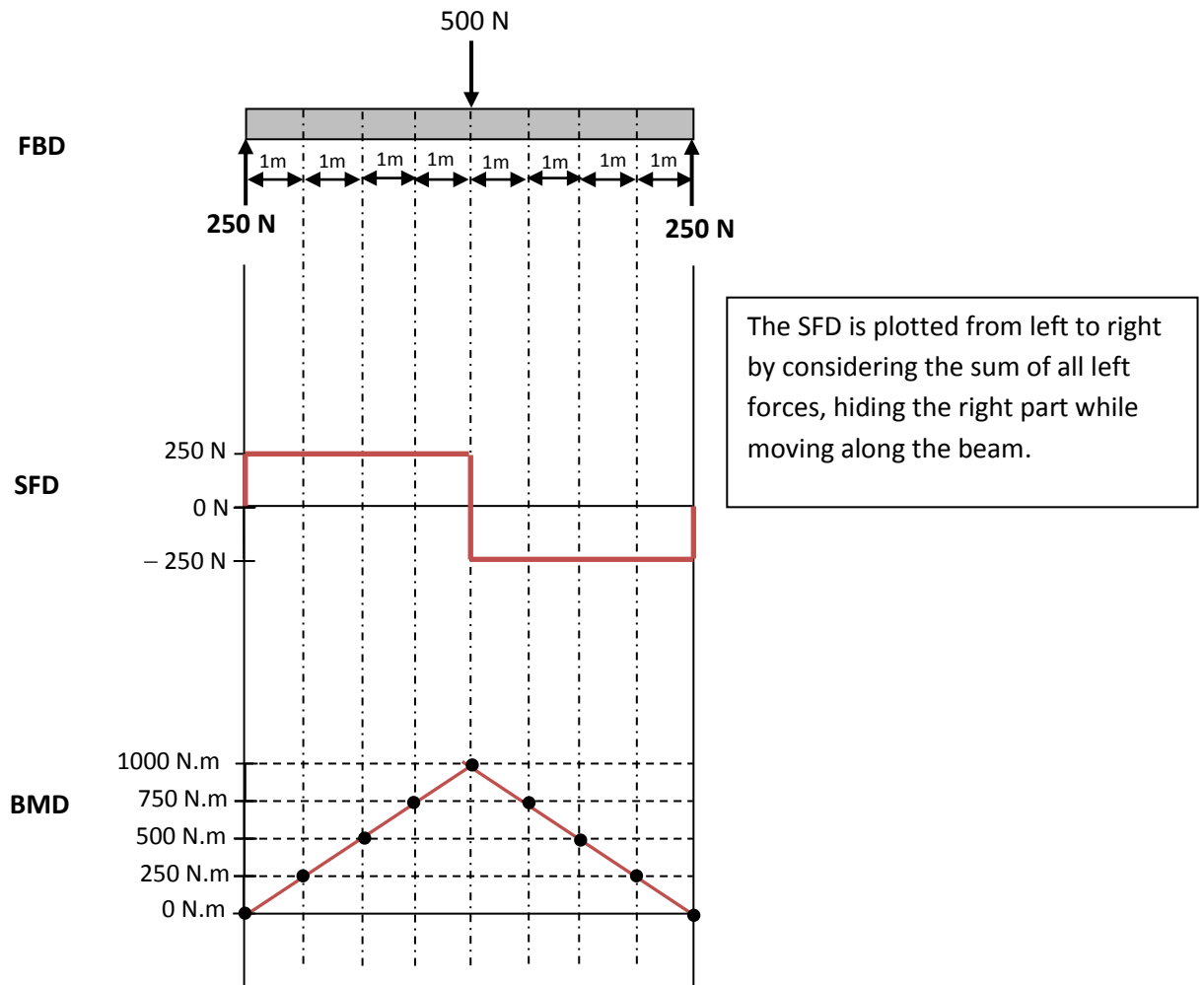
NEGATIVE
SHEAR FORCE

It exists a rule that states that,

$$M = \text{Area under SFD}$$

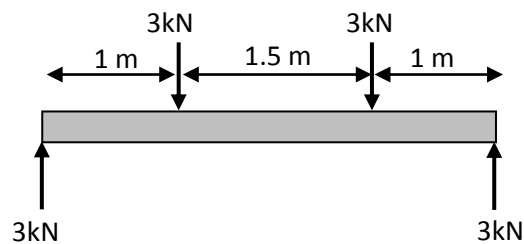
That rule allows to define the BMD really easily once the SFD has been plotted, which is always straightforward to do.

