

Maths (Lecture 10)

Vectors

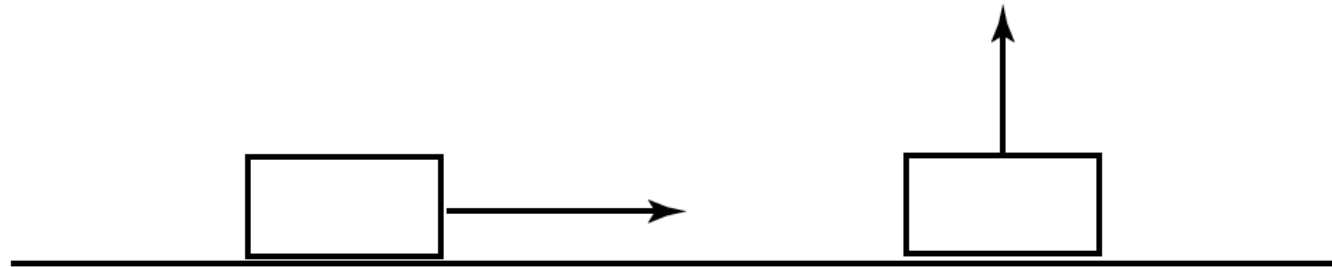
Dr Mohamed Bennasar

Introduction

vectors are quantities which require the specification of a magnitude and a direction.

There are many engineering applications where vectors are important, for example:

vector quantity is the force applied to an object to make it move, the effects depending on the direction of the force.



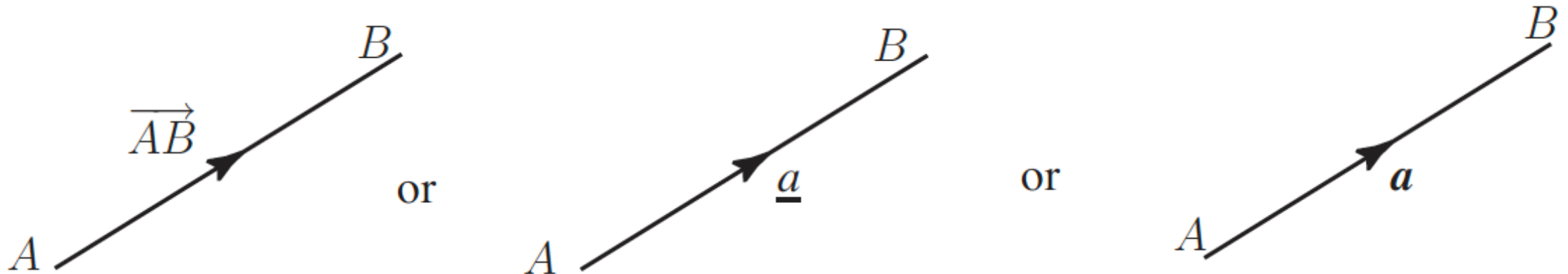
Some common scalar and vector quantities

quantity	type	S.I. unit
distance	scalar	metre, m
mass	scalar	kilogramme, kg
temperature	scalar	kelvin, K
pressure	scalar	pascal, Pa
work	scalar	joule, J
energy	scalar	joule, J
displacement	vector	metre m
force	vector	newton, N
velocity	vector	metres per second, m s^{-1}
acceleration	vector	metres per second per second, m s^{-2}

The mathematical description of vector quantities

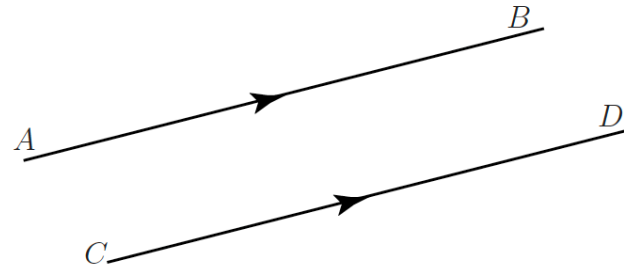
The vector has a direction as well as a magnitude, we can represent a vector by drawing a line. The length of the line represents the **magnitude** of the vector given some appropriate scale, and the **direction** of the line represents the direction of the vector.

Various notations are used to represent a vector, we emphasise that we are dealing with the vector from A to B by using an arrow and writing $\overrightarrow{AB}$.

