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A Stochastic Capital Rationing Model

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This paper addresses the issue of optimal project selection for capital expenditures assuming uncertain budgetary allocations. A critical review of the historical development of capital budgeting models indicates two major deficiencies: (i) deterministic models ignore the uncertain nature of capital budgeting problems; and (ii) those models which do incorporate the concept of uncertainty have serious computational problems when applied to larger problems. A stochastic capital rationing model (SCRM) is proposed which makes use of recent developments in stochastic programmes with recourse. This model remains computationally tractable despite the explicit incorporation of uncertainty and the application of theoretically sound penalties for constraint violations. Two problems are introduced which illustrate the model's superiority over comparable deterministic formulations. By varying both the probability distributions of the stochastic constraints and the borrowing rates, it was possible to identify the impact these factors have on project selections.

Key words: capital budget, stochastic linear programme, uncertainty

INTRODUCTION

The issue of capital rationing constitutes one of the major dichotomies between the prescriptions of financial theory and actual capital budgeting activities as practised by many firms in North America. Financial theory states that there are no interdependencies of financial decisions in a perfect market environment, and that the use of a discounted cash-flow criterion for capital investments, as computed using the firm's cost of capital, implies that firms should undertake *all* projects with a positive net present value. This view seems inconsistent with observed corporate capital budgeting practice. Ross¹ found that most firms employ some form of capital rationing, and Mukherjee and Henderson's² survey indicates that 'capital rationing is commonplace' in the 14 studies they have cited. This practice is supported by Hirshleifer³ and Weingartner,⁴ who see capital rationing as a market-imposed limitation or relate it to organizational problems such as the 'Penrose effect', where unrestricted growth rates lead to organizational inefficiencies. In reality, as with most phenomena, capital rationing exists somewhere on a continuum—in this case, between hard rationing as visualized by Weingartner and the unlimited supply of capital of perfect capital market assumptions. The position attained on this continuum, at any given time, is contingent on firm-specific factors, such as financial position and policies, and on various economic factors, such as inflation and domestic and/or foreign money supplies.

The stochastic linear-programming recourse model introduced in this paper recognizes self-imposed capital rationing, such as intrafirm (e.g. divisional) resource allocation, and the existence of realistic borrowing limitations placed on a firm by the market. The concept of capital rationing used in this model is not that of hard capital rationing as it appears in Weingartner's deterministic model. Rather, the stated budgetary constraints are treated in a more realistic manner—that is, as 'hurdles' beyond which the firm will be 'penalized' by increased borrowing-related costs such as higher interest charges and transaction costs.

Prior to the introduction of Weingartner's linear programming model of the capital investment problem in 1963, most models assumed an unlimited supply of capital. Discussions concerning Weingartner's capital expenditures model have focused on three main issues:

- (1) What is the appropriate objective function?
- (2) What is the appropriate discount rate?
- (3) How to accommodate the uncertain nature of capital budgeting.

The objective function contained in Weingartner's model (i.e. maximization of net present value, or NPV) is supported by financial theory, and there is general support of the use of the firm's cost of capital as a discount rate, but attempts to date to develop an operational model which explicitly incorporates uncertainty have been largely unsuccessful. Thus the stochastic capital rationing (SCR) model developed in this paper represents an incremental step in the evolution of capital budgeting models towards the ultimately desirable integration of capital budgeting practice and theory.

The development of a computationally tractable programming algorithm to solve stochastic linear programmes with simple recourse^{5,6} has allowed us to formulate the SCR model which incorporates the uncertain features of corporate capital expenditure decisions within a capital rationing environment. This model explicitly recognizes the uncertainty of cash flows for capital expenditure decisions, as well as the potential violation of stochastic constraints (such as those of a self-imposed budget), and applies an appropriate penalty for such a violation. The penalty for infeasibility is based on the recognition that when a firm borrows beyond certain market-set limits, its cost of capital will be affected. The incremental cost of borrowing the additional funds and the added cost of capital for the firm, which decrease the net present value of all discounted projects, are both incorporated in the calculation of this penalty.

We demonstrate the decision-support capabilities of the SCR model in this paper. There follows a review of capital rationing models and a discussion of their development. The stochastic capital rationing model (SCRM) is developed and formulated, and we demonstrate the effectiveness of SCRM using two problem examples.