V. SHEAR FORCE DIAGRAM and BENDING MOMENT DIAGRAM

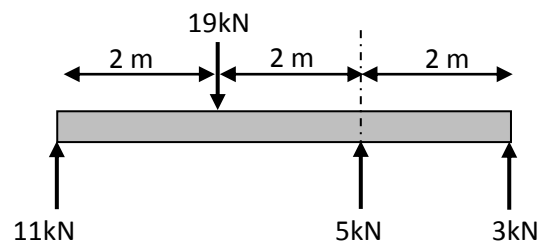


Plot the BDM based on the SFD to determine value and location of $M_{\max}$ for the following FBD,

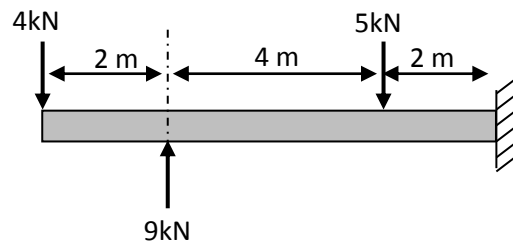
Example 1:



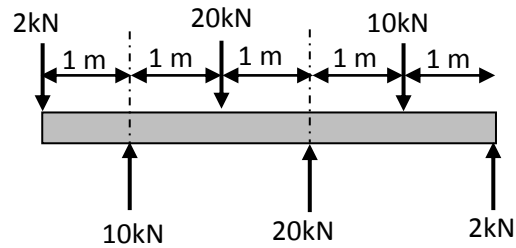
Example 2:



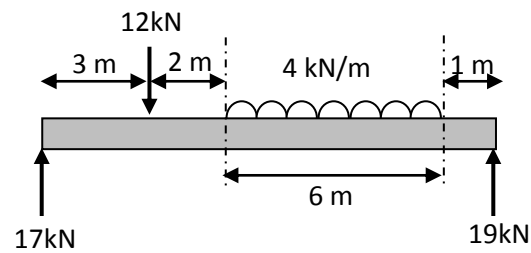
Example 3:



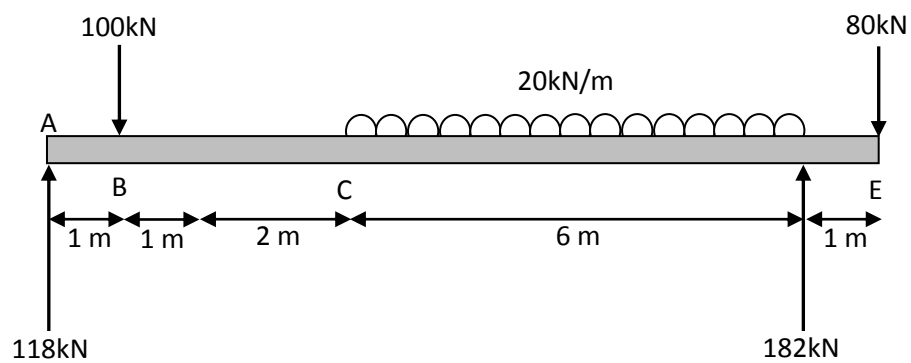
Example 4:



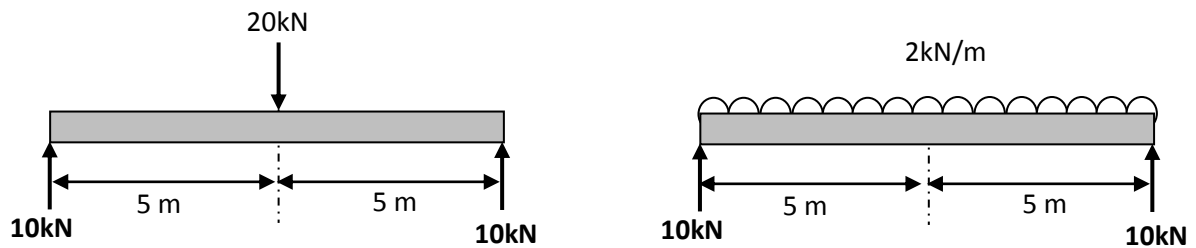
Example 5:



Example 6:



Example 7: Is $M_{\max}$ equivalent in the following two FBD cases?