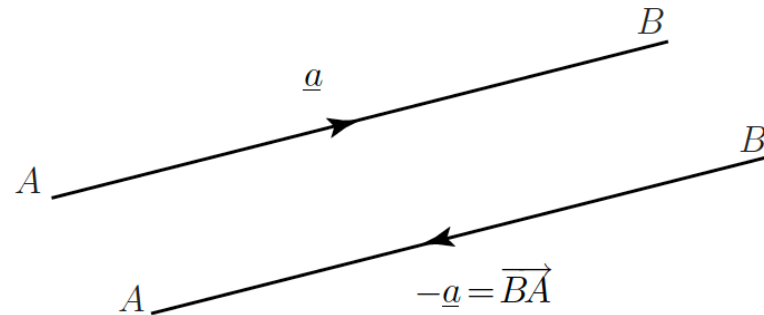


Properties of a vector

- Two vectors are said to be equal vectors if they have the same magnitude and same direction



- The vector $-a$ is a vector in the opposite direction to a .



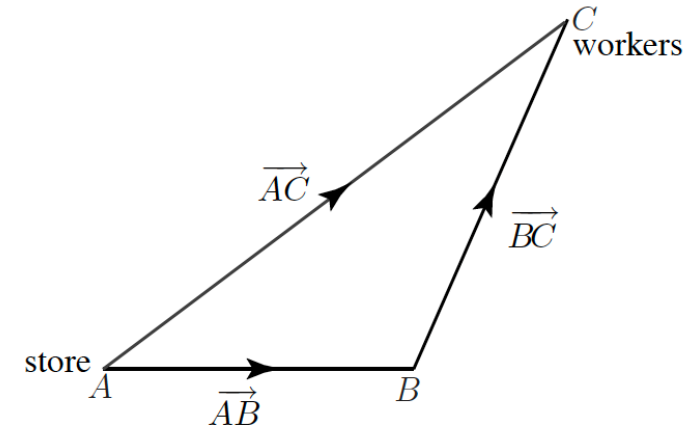
Addition of vectors

Vectors are added in a particular way known as the triangle law

Example: The route taken by an automated vehicle:

The vehicle may arrive at C either directly or via an intermediate point B. The movement from A to B can be represented by a displacement vector $\vec{AB}$. Similarly, movement from B to C can be represented by $\vec{BC}$.

$$\vec{AC} = \vec{AB} + \vec{BC}$$



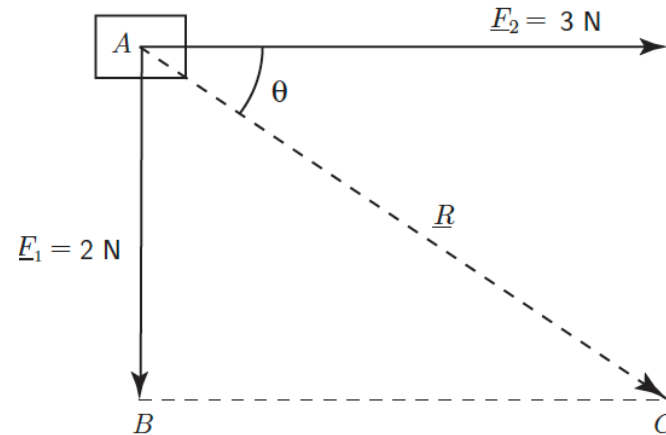
This is an example of the triangle law for adding vectors. We add vectors $\vec{AB}$ and $\vec{BC}$ by placing the tail of $\vec{BC}$ at the head of $\vec{AB}$ and completing the third side of the triangle so formed $\vec{AC}$.

Algebraic Properties of Vectors

- Commutative (vector) $P + Q = Q + P$.
- Associative (vector) $(P + Q) + R = P + (Q + R)$
- Additive identity There is a vector 0 such that $(P + 0) = P = (0 + P)$
- Additive inverse For any P there is a vector $-P$ such that $P + (-P) = 0$.