LITERATURE REVIEW

The foundation of current research in the area of capital expenditure decisions is based on the works of Lorie and Savage,⁷ Hirshleifer,³ Weingartner,⁸ and Baumol and Quandt.⁹ Weingartner was the first to formulate multi-period capital rationing as a mathematical programming problem maximizing the net discount cash inflows for projects subject to the maintenance of cash outflows and available resources in each period. Weingartner's model can be written as

$$\text{maximize } \sum \sum \{a_{tj}(1+k)^{-t}\}x_j,$$

subject to

$$\begin{aligned} \sum a_{tj}x_j &\leq M_t & t = 0, 1, \dots, T \\ x_j &\geq 0 & j = 1, 2, \dots, J, \end{aligned}$$

where k is the known constant discount rate; a_{tj} is the net cash flow of project j in period t ; x_j is that portion of project j that is accepted, and M_t is the available external cash resources in period t .

Criticisms of this model centre on three factors: selection of the objective function, the appropriate discount rate, and the model's failure to account adequately for uncertainty. Baumol and Quandt,⁹ among the strongest critics of this model, argue that the appropriate discount rate to be used in calculating the net present value of the objective function cannot be determined *a priori* since this rate should be the firm's marginal internal opportunity cost of capital (i.e. Hirshleifer's³ paradox). Baumol and Quandt proposed an objective function which considered the utility of owners' consumption levels but failed to resolve the problem of how to discount the utility of a dollar received in future periods according to Myers.¹⁰ Myers provides important insights into the controversy over an appropriate objective function for the multiperiod capital budgeting problem. He suggests that Hirshleifer's paradox does not apply to corporate capital budgeting decisions; if this is assumed, then Baumol and Quandt's model is essentially the same as Weingartner's model. He further maintains that the firm's objective remains the maximization of its market value (surrogate value), which supports the use of NPV as the appropriate investment criterion.

Myers suggests that the firm's cost of capital can be used as the discount rate in the case of 'hard capital rationing' problems. This view was stated earlier by Lorie and Savage,⁷ and later by Elton.¹¹ Hirshleifer and more recent contributors (Freeland and Rosenblatt¹²) feel that the marginal productivity of capital is the appropriate discount rate. These two seemingly disparate viewpoints have both then integrated within the SCR model. 'Hirshleifer's paradox' has been resolved through the penalty imposed for exceeding the budgeted funds. The calculation of this penalty includes an estimated 'loss' of NPV of the chosen projects, which should be further discounted due to the increased marginal cost of capital, as Hirshleifer has convincingly argued. The firm's average cost of capital, used to calculate the objective-function NPV values in the model, has been calculated as the weighted average cost of capital (WACC). This is the discount rate most frequently used by North American corporations in the last two decades.

Since the capital investment planning process operates in conditions where uncertainty is the rule rather than the exception, the primary weakness of the earlier models is their inability to cope with this uncertainty. Important contributions have been made by Hillier,¹³ Byrne *et al.*¹⁴ and Naslund¹⁵ toward correcting this drawback. One of the early attempts to deal with uncertainty was through the addition of constraints deemed to reduce the uncertain aspects of capital budgeting, such as liquidity requirements and payback period, to Weingartner's formulation. Lockett and Gear¹⁶ formulated the problem using decision trees and stochastic integer programming methods. However, the size of problems that can be considered in their model is severely limited given that the decision tree increases geometrically and becomes unmanageable when solving realistically sized problems. The explicit consideration of uncertainty in project cash flows tends to create an enormous computational burden, and solutions for real problems are difficult to obtain.^{17,18} Those models which incorporate uncertainty and are computationally tractable tend to focus on the individual issues as a series of independent events rather than as interdependent factors when handling capital rationing problems under uncertainty.^{19,20}

Another approach to explicitly recognizing uncertainty in the capital investment planning process is to formulate the problem as a chance-constrained programme. The drawback to these chance-constrained formulations is their inability to handle differential penalties for varying magnitudes of constraint violations and/or for different types of constraints.²¹

Uncertainty has been incorporated into the discounting method with some success; this includes: (1) RADR: a risk-adjusted discount rate,²² and (2) a certainty equivalent (CE) method to reflect investment risk introduced for capital rationing.^{23,24} In general, it has been shown that the CE method is more flexible than RADR since RADR procedures contain an implicit assumption that the degree of uncertainty varies directly with time. Since uncertainty, or risk, is not resolved smoothly over time, RADR is limited in its practical and theoretical application.²⁵ We can summarize the development of capital rationing models by stating that there has been little substantial improvement to the original Weingartner model. Myers has addressed the critical theoretical questions that have been raised concerning Weingartner's model and has demonstrated that Weingartner's model is indeed consistent with financial theory. The SCR model retains many of the essential elements of Weingartner's model and builds on this deterministic framework through the addition of stochastic constraints and penalties for infeasibility. Recent developments in stochastic linear programming have enabled us to approach the problem of capital rationing and issue of uncertainty with explicit consideration of the financial implications of exceeding capital budgeting constraints.