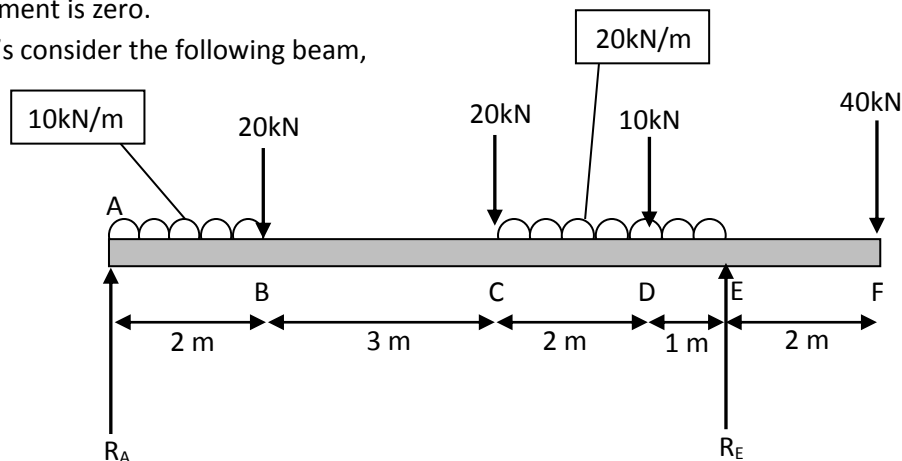


The bending moment is maximum when the shear force is zero as illustrated in these examples.

VI. POINTS OF INFLEXION or CONTRAFLEXURE

A point of inflexion (also called point of contraflexure) is a point where the curvature of the beam changes sign. It will be shown later to occur at the point, or points, on the beam where the bending moment is zero.

Let's consider the following beam,



From the SFD + BMD (to be plotted) the point of inflexion lies between C and D (BM at C positive, BM at D negative). If the required point is a distance x from C, then at that point,

$$M = 42.4(5 + x) - (10 \times 2)(4 + x) - 20(3 + x) - 20x - \frac{20x^2}{2}$$

$$\therefore M = -10x^2 - 17.5x + 72.5$$

Thus the bending moment BM is zero where,

$$-10x^2 - 17.5x + 72.5 = 0$$

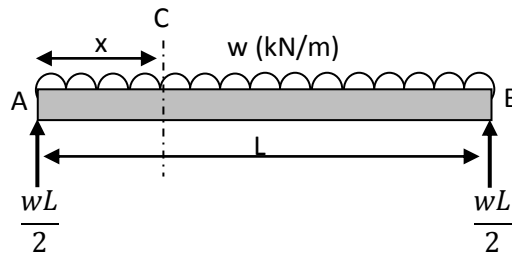
That is,

$$x = 1.96 \text{ or } -3.7$$

Since the last answer can be ignored (being outside of the beam), the point of inflexion must be situated at 1.96 m to the right of C.

VII. RELATIONSHIP BETWEEN SHEAR FORCE Q, BENDING MOMENT M and INTENSITY OF LOADING W

Consider the beam AB carrying a uniformly distributed loading intensity of w kN/m simply supported at A and B. By symmetry, each reaction takes half the total load, that is $wL/2$.



The BM at any point C, distance from A, is given by

$$M = \frac{wL}{2} \cdot x - (wx) \cdot \frac{x}{2}$$

$$\therefore M = \frac{1}{2}wLx - \frac{1}{2}wx^2$$

Differentiating,

$$\frac{dM}{dx} = \frac{1}{2}wL - wx$$

Now, the SF Q at C is,

$$Q = \frac{wL}{2} - wx$$

$$\therefore \frac{d^2M}{dx^2} = \frac{dQ}{dx} = -w$$

Thus, the rate of change of the BM with respect to x is equal to the shear force Q , or the slope of the BM diagram at a given point is the shear at that point.

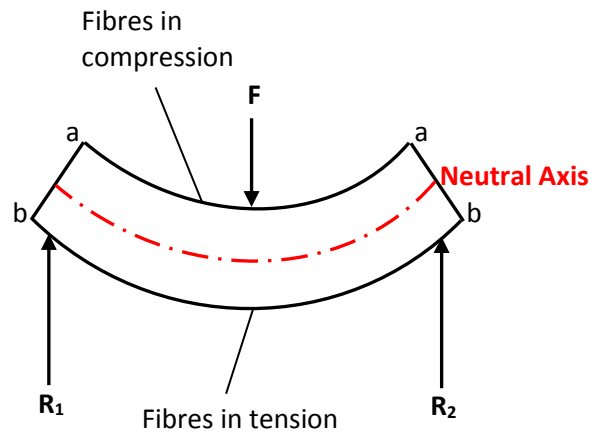
$$\boxed{\frac{dM}{dx} = Q \quad \text{or} \quad M = \int Q \, dx}$$

