

Maths (Lecture 11)

Vectors

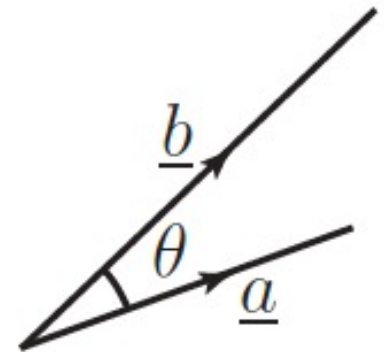
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The Scalar Product

Consider the two vectors $\underline{a}$ and $\underline{b}$, Note that the tails of the two vectors coincide and that the angle between the vectors is labelled θ .

Their scalar product, denoted by $\underline{a} \cdot \underline{b}$, is defined as the product

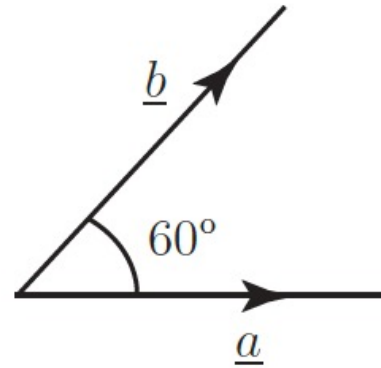
$$|\underline{a}| |\underline{b}| \cos \theta.$$



It is very important to use the **dot** in the formula, that is why the scalar product is also known as the **dot product**.

Example

Vectors $\underline{a}$ and $\underline{b}$ are shown below. Vector $\underline{a}$ has modulus 6 and vector $\underline{b}$ has modulus 7 and the angle between them is 60° . Calculate $\underline{a} \cdot \underline{b}$.



Solution

The angle between the two vectors is 60° . Hence

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta = (6)(7) \cos 60^\circ = 21$$

The scalar product of $\underline{a}$ and $\underline{b}$ is 21. Note that a scalar product is always a scalar.

$$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$$

$$\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{i} = 0$$

$$\underline{i} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0$$

$$\underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{j} = 0$$

A formula for finding the scalar product

We consider vectors in the $x y$ plane. Suppose

$\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$. Then

$$\begin{aligned}\underline{a} \cdot \underline{b} &= (a_1\underline{i} + a_2\underline{j}) \cdot (b_1\underline{i} + b_2\underline{j}) \\ &= a_1\underline{i} \cdot (b_1\underline{i} + b_2\underline{j}) + a_2\underline{j} \cdot (b_1\underline{i} + b_2\underline{j}) \\ &= a_1b_1\underline{i} \cdot \underline{i} + a_1b_2\underline{i} \cdot \underline{j} + a_2b_1\underline{j} \cdot \underline{i} + a_2b_2\underline{j} \cdot \underline{j}\end{aligned}$$

If $\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$ then

$$\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2$$

Example

If $\underline{a} = 7\underline{i} + 8\underline{j}$ and $\underline{b} = 5\underline{i} - 2\underline{j}$, find the scalar product $\underline{a} \cdot \underline{b}$.

Solution

We use Key Point 12:

$$\underline{a} \cdot \underline{b} = (7\underline{i} + 8\underline{j}) \cdot (5\underline{i} - 2\underline{j}) = (7)(5) + (8)(-2) = 35 - 16 = 19$$

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Exercise

If $\underline{a} = 5\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{b} = 8\underline{i} - 9\underline{j} + 11\underline{k}$, find $\underline{a} \cdot \underline{b}$.

Solution

We use the previous formula in

$$\underline{a} \cdot \underline{b} = (5)(8) + (3)(-9) + (-2)(11) = 40 - 27 - 22 = -9$$

Note again that the result is a scalar: there are no $\underline{i}$'s, $\underline{j}$'s, or $\underline{k}$'s in the answer.

Example

If $\underline{r} = 3\underline{i} + 2\underline{j} + 9\underline{k}$ find $\underline{r} \cdot \underline{r}$. Show that this is the same as $|\underline{r}|^2$.