SCRM MODEL

The proposed stochastic capital rationing model (SCRM) is systematic in approach, realistic in formulation and tractable in terms of computation. The main feature of the SCRM is the model's explicit consideration of uncertainty in corporate capital budgets under capital rationing. We have applied theoretical and computational advances to develop an operational stochastic programme with recourse.^{5,6,26–28}

The SCR model has as its objective function the maximization of expected net present values minus the expected minimum recourse penalty for stochastic constraint violations. Both stochastic and deterministic resources are considered in the constraint formulation. The incorporation of the inherent uncertainty characteristic to budgets increases the credibility of the solution and is therefore intuitively appealing to practitioners.

A general formulation of this stochastic linear programme with simple recourse (SLPSR) can be written as

$$\text{maximize } cx - E_{\beta}\{\min(p^+y^+ + p^-y^-)\},$$

subject to

$$Ax = b$$

$$Tx + I(y^+ - y^-) = \beta$$

$$x, y^+, y^- \geq 0,$$

where $c, x \in R^n$, $y^+, y^-, p^+, p^- \in R^m$, A is m_1 by n , T is m_2 by n , I is an m_2 -dimensional identity matrix, and β is an m_2 -dimensional random variable distributed independently of x on the probability space (Ω, F, μ) .^{5,6,26,27}

For the proposed model we have

$$\text{maximize } \sum_i \left\{ \sum_j [c_{ij}(1+k)^{-i}]x_j - \min \sum_l (p_i^+ y_{il}^+ + p_i^- y_{il}^-)P(\beta_{il}) \right\},$$

subject to

$$\begin{aligned} \sum a_{hj}x_j &= b_h & \text{for } h = 1, \dots, m_1, \\ \sum t_{ij}x_j + y_{il}^+ - y_{il}^- &= \beta_{il}, & \text{for } i = 1, \dots, m_2, \quad l = 1, \dots, L_i, \\ x_j, y_{il}^+, y_{il}^- &\geq 0, & j = 1, \dots, n, \end{aligned}$$

where the summation over i in the objective function is over the time periods only;

- y_{il}^+ = the shortage variable for the l th realization of the i th resource;
- y_{il}^- = the surplus variable for the l th realization of the i th resource;
- β_{il} = the l th realization of the i th resource;
- c_{ij} = the net cash flow for the j th project in the i th time period;
- a_{hj} = the coefficient for the j th variable in the h th deterministic constraint;
- b_h = the resource constraint for the h th deterministic constraint;
- t_{ij} = the coefficient for the j th variable in the i th stochastic constraint; and
- k = the discount rate.

The first m_1 constraints are deterministic. They represent limitations which cannot be altered within the planning horizon, such as physical capacity and human resources, inflexible policy-driven constraints and 0, 1 constraints, etc. The constraints $(m_1 + 1)$ to $(m_1 + m_2)$ are stochastic and represent such uncertain limitations as cash-flow limitations, flexible policy decisions and other forecast-dependent variables such as economic conditions, expected revenues, sales, etc. The above problem can be solved using Wets' algorithm under the assumption that $p_{il}^+ + p_{il}^- > 0$.

We note that, with this formulation, one can deal with the violation of budget constraints directly, but that the appropriate penalty costs associated with the shortage and/or surplus of funds must still be determined. In addition, we also note that inherent in the LP formulation is the possibility of undertaking a fraction of a project. Examples of these types of projects include capital investment decisions by pension funds, the partial construction of warehouse space, etc. (see Brealey and Myers,²⁹ p. 105 for a detailed discussion of these types of investments). Although there are many instances where the integrality of investment decisions is essential—such as purchasing a crane or building a bridge—a linear programming approach is desirable for two reasons: the number of potential projects accepted is bounded by the number of budget constraints, and by the efficiency of LP algorithms in terms of computer time.⁸