And the rate of change of the shearing force with respect to x is equal to the load or the slope of the SF diagram at a given point equals the load at that point.

$$\boxed{\frac{d^2M}{dx^2} = \frac{dQ}{dx} = -w \quad \text{or} \quad Q = \int w \, dx}$$

VIII. BENDING STRESS

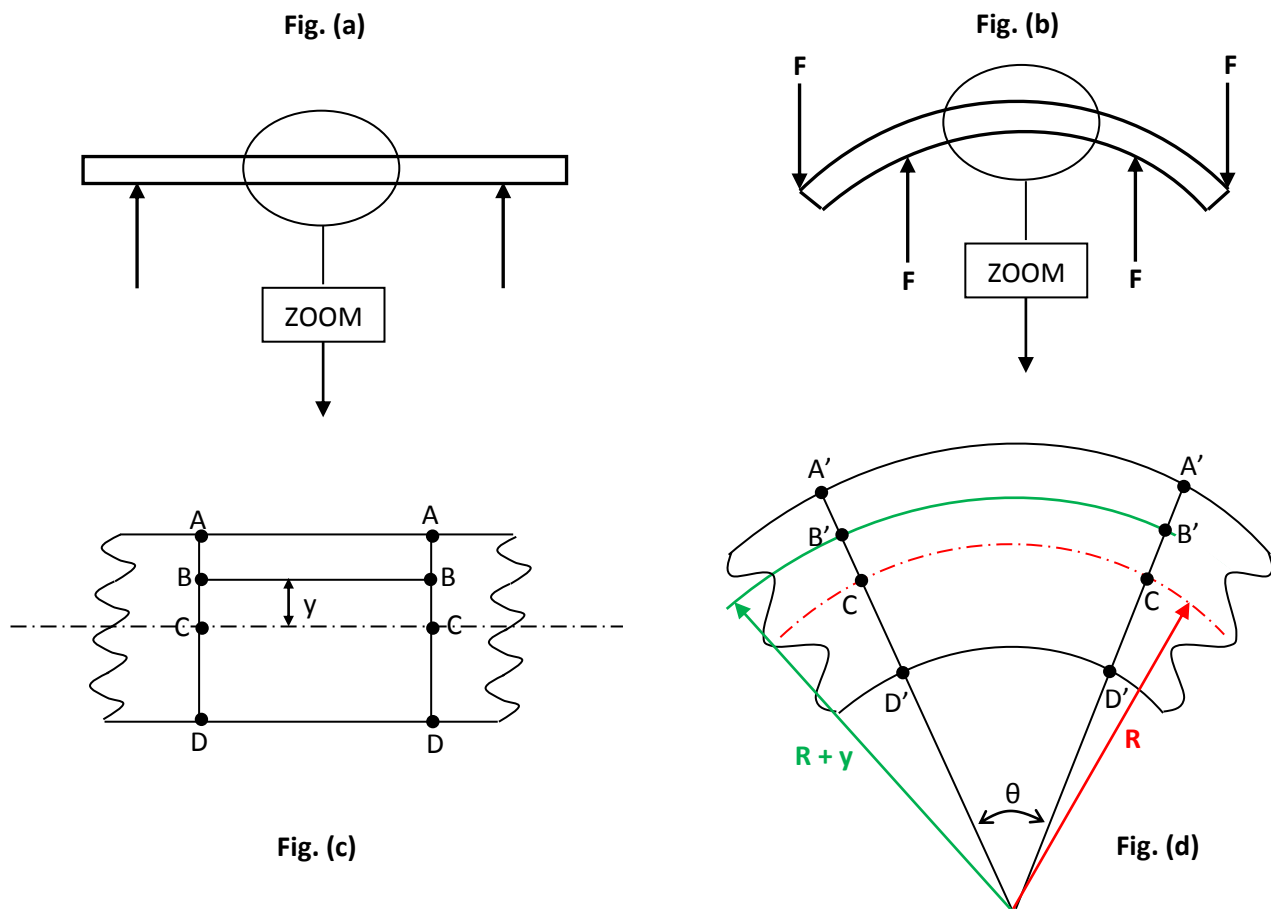
1. Definition:



The above figure shows a simply supported beam with a transverse force F at mid-span. The beam bends under the force, causing the fibres in the plane of a-a to be compressed and the fibres in the plane of b-b to be stretched, setting up associated compressive and tensile stresses in the beam, called **bending stresses**.