Answer

$$\underline{r} \cdot \underline{r} = (3\underline{i} + 2\underline{j} + 9\underline{k}) \cdot (3\underline{i} + 2\underline{j} + 9\underline{k}) = 3\underline{i} \cdot 3\underline{i} + 3\underline{i} \cdot 2\underline{j} + \dots = 9 + 0 + \dots = 94.$$

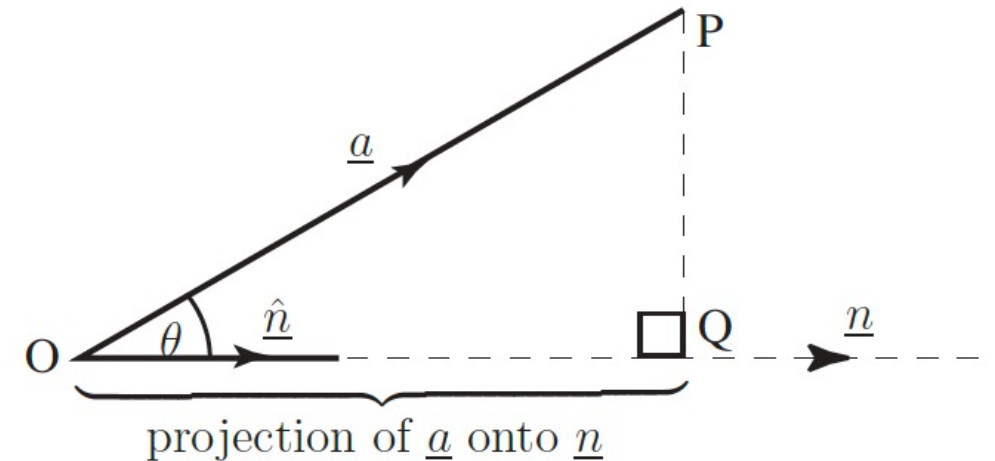
$$|\underline{r}| = \sqrt{9 + 4 + 81} = \sqrt{94}, \text{ hence } |\underline{r}|^2 = \underline{r} \cdot \underline{r}.$$

For any vector $\underline{r}$, $|\underline{r}|^2 = \underline{r} \cdot \underline{r}$

Resolving one vector along another

The scalar product can be used to find the component of a vector in the direction of another vector. Let $\underline{n}$ be a unit vector in the direction of $\underline{n}$.

Study the figure carefully and note that a perpendicular has been drawn from P to meet $\underline{n}$ at Q . The distance OQ is called the **projection** of $\underline{a}$ onto $\underline{n}$.



$$\underline{a} \cdot \underline{\hat{n}} = |\underline{a}| |\underline{\hat{n}}| \cos \theta = |\underline{a}| \cos \theta \quad (\text{since } |\underline{\hat{n}}| = 1)$$

$\underline{a} \cdot \underline{\hat{n}}$ is the component of $\underline{a}$ in the direction of $\underline{n}$

Using the scalar product to find the angle between vectors

We have two distinct ways of calculating the scalar product of two vectors.

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta \text{ whilst from Key Point 13 } \underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Example

Find the angle between the vectors $\underline{a} = 5\underline{i} + 3\underline{j} - 2\underline{k}$ and $\underline{b} = 8\underline{i} - 9\underline{j} + 11\underline{k}$.

Solution

modulus of $\underline{a}$ is $\sqrt{5^2 + 3^2 + (-2)^2} = \sqrt{38}$. The modulus of $\underline{b}$ is $\sqrt{8^2 + (-9)^2 + 11^2} = \sqrt{266}$.
Substituting these values for $\underline{a} \cdot \underline{b}$, $|\underline{a}|$ and $|\underline{b}|$ into the formula for the scalar product we find

$$\begin{aligned} \underline{a} \cdot \underline{b} &= |\underline{a}| |\underline{b}| \cos \theta \\ -9 &= \sqrt{38} \sqrt{266} \cos \theta \end{aligned}$$