COST OF CAPITAL AND PENALTY COSTS

Cost of capital

If a firm is faced with an abundance of potential investment projects but insufficient funds to invest in all of them, under the perfect market assumption the appropriate decision for the firm is to seek outside funding and continue to invest, provided that the cost of these funds is less than the anticipated return of the investment projects (i.e. they all have positive NPV). The firm's appropriate cost of equity is K_e , as derived using the capital asset pricing model, or CAPM.³⁰ However, since a capital rationer does not operate in a perfect market, the firm's capital structure does impact on its ability to raise funds. The cost of financing individual projects is not calculated solely from the cost of debt or from the cost of equity, but is derived from a pool of funds whose weighted average cost of capital (WACC) is determined by the firm's capital structure. In cases where the adoption of a project (or projects) radically alters the firm's Beta, the effects of the increased perceived risk should be incorporated into the firm's capital cost. Note that in this

formulation, we assume that the potential projects have the same inherent risk as the firm. Also we assume that the WACC remains constant throughout the planning horizon. Although this is not an ideal way of treating WACC and its impact on selection of investment projects during the planning horizon, the explicit incorporation of WACC into the model will result in considerable computational difficulties. Therefore, the constant value is used as an estimate of the WACC over time.

Penalties for the recourse variables

Penalties are incurred where the stochastic (budget) constraint is either under- or over-utilized; that is, either where the expenditures for a given period required for the investments chosen in the solution vector exceed the maximum cash outflows (i.e. a surplus condition), or where there are insufficient planned outflows for the chosen investments given the funds available for that period (i.e. a shortage condition exists).

Penalty for the Surplus Variable. The 'surplus' condition is defined as that situation where the planned cash outflows exceed the budget: that is, there exists a 'surplus' of cash outflows for the constraint imposed by capital rationing, i.e. $y_{ii}^+ = 0$, and

$$y_{ii}^- = \sum t_{ij}x_j - \beta_{ii} > 0.$$

When a surplus occurs, the firm's cost for the additional funds required to support these projects is a function of the marginal interest rate of borrowing for the firm. We assume that in practice, the marginal cost of these funds will be greater than WACC, given that the funds available to the firm without increasing risk to the overall corporation have already been included in the budgetary constraint. Having already exhausted the low cost sources, the firm would be forced to seek more expensive methods of financing additional expenditures. This penalty cost must reflect the fact that the firm has access to these additional funds for the duration of the chosen planning horizon of the model. All net cash flows for this loan, including the calculated interest payments to be paid in each period, will be discounted (using WACC) to present value. The penalty per dollar borrowed in period i will therefore be

$$p_i^- = \sum_{t=i}^T \frac{(k+a)}{(1+k)^t} + \frac{1}{(1+k)^T} - \frac{1}{(1+k)^{i-1}}$$

(i.e. the interest paid per period $(k+a)$ plus the principal paid back at the end of the horizon are treated as cash outflows at the end of the periods in question and the original cash inflow is assumed to be received at the beginning of period i). All of these net cash flows are discounted at k , the cost of capital of the firm, where a is the premium cost over k (WACC) to borrow the funds, i is the period in which the surplus occurs, and T is the number of periods in the planning horizon.

The total surplus penalty for the entire planning horizon will be

$$\sum_i \sum_t p_i^- y_{ii}^- P(\beta_{ii}).$$

Penalty for the Shortage Variable. The 'shortage' condition is defined as that situation where the budget exceeds the actual cash outflows of the solution vector: that is, there exists a 'shortage' of cash outflows for the constraint imposed by capital rationing, i.e. $y_{ii}^- = 0$, and

$$y_{ii}^+ = \beta_{ii} - \sum t_{ij}x_j > 0.$$

In this case, the firm has the option to invest these excess funds in a short-term investment. The 'penalty' to the firm, therefore, for not investing these funds which were allocated for a particular period is a net 'gain' to the firm in the period in which they are invested. This net gain must be added to the objective value to determine the overall net profit to the firm. The rate of return for this short-term investment is assumed to be at least the risk-free rate but lower than WACC. Hence it will be less profitable than an investment with a positive NPV. The penalty associated

with not utilizing the available funds in period i owing to a lack of appropriate projects must be calculated as a function of the return that those funds generate in a short-term investment. It is assumed that this will provide a return comparable to the market risk-free rate of return, compounded annually. It is also assumed that this shortage condition will continue for the rest of the planning horizon, that is, funds that are not utilized for project investment in any given period will be invested in a lower-return alternative investment and will remain in that investment for the remainder of the planning horizon. The 'penalty' will consist of the present value of the compound interest earned on these funds for the remainder of the planning horizon less the present value of the original cash outflow for the 'loan'. As an approximation, one can consider the following:

$$p_i^+ = \frac{1}{(1+k)^{i-1}} - \frac{(1+k-b)^{T-i+1}}{(1+k)^T}$$

(i.e. the original cash outflow of the shortage amount is assumed to occur at the beginning of period i and the cash inflow, including compounded interest at $(k-b)$, is received at the end of the planning horizon). They are both discounted to the present value, where $k-b \geq r_f$, r_f is the risk-free rate and b is the discount (below WACC) which gives the rate at which the excess funds can be invested. The total shortage penalty for the entire planning horizon will therefore be

$$\sum_i \sum_j p_i^+ y_{ij}^+ P(\beta_{ij}).$$

APPLICATIONS OF THE SCRM MODEL

In this section we demonstrate the ease of use of the SCR model and present two problems where we compare the solutions provided by the SCR model to those of an equivalent deterministic capital rationing model. The larger, more realistically-sized problem (2) illustrates that, from both a modelling and a computational point of view, the SCR formulation does not result in an 'explosion' of difficulties generally associated with problems of a larger magnitude, unlike most other realistic stochastic formulations. Both problems have been formulated using cash-flow constraints alone. Other types of constraints, such as other resource limitations or internal policies, have been omitted in order to observe the impact of fluctuating borrowing rates and varying probability distributions on the solution.

Problems 1 and 2 are solved using various combinations of these parameters, including the following:

- (i) Symmetric and skewed (right and left) probability distributions for the cash-flow constraints. Each element of the random vector has a five-point probability distribution. The mean and standard deviation for each element are held constant in order to determine the effect of various probability distributions on the solution.
- (ii) Penalties reflecting differing market conditions, such as money supply and prevailing interest rates, and firm-specific conditions, namely borrowing ability.

Our initial conjectures concerning the economic impact of the varying parameters on the SCR model solutions were as follows:

- (i) With regard to the varied probability distributions, the symmetrical distribution was expected to produce a net value nearest to that of the deterministic model, since the greatest mass is placed on the expected value in this scenario. We also conjectured that the left-skewed probability distribution will give a solution with the greatest improvement over the deterministic model, given the high mass placed on higher cash-flow values.
- (ii) The various borrowing rates with which the penalty sets are calculated were also expected to impact the 'spread' between the SCR model optimal net value and that of the deterministic model. We conjecture that the lower the borrowing rate, the greater the improvement of the stochastic solution over the equivalent deterministic solution when the 'net' value (i.e. net of the predicted penalty) of the two solutions is compared. The lower borrowing rate 'softens' the hard rationing approach of the deterministic model and renders a more aggressive investment solution when a low borrowing rate makes this economically viable for the firm. Hence, the stochastic model which optimizes an investment decision reflecting a lower penalty for

borrowing can capitalize on this opportunity, unlike an inflexible deterministic model which cannot explicitly incorporate information such as borrowing opportunities.

The optimal portfolio for each scenario provided by the SCR model is compared to the optimal portfolio generated by an equivalent deterministic model. Both solutions are subjected to the same probability distribution and penalties. The difference in the resulting net values (the objective value less the corresponding penalty) determines the degree to which the SCR model provides a superior solution.

To summarize, we shall demonstrate in this section:

- (i) the ease of use of the SCR model;
- (ii) the superiority of the SCR model results over those obtained using an equivalent deterministic linear programming model; and
- (iii) that the order of computing time of the SCR model is comparable to the deterministic equivalent model.

Problem 1

The firm is faced with a pool of 12 capital investment projects which are examined over a 5-year horizon and have as their constraints a limited capital budget in periods 1 and 2 as well as a target cash inflow for periods 3, 4 and 5. The objective function coefficients (NPV) have been calculated using a cost of capital of 10%. Table 1 presents the relevant data for each project, including the IRR (internal rate of return), the net cash outflows or inflows for each period. Table 2 displays the penalties which have been calculated for each period based on differing borrowing rates of 13%, 12%, 11% and 10.5% (note that p^+ values are based on lending rates (8%) for periods 1 and 2 where cash outflows are constrained and on borrowing rates for periods 3 to 5 where minimum targets have been established). The cash-flow constraints in periods 1 and 2 have an expected value of 350 and 50 respectively and represent the capital rationing imposed for these 2 years on capital investment expenditures. The stochastic cash-flow constraints for periods 3–5, with target cash inflows of 90, 290 and 245 respectively, represent the minimum required cash needed by the firm for other expenditures which will be financed by these projects.