Example:

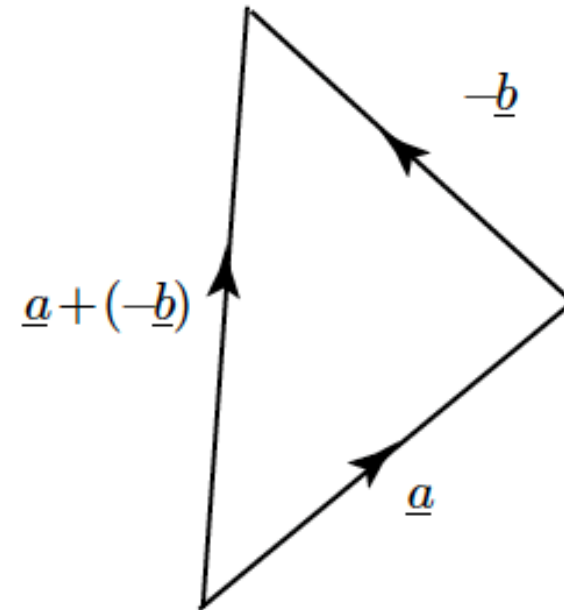
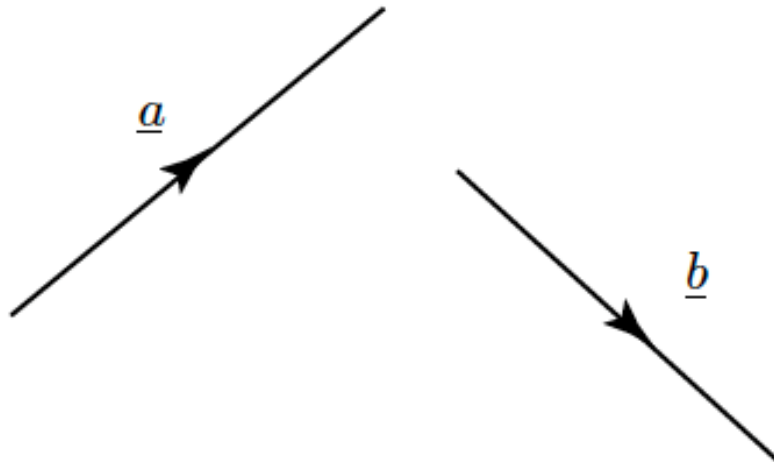
A force F_1 of 2 N acts vertically downwards, and a force F_2 of 3 N acts horizontally to the right, upon the body shown below



Using Pythagoras' theorem, The magnitude of the R force : $R = \sqrt{F_1^2 + F_2^2} = \sqrt{13}\text{ N}$
The direction of the vector is identified by the angle $\theta = \tan^{-1} 2/3$

Subtraction of vectors

Subtraction of one vector from another is performed by adding the corresponding negative vector, if we seek $\underline{a} - \underline{b}$ we form $\underline{a} + (-\underline{b})$.



Multiplying a vector by a scalar

If k is any positive scalar and $\underline{a}$ is a vector then $k\underline{a}$ is a vector in the same direction as $\underline{a}$ but k times as long.

$$k(\underline{a} + \underline{b}) = k\underline{a} + k\underline{b}$$

$$(k + l)\underline{a} = k\underline{a} + l\underline{a}$$

$$k(l)\underline{a} = (kl)\underline{a}$$

Example

(a) $3\underline{a} + 7\underline{a}$

(b) $2(7\underline{b})$

(c) $4\underline{q} + 4\underline{r}$

Answer

(a) Using the second rule, $3\underline{a} + 7\underline{a}$ can be simplified to $(3 + 7)\underline{a} = 10\underline{a}$.

(b) Using the third rule $2(7\underline{b}) = (2 \times 7)\underline{b} = 14\underline{b}$.

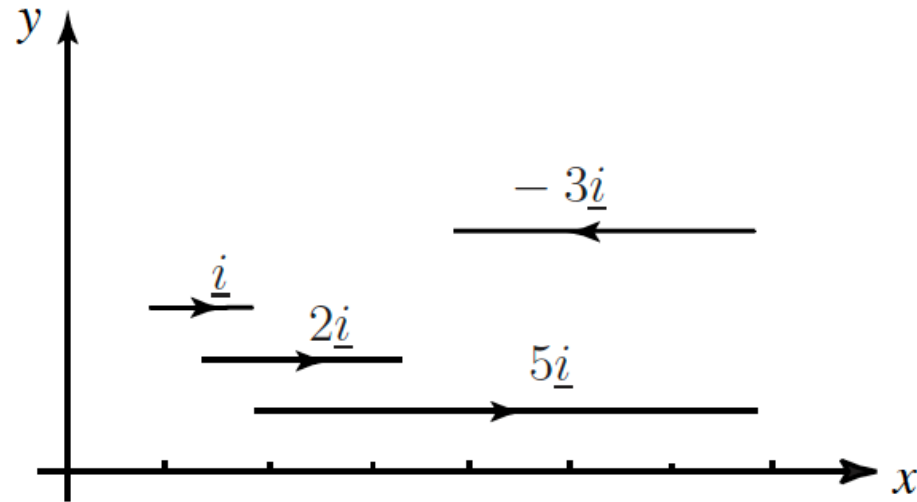
(c) Using the first rule $4\underline{q} + 4\underline{r} = 4(\underline{q} + \underline{r})$.