from which

$$\cos \theta = \frac{-9}{\sqrt{38}\sqrt{266}} = -0.0895$$

so that $\theta = \cos^{-1}(-0.0895) = 95.14^\circ$

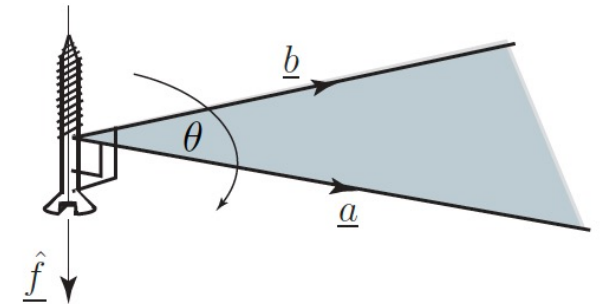
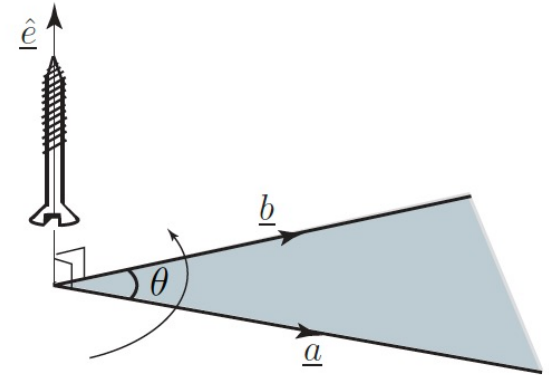
$$\cos \theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|}$$

The Vector Product

The right-handed screw rule:

The two vectors $\underline{a}$ and $\underline{b}$ lie in a plane; this plane, denoted $\underline{e}$, which is perpendicular to this plane.

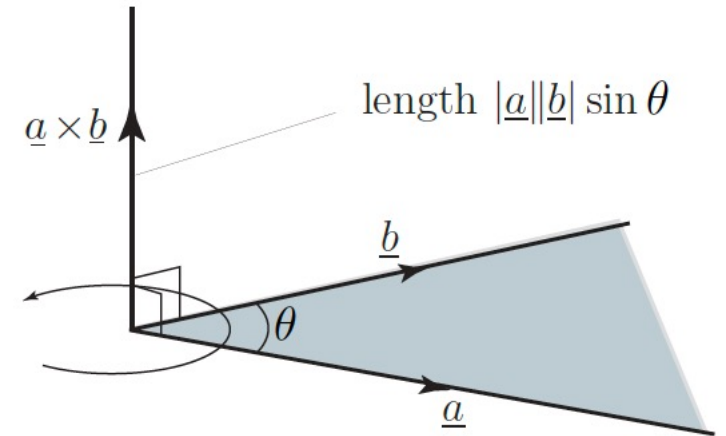
On the other hand, if the right-handed screw is turned from $\underline{b}$ towards $\underline{a}$ the screw will retract in the direction of $\underline{f}$



Definition of the vector product

We define the vector product of $\underline{a}$ and $\underline{b}$, written $\underline{a} \times$

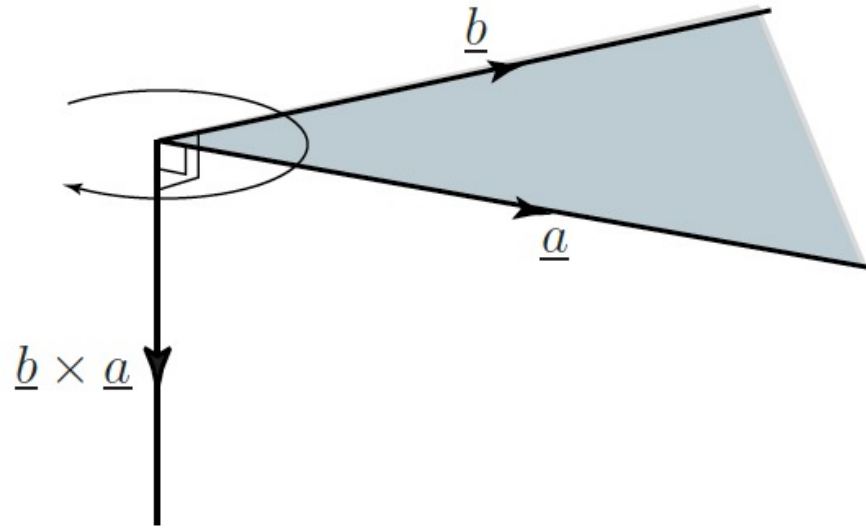
$$\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \theta$$



Where $\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \theta$ Is the magnitude of a vector in the direction of unit vector $\underline{e}$

The vector product of $\underline{a}$ and $\underline{b}$ is: $\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \theta \hat{e}$
The modulus of the vector product is: $|\underline{a} \times \underline{b}| = |\underline{a}| |\underline{b}| \sin \theta$

- $\underline{a} \times \underline{b}$ is not equal to $\underline{b} \times \underline{a}$ because their directions are opposite. In fact $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$.