At a plane in the beam cross-section known as the **neutral axis**, there will be no change in length of the fibres and thus no bending stress, i.e. the neutral axis is a plane of zero bending stress.

2. Distribution of bending stress:



If the simply supported beam in fig. (a) is loaded as shown, it will bend to the shape of fig. (b), i.e. between the supports the bent shape of the beam is a circular arc of mean radius **R**. An enlarged view of a small portion of the beam between the supports is shown in figs. (c) and (d).

➤ Before bending:

$$AA = BB = CC = DD \quad \text{initial length}$$

➤ Consider the planes BB and CC after bending:

- After bending CC remains unchanged (neutral axis), i.e.

$$CC = R \cdot \theta$$

- And BB increases to B'B':

$$B'B' = (R + y) \cdot \theta$$

➤ Increase in length of BB = B'B' – BB

$$\begin{aligned} &= (R + y) \cdot \theta - R \cdot \theta && \text{(since } BB = CC = R \cdot \theta \text{)} \\ &= R \cdot \theta + y \cdot \theta - R \cdot \theta \\ &= y \cdot \theta \end{aligned}$$

➤ Because the length BB has been increased, a strain is induced in the beam:

$$\text{strain} = \frac{\text{change in length}}{\text{original length}}$$

$$\therefore \text{strain} = \frac{y \cdot \theta}{R \cdot \theta} = \frac{y}{R}$$

➤ If Hooke's law is assumed to be obeyed then:

$$\text{modulus of Elasticity } E = \frac{\text{stress } \sigma}{\text{strain } \epsilon}$$

$$\therefore \sigma = E \cdot \epsilon$$

Therefore the stress acting on the bent layer B'B' is:

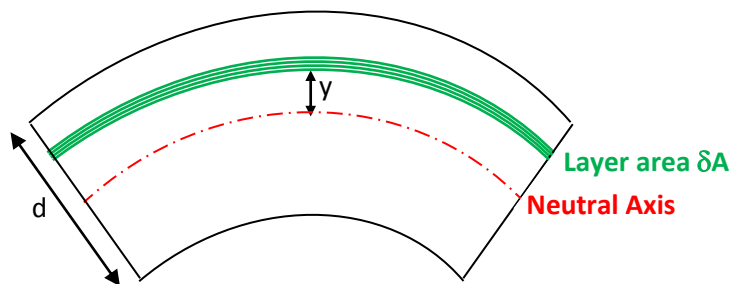
$$\sigma = \frac{y \cdot E}{R}$$

Thus the bending stress at any plane in the beam section is directly proportional to the distance y from the neutral axis.

3. Position of the neutral axis:

We have just defined that the stress acting on a layer in a beam is given by:

$$\sigma = \frac{y \cdot E}{R}$$



y = distance of the layer from the neutral axis

R = radius to which the beam is bent

E = modulus of Elasticity

Before this equation can be used for stress calculation we need to know the location of the neutral axis, that axis along which the stress is zero.

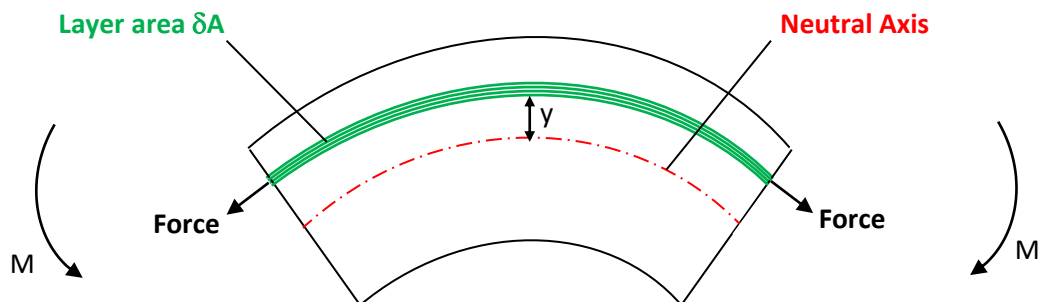
We will only consider beams that have cross-sections that are symmetric about both the horizontal and vertical axis, i.e. rectangular and circular cross-section beams and I beam with flanges of equal size. For all beams of these types, the neutral axis is at mid-height of the section $d / 2$.