Unit vectors:

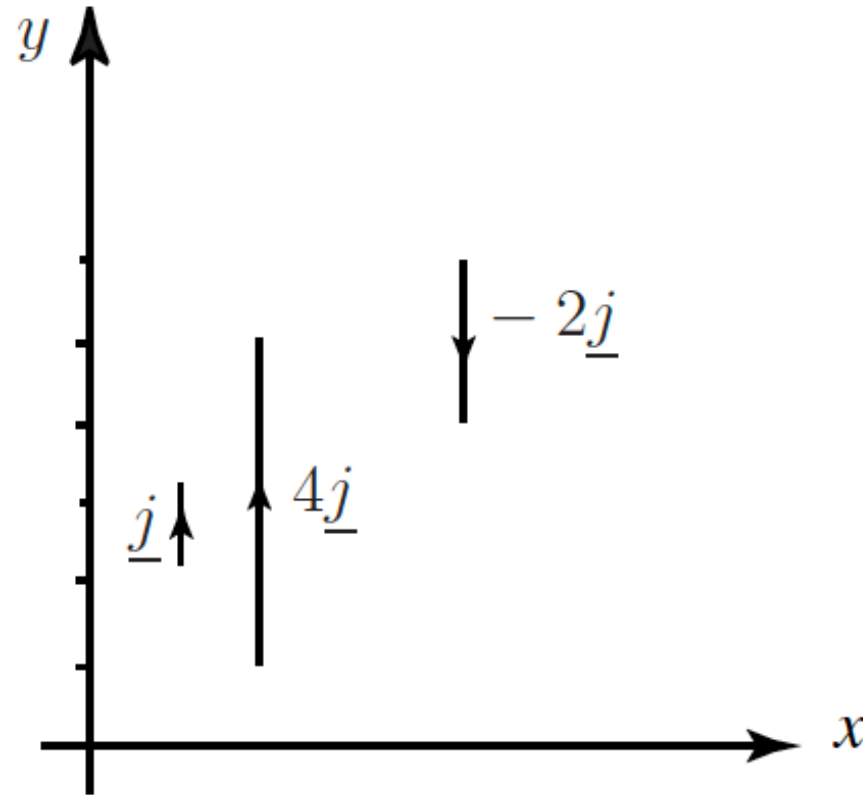
A vector which has a magnitude of 1 is called a unit vector.

Cartesian Components of Vectors

- Two-dimensional coordinate frames
 - A unit vector pointing in the positive direction of the x-axis is denoted by $\underline{i}$, any vector in the direction of the x-axis will be a multiple of $\underline{i}$.