- If $\underline{a}$ and $\underline{b}$ are parallel, show that $\underline{a} \times \underline{b} = \underline{0}$. Why?

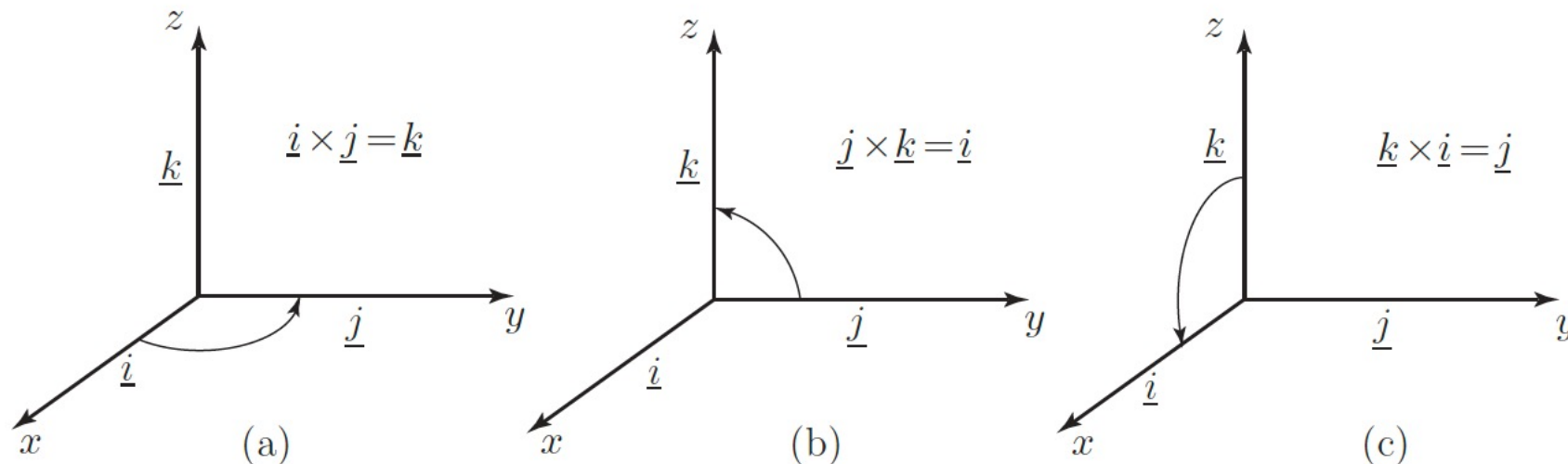
$$\underline{i} \times \underline{i} = \underline{0} \quad \underline{j} \times \underline{j} = \underline{0} \quad \underline{k} \times \underline{k} = \underline{0}$$

Example

Show that $\underline{i} \times \underline{j} = \underline{k}$ and find expressions for $\underline{j} \times \underline{k}$ and $\underline{k} \times \underline{i}$.

Solution

Note that $\underline{i}$ and $\underline{j}$ are perpendicular so that the angle between them is 90° . Also, the vector $\underline{k}$ is perpendicular to both $\underline{i}$ and $\underline{j}$. Using Key Point 17, the modulus of $\underline{i} \times \underline{j}$ is $(1)(1) \sin 90^\circ = 1$. So $\underline{i} \times \underline{j}$ is a unit vector. The unit vector perpendicular to $\underline{i}$ and $\underline{j}$ in the sense defined by the right-handed screw rule is $\underline{k}$ as shown in Figure



$$\underline{i} \times \underline{j} = \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j} \quad \underline{j} \times \underline{i} = -\underline{k}, \quad \underline{k} \times \underline{j} = -\underline{i}, \quad \underline{i} \times \underline{k} = -\underline{j}$$

To help remember these results you might like to think of the vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$ written in alphabetical order like this:

$$\underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{i} \quad \underline{j} \quad \underline{k}$$

Moving left to right yields a positive result: e.g. $\underline{k} \times \underline{i} = \underline{j}$.