TABLE 1. IRR, NPV and cash flows for problem 1

Project	IRR	NPV	Net cash flows by period				
			1	2	3	4	5
1	0.24	19.50	-45	-15	40	60	0
2	0.17	14.24	-90	25	35	70	0
3	0.20	12.36	-50	0	30	50	0
4	0.16	11.57	-60	-10	10	60	40
5	0.13	11.29	-100	-10	25	60	80
6	0.16	11.03	-40	-35	30	50	30
7	0.15	9.50	-60	5	10	30	50
8	0.12	6.32	-80	-15	35	40	60
9	0.12	5.53	-75	0	-10	50	75
10	0.13	5.06	-30	-20	-5	40	40
11	0.11	1.64	-35	-10	-5	30	40
12	0.10	0.80	-54	-20	-15	50	70
Totals		108.84	-719	-105	180	590	485

TABLE 2. Surplus and shortage penalties for problem 1

Period	Borrowing rates							
	13%		12%		11%		10.5%	
	P^+	P^-	P^+	P^-	P^+	P^-	P^+	P^-
1	0.0877	0.1137	0.0877	0.0758	0.0877	0.0379	0.0877	0.0190
2	0.0643	0.0865	0.0643	0.0576	0.0643	0.0288	0.0643	0.0144
3	0.0617	0.0443	0.0411	0.0443	0.0206	0.0443	0.0103	0.0443
4	0.0391	0.0271	0.0261	0.0271	0.0130	0.0271	0.0065	0.0271
5	0.0186	0.0124	0.0124	0.0124	0.0062	0.0124	0.0031	0.0124

All cash flows discounted at $k = 10\%$.

Lending rate set at 8%.

Problem 2

In this problem the firm is faced with a pool of 20 capital investment projects and has as its constraints a limited capital budget for periods 1, 2 and 3 of 450, 250 and 60. Target cash inflows for periods 4–10 have been established as 235, 250, 280, 210, 130, 125 and 120 respectively. Table 3 contains the relevant data included in this problem, and Table 4 indicates the penalty set established for each period in accordance with the pattern established for problem 1.

TABLE 3. IRR, NPV and cash flows for problem 2

Project	IRR	NPV	Net cash flows by period									
			1	2	3	4	5	6	7	8	9	10
1	0.17	27.16	−50	−50	15	25	35	45	35	25	15	0
2	0.14	20.10	−80	−35	−35	40	70	70	70	0	0	0
3	0.13	19.87	−70	−60	−60	45	45	50	50	50	50	50
4	0.17	16.17	−35	−45	25	25	25	30	30	0	0	0
5	0.12	16.03	−50	−30	−30	0	0	0	60	60	60	60
6	0.18	15.99	−5	−45	−10	30	15	15	15	20	20	0
7	0.12	13.99	−100	−30	−10	35	35	40	40	40	40	30
8	0.22	13.40	−50	20	20	20	20	0	0	0	0	0
9	0.12	10.22	−20	−30	−40	−15	20	25	30	35	40	45
10	0.15	8.41	−55	0	25	35	15	10	0	0	0	0
11	0.13	8.22	−40	−10	5	10	10	15	15	15	15	15
12	0.14	7.88	−30	−50	25	25	25	25	20	0	0	0
13	0.13	7.42	−30	−20	−20	10	20	30	30	30	0	0
14	0.13	7.34	0	−50	−25	−25	40	50	60	0	0	0
15	0.11	3.46	−60	−25	5	20	25	35	50	0	0	0
16	0.12	2.60	−10	−20	−20	20	20	30	0	0	0	0
17	0.10	1.52	−100	10	15	15	15	15	25	25	25	25
18	0.11	1.28	−30	−10	−10	15	20	20	20	0	0	0
19	0.11	0.84	−55	5	10	30	30	0	0	0	0	0
20	0.10	−1.92	−75	−50	−30	20	30	40	40	40	40	50
Totals		199.97	−945	−525	−145	380	515	545	590	340	305	275

TABLE 4. Surplus and shortage penalties for problem 2

Period	Penalties for each period							
	13%		12%		11%		10.5%	
	P ⁺	P [−]	P ⁺	P [−]	P ⁺	P [−]	P ⁺	P [−]
1	0.1676	0.1843	0.1676	0.1229	0.1676	0.0614	0.1676	0.0307
2	0.1384	0.1571	0.1384	0.1047	0.1384	0.0524	0.1384	0.0262
3	0.1128	0.1323	0.1128	0.0882	0.1128	0.0441	0.1128	0.0220
4	0.1097	0.0906	0.0732	0.0906	0.0366	0.0906	0.0183	0.0906
5	0.0892	0.0712	0.0595	0.0712	0.0297	0.0712	0.0149	0.0712
6	0.0706	0.0544	0.0471	0.0544	0.0235	0.0544	0.0118	0.0544
7	0.0537	0.0399	0.0358	0.0399	0.0179	0.0399	0.0089	0.0399
8	0.0383	0.0275	0.0255	0.0275	0.0128	0.0275	0.0064	0.0275
9	0.0243	0.0168	0.0162	0.0168	0.0081	0.0168	0.0040	0.0168
10	0.0116	0.0077	0.0077	0.0077	0.0039	0.0077	0.0019	0.0077

All cash flows discounted at $k = 10\%$.
Lending rate set at 8% .