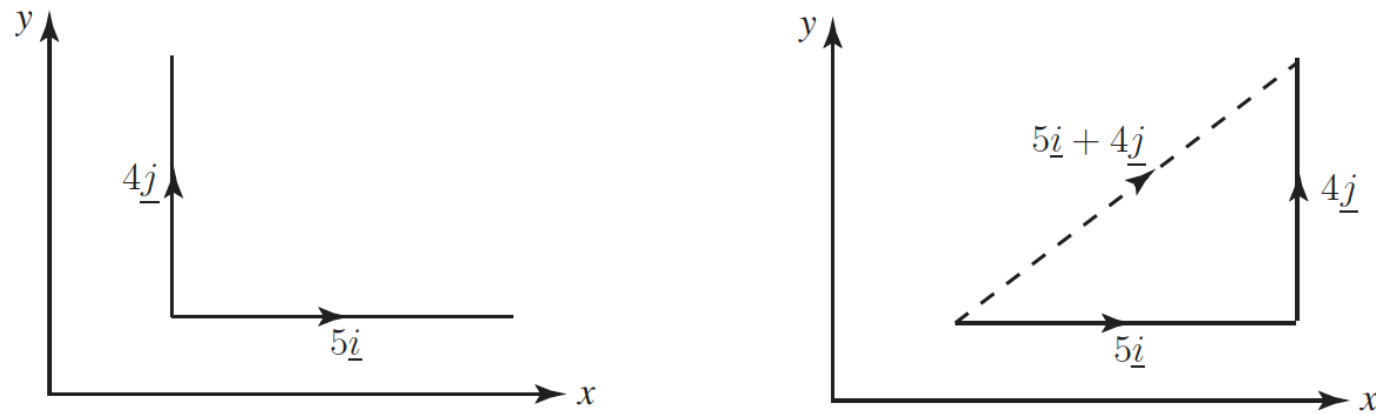


Similarly, a unit vector pointing in the positive y-axis is denoted by $\underline{j}$. So any vector in the direction of the y-axis will be a multiple of $\underline{j}$



Example

Draw the vectors $5\underline{i}$ and $4\underline{j}$. Use your diagram and the triangle law of addition to add these two vectors together. First draw the vectors $5\underline{i}$ and $4\underline{j}$. Then, by translating the vectors so that they lie head to tail, find the vector sum $5\underline{i} + 4\underline{j}$.



It shows a vector $\underline{r} = \overrightarrow{AB}$. We can regard $\underline{r}$ as being the resultant of the two vectors $\overrightarrow{AC} = a\underline{i}$, and $\overrightarrow{CB} = b\underline{j}$. From the triangle law of vector addition

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AC} + \overrightarrow{CB} \\ &= a\underline{i} + b\underline{j}\end{aligned}$$

- We conclude that any vector in the xy plane can be expressed in the form $\underline{r} = \underline{a}i + \underline{b}j$. The numbers a and b are called the components of r in the x and y directions.

Column vector notation

An alternative, useful, and often briefer notation is to write the vector $\underline{r} = \underline{a}i + \underline{b}j$ in column vector notation as

$$\underline{r} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Example

$$\text{Let } \underline{p} = 2\underline{i} + 3\underline{j}, \text{ and } \underline{q} = 5\underline{i} + \underline{j}.$$

- 1- Express $\underline{p}$ and $\underline{q}$ using column vector notation.
- 2- Express the resultant $\underline{p} + \underline{q}$ in terms of $\underline{i}$ and $\underline{j}$.

Answer

$$\underline{p} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \underline{q} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$7\underline{i} + 4\underline{j}$$

Exercise

If $\underline{a} = 9\underline{i} + 7\underline{j}$ and $\underline{b} = 8\underline{i} + 3\underline{j}$ find (a) $\underline{a} + \underline{b}$ (b) $\underline{a} - \underline{b}$

Answer

(a) Simply add the respective components: $17\underline{i} + 10\underline{j}$,

(b) Simply subtract the respective components: $\underline{i} + 4\underline{j}$

Example

State the position vectors of the points with coordinates

(a) $P(2, 4)$, (b) $Q(-1, 5)$, (c) $R(-1, -7)$, (d) $S(8, -4)$.