Moving right to left yields a negative result: e.g. $\underline{j} \times \underline{i} = -\underline{k}$.

A formula for finding the vector product

Develop a formula for finding the vector product of two vectors given in Cartesian form: Suppose $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\begin{aligned}\underline{a} \times \underline{b} &= (a_1\underline{i} + a_2\underline{j} + a_3\underline{k}) \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) \\&= a_1\underline{i} \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) \\&\quad + a_2\underline{j} \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) \\&\quad + a_3\underline{k} \times (b_1\underline{i} + b_2\underline{j} + b_3\underline{k}) \\&= a_1b_1(\underline{i} \times \underline{i}) + a_1b_2(\underline{i} \times \underline{j}) + a_1b_3(\underline{i} \times \underline{k}) \\&\quad + a_2b_1(\underline{j} \times \underline{i}) + a_2b_2(\underline{j} \times \underline{j}) + a_2b_3(\underline{j} \times \underline{k}) \\&\quad + a_3b_1(\underline{k} \times \underline{i}) + a_3b_2(\underline{k} \times \underline{j}) + a_3b_3(\underline{k} \times \underline{k})\end{aligned}$$

$$\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\underline{a} \times \underline{b} = (a_2b_3 - a_3b_2)\underline{i} - (a_1b_3 - a_3b_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

Example

Evaluate the vector product $\underline{a} \times \underline{b}$ if $\underline{a} = 3\underline{i} - 2\underline{j} + 5\underline{k}$ and $\underline{b} = 7\underline{i} + 4\underline{j} - 8\underline{k}$.

Solution

Identifying $a_1 = 3$, $a_2 = -2$, $a_3 = 5$, $b_1 = 7$, $b_2 = 4$, $b_3 = -8$ we find

$$\begin{aligned}\underline{a} \times \underline{b} &= ((-2)(-8) - (5)(4))\underline{i} - ((3)(-8) - (5)(7))\underline{j} + ((3)(4) - (-2)(7))\underline{k} \\ &= -4\underline{i} + 59\underline{j} + 26\underline{k}\end{aligned}$$

Using determinants to evaluate a vector product

A more convenient and easily remembered method is to use determinants

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

the vector product of two vectors given in Cartesian form: Suppose $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Using determinants to evaluate a vector product

If $\underline{a} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j} + b_3\underline{k}$ then

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}(a_2b_3 - a_3b_2) - \underline{j}(a_1b_3 - a_3b_1) + \underline{k}(a_1b_2 - a_2b_1)$$

Example

Find the vector product of $\underline{a} = 3\underline{i} - 4\underline{j} + 2\underline{k}$ and $\underline{b} = 9\underline{i} - 6\underline{j} + 2\underline{k}$.

Solution

We have $\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & 2 \\ 9 & -6 & 2 \end{vmatrix}$ which, when evaluated, gives

$$\underline{a} \times \underline{b} = \underline{i}(-8 - (-12)) - \underline{j}(6 - 18) + \underline{k}(-18 - (-36)) = 4\underline{i} + 12\underline{j} + 18\underline{k}$$

Lines and Planes

Consider the point $P(4, 5)$ and its position vector $4\mathbf{i} + 5\mathbf{j}$ shown

The **direction ratio** of the vector $\vec{OP}$ is defined to be 4:5.