ANALYSIS OF THE RESULTS

Tables 5 and 6 contain the resulting solution vectors of problems 1 and 2 respectively. Tables 7 and 8 contain the improvement of the stochastic solution over the deterministic solution in terms of absolute dollars and percentage increase of the objective value net of expected penalties for each set of parameters. The results were obtained by running the SCR model on a CDC 830D main-frame computer, and the processing time averaged approximately 3.5 c.p.u. seconds per solution for problem 1 and approximately 8 c.p.u. seconds for problem 2. This is in the order of twice the time required for the equivalent deterministic problems.

Prior to analysing the economic impact of varying the parameters (i.e. penalties and probability distributions), we note that there are significant improvements in the optimal values attained by

TABLE 5. Solution vectors for problem 1

Det. soln borrowing rate		Right-skewed				Symmetrical				Left-skewed			
		13	12	11	10.5	13	12	11	10.5	13	12	11	10.5
X1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	0.804	1	1	1	1	1	1	1	1	1	1	1	1
X3	1	1	1	1	1	1	1	1	1	1	1	1	1
X4	1	1	1	1	1	1	1	1	1	1	1	1	1
X5	0	0	0.05	0.52	0.52	0	0.05	0.52	0.52	0	0.2	0.52	0.52
X6	1	1	1	1	1	1	1	1	1	1	1	1	1
X7	1	1	1	1	1	1	1	1	1	1	1	1	1
X8	0	0	0	0	0	0	0	0	0	0	0	0	0
X9	0	0	0	0	0	0	0	0	0	0	0	0	0
X10	0.755	1	1	1	1	1	1	1	1	1	1	1	1
X11	0	0	0	0	0	0	0	0	0	0	0	0	0
X12	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 6. Solution vectors for problem 2

Det. soln borrowing rate		Right-skewed				Symmetrical				Left-skewed			
		13	12	11	10.5	13	12	11	10.5	13	12	11	10.5
X1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	1	1	1	1	1	1	1	1	1	1	1	1	1
X3	1	1	0.99	0.96	1	1	1	0.97	1	1	1	0.97	1
X4	1	1	1	1	1	1	1	1	1	1	1	1	1
X5	0.37	0.74	1	1	1	0.6	1	1	1	0.54	1	1	1
X6	1	1	1	1	1	1	1	1	1	1	1	1	1
X7	0	0	0	0	0	0	0	0	0	0	0	0	0
X8	1	1	1	1	1	1	1	1	1	1	1	1	1
X9	0.37	0.52	0.72	1	1	0.35	0.74	1	1	0.29	0.77	1	1
X10	1	1	1	1	1	1	1	1	1	1	1	1	1
X11	1	1	1	1	1	1	1	1	1	1	1	1	1
X12	0	0.19	0.45	0.83	1	0.1	0.51	0.85	1	0.01	0.59	0.87	0.98
X13	0.85	0.34	0	0	0.17	0.77	0	0	0.16	0.92	0	0	0.02
X14	0	0	0	0	0	0	0	0	0	0	0	0	0
X15	0	0	0	0	0	0	0	0	0	0	0	0	0
X16	0	0	0	0	0	0	0	0	0	0	0	0	0
X17	0	0	0	0	0	0	0	0	0	0	0	0	0
X18	0	0	0	0	0	0	0	0	0	0	0	0	0
X19	0	0	0	0	0	0	0	0	0	0	0	0	0
X20	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 7. Problem 1: stochastic vs deterministic net values

Distribution		Borrowing rates			
		13	12	11	10.5
Symmetrical	\$	2.03	2.70	5.22	6.54
	%	2.7	3.6	7.0	8.7
Right-skewed	\$	2.40	3.00	5.46	6.75
	%	3.2	4.0	7.3	9.0
Left-skewed	\$	2.40	3.02	5.46	6.75
	%	3.2	4.0	7.3	9.0

TABLE 8. Problem 2: stochastic vs deterministic net values

Distribution		Borrowing rates			
		13	12	11	10.5
Symmetrical	\$	1.35	3.23	7.50	10.23
	%	1.0	2.4	5.4	7.4
Right-skewed	\$	1.74	3.63	7.73	10.43
	%	1.3	2.7	5.6	7.5
Left-skewed	\$	1.91	3.55	7.77	10.52
	%	1.4	2.6	5.6	7.6

the SCR model compared with the values generated by the equivalent deterministic formulation. This improvement was achieved without an increase in the order of computational difficulty. In addition, we note that the solution vectors generated by the SCR model under varying sets of parameters are relatively robust. However, when these are compared with the solutions generated by the deterministic problem, there appears to be a significant difference.