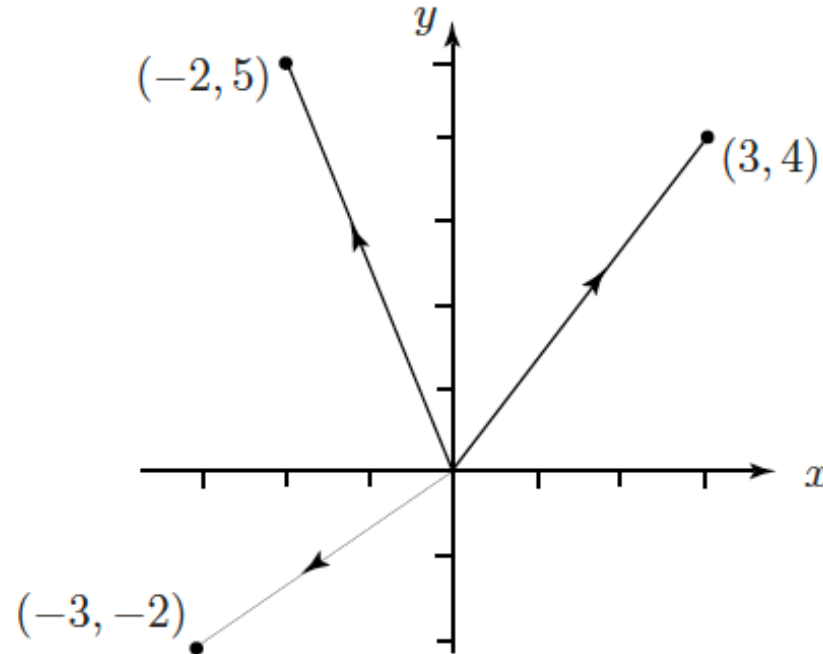
Solution

(a) $2\underline{i} + 4\underline{j}$. (b) $-\underline{i} + 5\underline{j}$. (c) $-\underline{i} - 7\underline{j}$. (d) $8\underline{i} - 4\underline{j}$.

The modulus of a vector

the modulus of $\underline{r}$ is usually denoted by $|\underline{r}|$. When $\underline{r} = \underline{a}i + \underline{b}j$ the modulus can be obtained using Pythagoras' theorem.

$$\text{If } \underline{r} = \underline{a}i + \underline{b}j \text{ then } |\underline{r}| = \sqrt{a^2 + b^2}$$



Example

Find the modulus of each of the vectors

$$\underline{r}_1 = 3\underline{i} + 4\underline{j}, \underline{r}_2 = -2\underline{i} + 5\underline{j}, \underline{r}_3 = -3\underline{i} - 2\underline{j}.$$

Solution

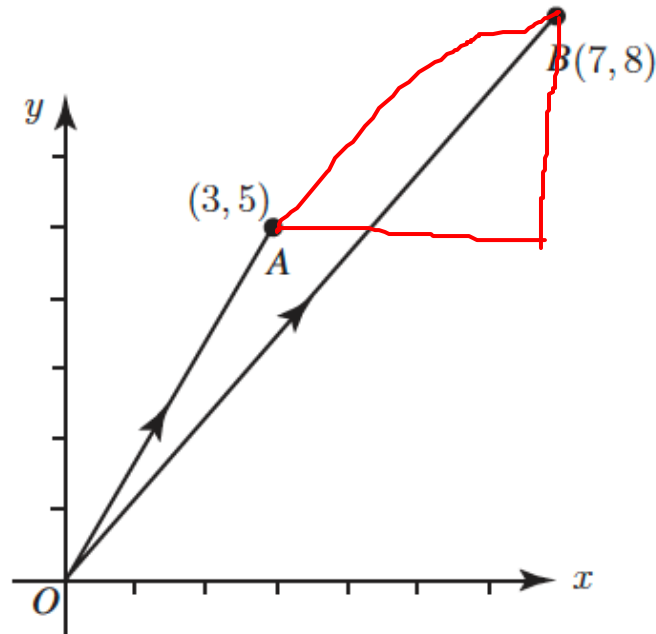
$$(a) |\underline{r}_1| = |3\underline{i} + 4\underline{j}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5. \quad (b) |\underline{r}_2| = \sqrt{(-2)^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}.$$

$$(c) |\underline{r}_3| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}$$

Exercise

Point A has coordinates (3, 5). Point B has coordinates (7, 8).

(a) Draw a diagram which shows points A and B and draw the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$:



(b) State the position vectors of A and B:

Answer

$$\overrightarrow{OA} = \underline{a} = 3\underline{i} + 5\underline{j}, \quad \overrightarrow{OB} = \underline{b} = 7\underline{i} + 8\underline{j}$$

(c) Referring to your figure and using the triangle law you can write $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ so that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. Hence write down an expression for $\overrightarrow{AB}$ in terms of the unit vectors $\underline{i}$ and $\underline{j}$:

Answer

$$\overrightarrow{AB} = (7\underline{i} + 8\underline{j}) - (3\underline{i} + 5\underline{j}) = 4\underline{i} + 3\underline{j}$$