The **direction cosines** of the vector $\vec{OP}$ are the cosines of the angles between the vector and each of the axes. Specifically, referring to Figure 46 these are

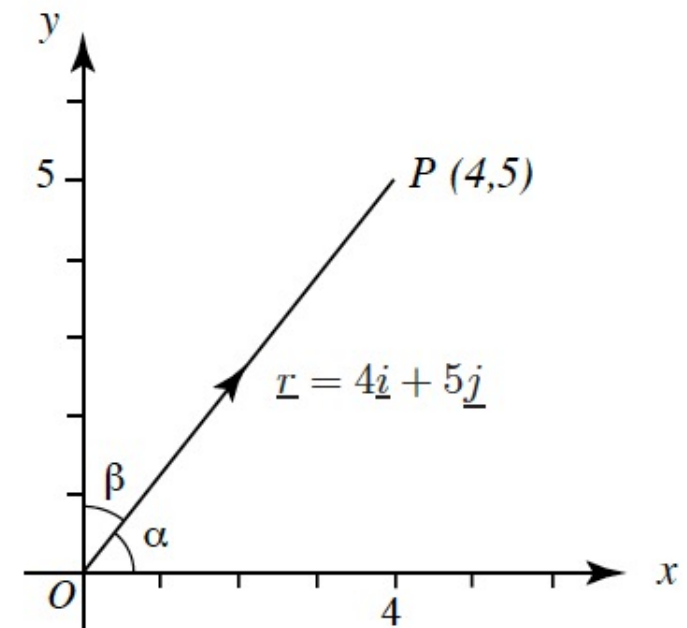
$$\cos \alpha \quad \text{and} \quad \cos \beta$$

Noting that the length of $\vec{OP}$ is $\sqrt{4^2 + 5^2} = \sqrt{41}$, we can write

$$\cos \alpha = \frac{4}{\sqrt{41}}, \quad \cos \beta = \frac{5}{\sqrt{41}}.$$

It is conventional to label the direction cosines as ℓ and m so that

$$\ell = \frac{4}{\sqrt{41}}, \quad m = \frac{5}{\sqrt{41}}.$$



For **any** vector $\underline{r} = a\underline{i} + b\underline{j}$, its direction ratio is $a : b$.

Its direction cosines are

$$\ell = \frac{a}{\sqrt{a^2 + b^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2}}$$

If ℓ and m are the direction cosines of a vector lying in the xy plane, then $\ell^2 + m^2 = 1$

Example

Point A has coordinates $(3, 5)$, and point B has coordinates $(7, 8)$.

- (a) Write down the vector $\overrightarrow{AB}$.
- (b) Find the direction ratio of the vector $\overrightarrow{AB}$.
- (c) Find its direction cosines, ℓ and m .
- (d) Show that $\ell^2 + m^2 = 1$.

Solution

(a) $\overrightarrow{AB} = \underline{b} - \underline{a} = 4\underline{i} + 3\underline{j}$.

(b) The direction ratio of $\overrightarrow{AB}$ is therefore 4:3.

(c) The direction cosines are

$$\ell = \frac{4}{\sqrt{4^2 + 3^2}} = \frac{4}{5}, \quad m = \frac{3}{\sqrt{4^2 + 3^2}} = \frac{3}{5}$$

(d)

$$\ell^2 + m^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$$

Direction ratios and cosines in three dimensions

The direction cosines are the cosines of the angles between the vector and each of the axes. It is conventional to label direction cosines as ℓ , m and n and they are given by

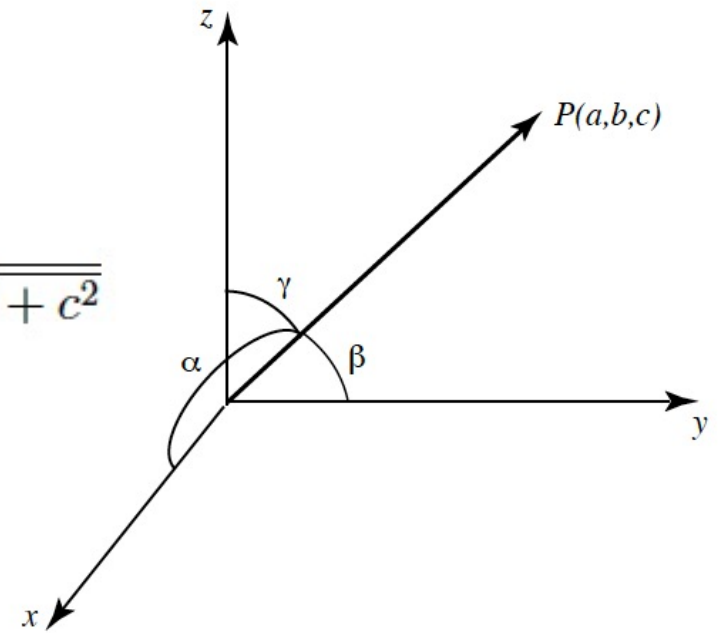
$$\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$$

For any vector $\underline{r} = a\underline{i} + b\underline{j} + c\underline{k}$ its direction ratios are $a : b : c$.