4. Examples:

- Example 1: Calculate the maximum stress that will be produced in a steel beam of rectangular cross-section when bent into the arc of a circle of radius 5 m. The depth of the beam is 6 mm, the modulus of Elasticity 210 kN/mm². Give your answer in kN/mm².
- Example 2: Calculate the radius to which a steel strip of thickness 4 mm can be bent if the maximum stress acting on the material is not to exceed 100 N/mm². The material has a modulus of elasticity of 210 kN/mm².

IX. THE GENERAL BENDING FORMULA

1. Simple bending equation:



When a beam is bent, there are forces acting on each layer of material within that beam as shown in the previous figure. The longitudinal force acting on a layer of cross-sectional area δA at a distance y from the neutral axis must be:

$$\begin{aligned}\text{Force} &= \text{stress} \times \text{area} \\ &= \sigma \times \delta A\end{aligned}$$

Using the relationship for the stress σ acting on that layer:

$$\sigma = \frac{y \cdot E}{R}$$

$$\therefore \text{Force} = \frac{E}{R} \cdot y \cdot \delta A \quad (\text{this is the force acting on just one layer})$$

The force is acting at a distance y from the neutral axis and so gives a moment. The moment of this force about the neutral axis is:

$$\begin{aligned}\text{Moment of layer} &= \text{Force} \times y \\ &= \left(\frac{E}{R} \cdot y \cdot \delta A \right) \times y \\ &= \frac{E}{R} \cdot y^2 \cdot \delta A\end{aligned}$$

Each layer will exert a moment about the neutral axis and thus the total moment is the sum of all the moments due to each layer:

$$\text{Total Moment} = \sum \frac{E}{R} \cdot y^2 \cdot \delta A$$

As E and R are the same for each layer:

$$\text{Total Moment} = \frac{E}{R} \cdot \sum y^2 \cdot \delta A$$

The quantity $\sum y^2 \cdot \delta A$ is called the **SECOND MOMENT OF AREA** about the neutral axis of the cross-section and is denoted by the capital letter I . Units for I are: $\text{m}^2 \times \text{m}^2 = \text{m}^4$.

Hence the total moment can be written:

$$\text{Total Moment} = \frac{E \cdot I}{R}$$

The Second Moment of Area I is a property of a cross-section that indicates how effective it will be in resisting bending.

At equilibrium the total moment is equal in magnitude (but opposite in direction) to the bending moment. Thus:

$$M = \frac{E \cdot I}{R}$$

$$\therefore \boxed{\frac{M}{I} = \frac{E}{R}}$$

The stress acting on a layer a distance y from the neutral axis is given by:

$$\sigma = \frac{y \cdot E}{R}$$

$$\therefore \frac{\sigma}{y} = \frac{E}{R}$$

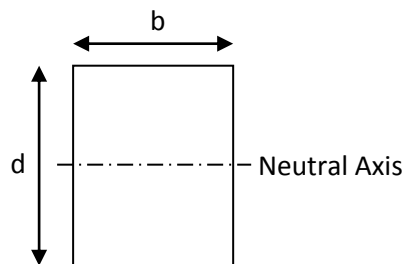
Therefore these two equations together give:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

This equation is known as the **Simple Bending Equation**.

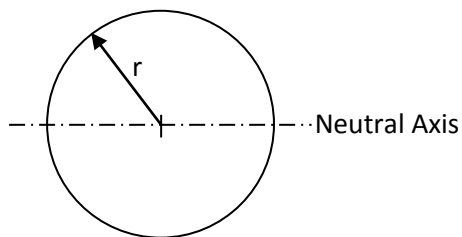
2. Second Moment of Area I of common shape:

a. Rectangular cross-section:



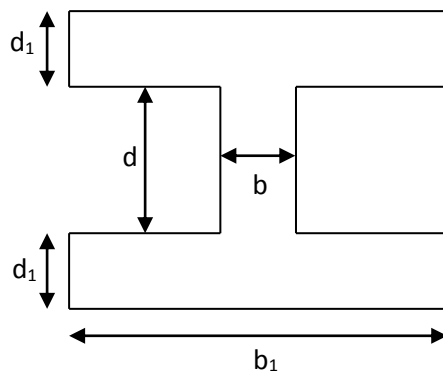
$$I = \frac{b \cdot d^3}{12}$$

b. Circular cross-section:



$$I = \frac{\pi \cdot r^4}{4}$$

c. I section:



$$I = \frac{b_1 \cdot (d + 2 \cdot d_1)^3}{12} - \frac{(b_1 - b) \cdot d^3}{12}$$