(d) Calculate the length of $\overrightarrow{AB} = 4\underline{i} + 3\underline{j}$:

$$|\overrightarrow{AB}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

Three-dimensional coordinate frames

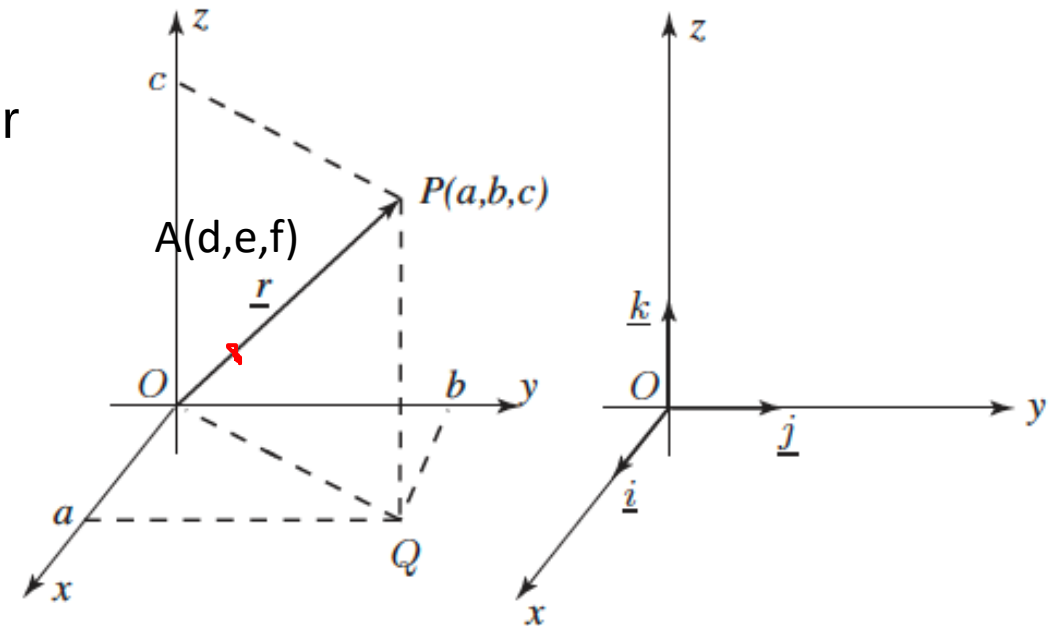
The real world is three-dimensional and in order to solve many engineering problems it is necessary to develop expertise in the mathematics of three-dimensional space.

Any point in the three dimensional space can be defined in terms of its x, y and z coordinates. Consider the point P with coordinates (a, b, c).

The vector from the origin to the point P is known as the position vector of P, denoted $\overrightarrow{OP}$ or $\underline{r}$.

$$\begin{aligned}\underline{r} &= \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP} \\ &= a\underline{i} + b\underline{j} + c\underline{k}\end{aligned}$$

$$|\underline{r}| = \sqrt{a^2 + b^2 + c^2}$$



The length of the vector $\overrightarrow{AP}$

$$|r| = \sqrt{(a - d)^2 + (e - b)^2 + (f - c)^2}$$

Example

Points A , B and C have coordinates $(-1, 1, 4)$, $(8, 0, 2)$ and $(5, -2, 11)$ respectively.

- (a) Find the position vectors of A , B and C .
- (b) Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
- (c) Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$.