Its direction cosines are

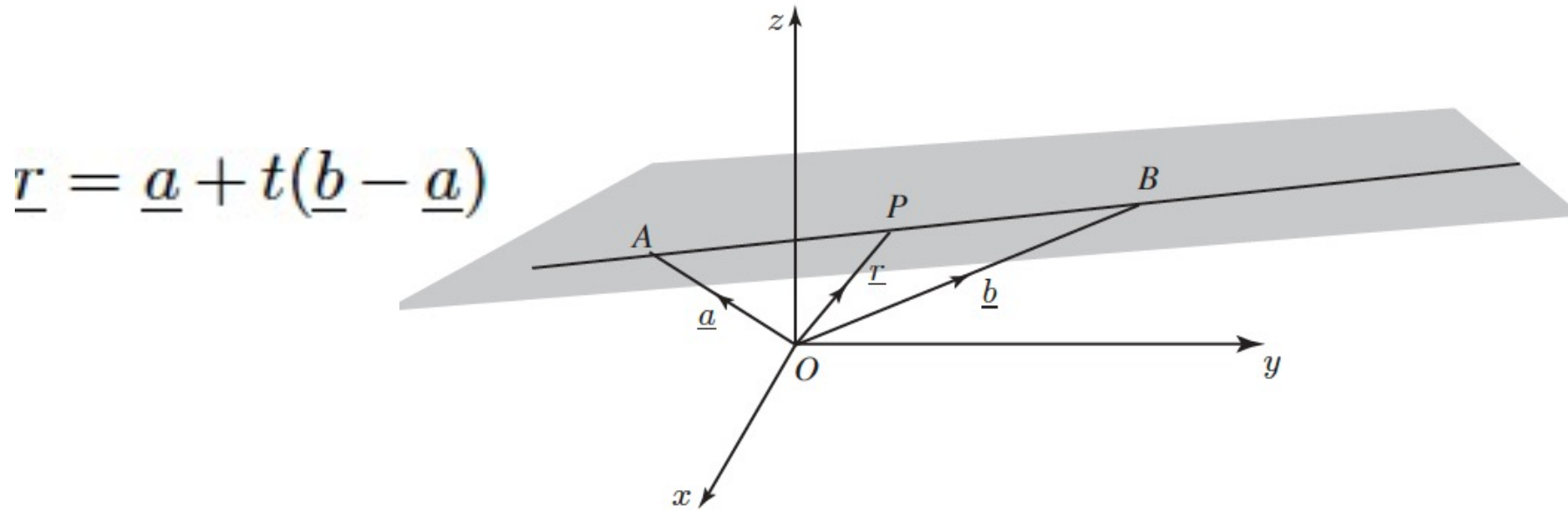
$$\ell = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where $\ell^2 + m^2 + n^2 = 1$



The vector equation of a line

Consider the straight line APB shown in the figure below. This is a line in three-dimensional space



If $0 < t < 1$ the point P lies on the line between A and B . If $t > 1$ the point P lies on the line beyond B (to the right in the figure). If $t < 0$ the point P lies on the line beyond A (to the left in the figure).

Example

Write down the vector equation of the line which passes through the points with position vectors $\underline{a} = 3\underline{i} + 2\underline{j}$ and $\underline{b} = 7\underline{i} + 5\underline{j}$. Also express the equation in column vector form.