Varying probability distributions

To assess differences in optimal portfolio composition and profitability due to differences in budgetary distributions, both symmetrical and asymmetrical distributions were considered. The results are summarized in Tables 5–8. The first observation is that asymmetrical distributions result in larger improvements in optimal values, regardless of the borrowing rates. This result is reasonable since the symmetrical distributions are ‘better’ represented than the asymmetrical distributions by mean values. The distribution was found to have a larger effect on the solution vectors in problem 2, where the project ‘pool’ is larger and more varied in character (i.e. cash-flow patterns, NPV and IRR) than in problem 1. The improvement in net value in problem 1 was derived mainly from the smaller penalties incurred with the asymmetrical distributions, whereas the improvement in problem 2 was based on both decreased penalties and differing solution vectors. As can be seen in Table 6, four different projects (5, 9, 12 and 13) are selected in differing proportions, depending on the different combinations of penalty sets and distributions.

Varying penalties

The SCR solution vector is quite sensitive to changes made on shortage and surplus penalties. The percentage of partially accepted projects tends to change considerably, as shown in Tables 5 and 6. However, those projects which have been accepted as a whole tend to remain in the solution vector in all cases in their entirety.

In this model, impacts due to changes made in the borrowing or lending rate are reflected implicitly in the calculation of penalties, as explained earlier. The penalty sets representing borrowing rates of 10.5% and 11% (i.e. $\frac{1}{2}$ to 1% above WACC) represent the situation most frequently encountered in practice. It is unlikely that the borrowing rate would exceed WACC by a very wide margin. As is indicated in Tables 7 and 8, the highest percentage improvement of the SCR model over the equivalent deterministic model solution was gained where the two lowest borrowing rates were incorporated in the penalties. The improvements range from 5.4% to 7.3% for the 11% borrowing rate, and from 7.4% to 9% where borrowing is assumed to be at 10.5%.

The most striking observation, however, is that the dominant factors influencing portfolio selection and profitability are those factors represented by the penalties. That is, the economic factors dominate the probability distributions, which is consistent with our earlier conjectures.

CONCLUSION

Upon review of capital budgeting literature, we identified two major weaknesses of the existing methodologies: either an absence of the explicit considerations of uncertainty, or, where uncertainty is considered, computations encountered in determining the optimal solution become intractable.

First, we formulated the capital rationing problem as an SLPSR problem and developed penalties for the recourse variables on the base of sound financial theory. After generating a set of probability distributions, coupled with the penalty sets established, we were able to analyse the results and identify the critical factors influencing the selection and structure of the optimal solution of the model.

The findings from the analyses of the empirical results indicate that the factors we have examined have a substantial influence on the marginal rate of improvements on the optimal solution of SCR over the corresponding deterministic LP solution:

- (i) The borrowing rate is the predominant factor influencing the solutions structures (i.e. investment project choices), and this result is consistent with both practice and the literature.
- (ii) The uncertainty distribution with the highest improvements of the stochastic solution over the deterministic solution results, net of penalties, was the negatively skewed probability

distribution. The symmetrical distributions yielded results most similar to the deterministic solution in problem 1, as was predicted. The greatest disparity between solution vectors and net economic impact on the firm is found in the results tabulated for the largest problem (2), where the larger number and heterogeneous nature of the projects led to more distinctive results for each of the differing uncertainty distributions.

It can therefore be concluded that the firm-specific (i.e. non-systematic) risk of a firm is an important determinant in optimal investment decisions. The life-cycle stage of the firm and industry is an important determinant of these factors and may be used to establish appropriate probability distributions for a given firm.

Given these observations, it is clear that the optimal investment decisions based on deterministic linear programming models are unable to reflect the complex, interdependent financing and risk factors. The SCR model presented in this paper has been able to incorporate some of the factors which occur in capital allocation decision-making problems in practice. As such, the SCR model is superior to these other methods in providing solutions in a manner which can cope with realistically sized problems and remain computationally tractable.

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