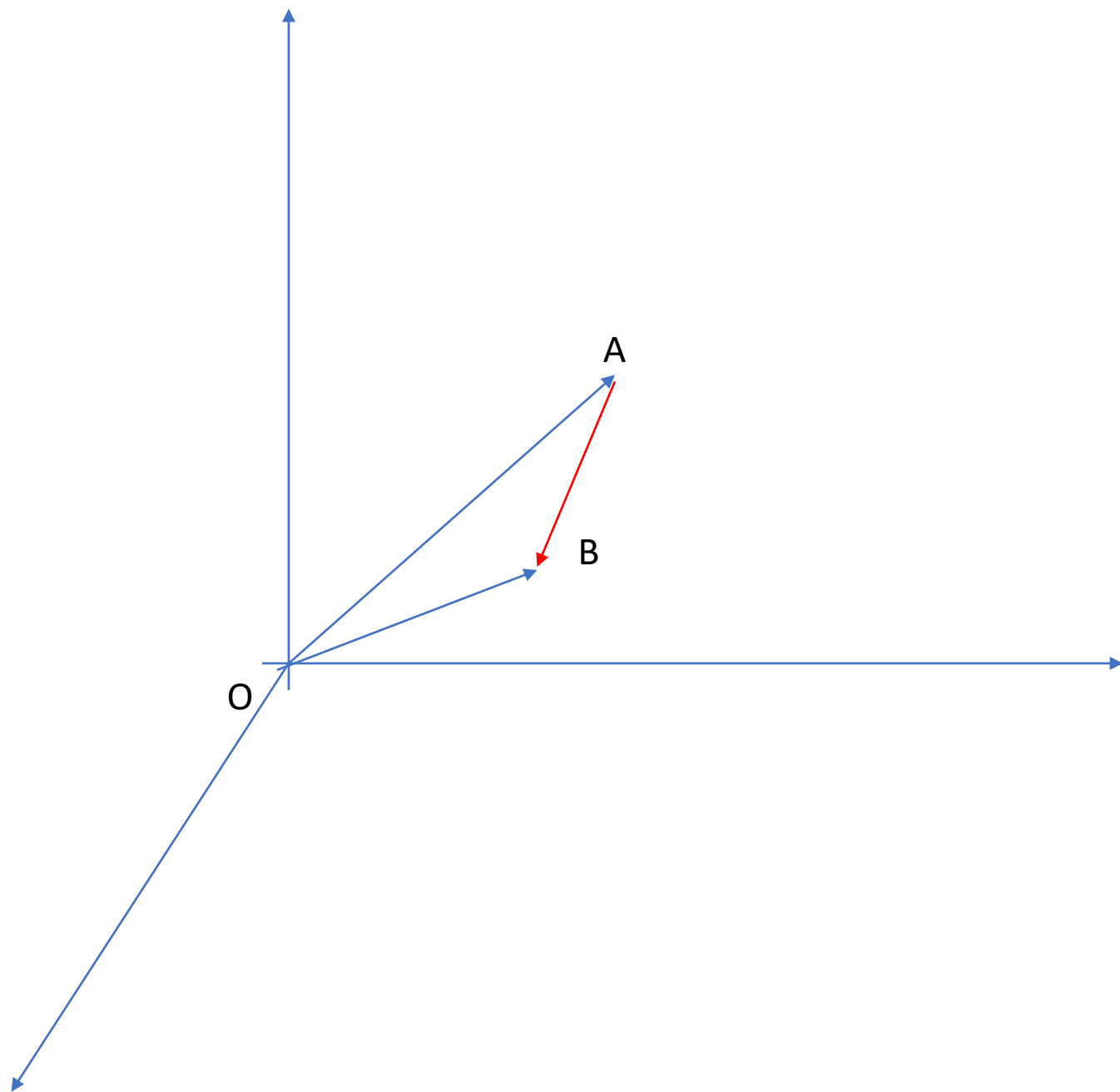
Solution

- (a) Denoting the position vectors of A , B and C by $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively, we find

$$\underline{a} = -\underline{i} + \underline{j} + 4\underline{k}, \quad \underline{b} = 8\underline{i} + 2\underline{k}, \quad \underline{c} = 5\underline{i} - 2\underline{j} + 11\underline{k}$$

(b) $\overrightarrow{AB} = \underline{b} - \underline{a} = 9\underline{i} - \underline{j} - 2\underline{k}$. $\overrightarrow{BC} = \underline{c} - \underline{b} = -3\underline{i} - 2\underline{j} + 9\underline{k}$.

(c) $|\overrightarrow{AB}| = \sqrt{9^2 + (-1)^2 + (-2)^2} = \sqrt{86}$. $|\overrightarrow{BC}| = \sqrt{(-3)^2 + (-2)^2 + 9^2} = \sqrt{94}$.



Exercise

Points P , Q and R have coordinates $(9, 1, 0)$, $(8, -3, 5)$, and $(5, 5, 7)$ respectively.

(a) Find the position vectors $\underline{p}$, $\underline{q}$, $\underline{r}$ of P , Q and R ,

(b) Find $\overrightarrow{PQ}$ and $\overrightarrow{QR}$

(c) Find $|\overrightarrow{PQ}|$ and $|\overrightarrow{QR}|$.

Answers

(a) $\underline{p} = 9\underline{i} + \underline{j}$, $\underline{q} = 8\underline{i} - 3\underline{j} + 5\underline{k}$, $\underline{r} = 5\underline{i} + 5\underline{j} + 7\underline{k}$.

(b) $\overrightarrow{PQ} = -\underline{i} - 4\underline{j} + 5\underline{k}$, $\overrightarrow{QR} = -3\underline{i} + 8\underline{j} + 2\underline{k}$

(c) $|\overrightarrow{PQ}| = \sqrt{42}$, $|\overrightarrow{QR}| = \sqrt{77}$