Answer

$$\underline{b} - \underline{a} = (7\underline{i} + 5\underline{j}) - (3\underline{i} + 2\underline{j}) = 4\underline{i} + 3\underline{j}$$

The equation of the line is then

$$\begin{aligned}\underline{r} &= \underline{a} + t(\underline{b} - \underline{a}) \\ &= (3\underline{i} + 2\underline{j}) + t(4\underline{i} + 3\underline{j})\end{aligned}$$

Using column vector notation we could write

$$\underline{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

The **Cartesian form** of the equation of the straight line which passes through the points with coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3) is

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

Example

- (a) Write down the Cartesian form of the equation of the straight line which passes through the two points $(9, 3, -2)$ and $(4, 5, -1)$.
- (b) State the equivalent vector equation.

Solution

(a)

$$\frac{x-9}{4-9} = \frac{y-3}{5-3} = \frac{z-(-2)}{-1-(-2)}$$

that is

$$\frac{x-9}{-5} = \frac{y-3}{2} = \frac{z+2}{1} \quad (\text{Cartesian form})$$

(b) The vector equation is

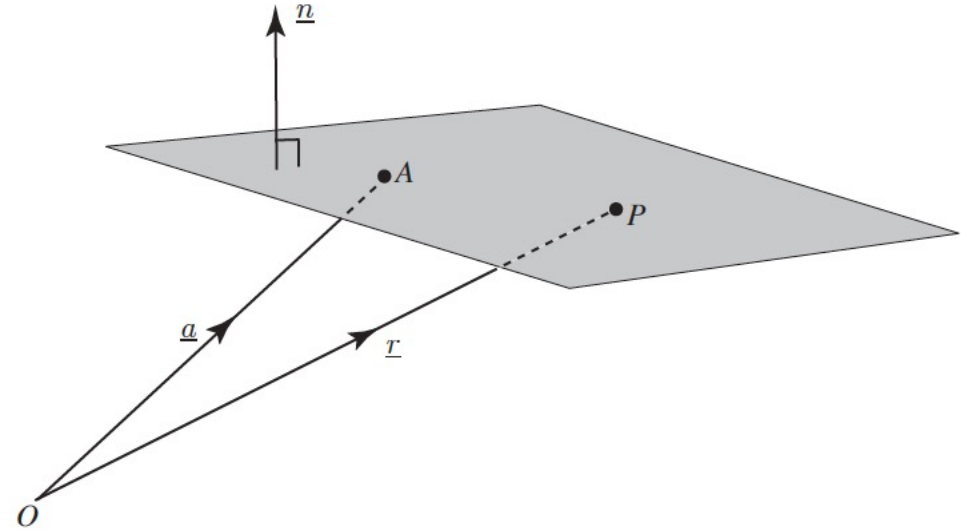
$$\begin{aligned} \underline{r} &= \underline{a} + t(\underline{b} - \underline{a}) \\ &= \begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix} + t \left(\begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix} + t \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} \end{aligned}$$

The vector equation of a plane

Consider this plane:

the vector $\overrightarrow{AP}$ in terms of $\underline{a}$ and $\underline{r}$

$$\underline{r} - \underline{a}$$



Suppose that A is a fixed point in the plane and has position vector $\underline{a}$. Suppose that P is any other arbitrary point in the plane with position vector $\underline{r}$. Clearly the vector $\overrightarrow{AP}$ lies in the plane.

A plane perpendicular to the vector $\underline{n}$ and passing through the point with position vector $\underline{a}$, has equation

$$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$$

A plane with unit normal $\hat{n}$, which is a perpendicular distance d from O is given by

$$\underline{r} \cdot \hat{n} = d$$

Example

- (a) Find the vector equation of the plane which passes through the point with position vector $3\underline{i} + 2\underline{j} + 5\underline{k}$ and which is perpendicular to $\underline{i} + \underline{k}$.
- (b) Find the Cartesian equation of this plane.

Solution

- (a) Using the previous results we can write down the equation

$$\underline{r} \cdot (\underline{i} + \underline{k}) = (3\underline{i} + 2\underline{j} + 5\underline{k}) \cdot (\underline{i} + \underline{k}) = 3 + 5 = 8$$

- (b) Writing $\underline{r}$ as $x\underline{i} + y\underline{j} + z\underline{k}$ we have the Cartesian form:

$$(x\underline{i} + y\underline{j} + z\underline{k}) \cdot (\underline{i} + \underline{k}) = 8$$

so that

$$x + z = 8$$