

Thin Cylinders and Shells

Thin cylinders under internal pressure

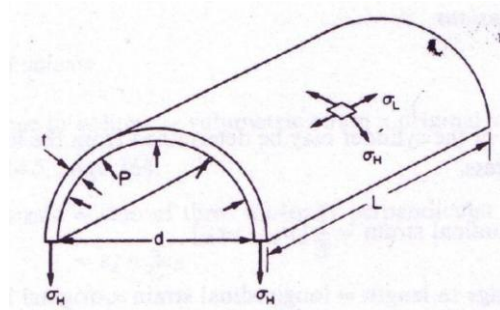
When a thin-walled cylinder is subjected to internal pressure, three mutually perpendicular stresses will be set-up

- Circumferential or Hoop stress
- Radial Stress
- Longitudinal Stress

Provided the ratio of thickness to internal diameter is less than 1/20 it can be assumed that the hoop and longitudinal stresses are constant across the wall thickness and that the radial stress is insignificant compared to the hoop and the longitudinal stress.

Circumferential or hoop stress

This is the stress that is set-up in resisting the bursting effect of the applied pressure. Consider the following;



The total force on half the cylinder due to the internal pressure = $p \times \text{projected area} = pdL$

The resisting force owing to hoop stress σ_H , set up in the cylinder walls = $2\sigma_H \times Lt$

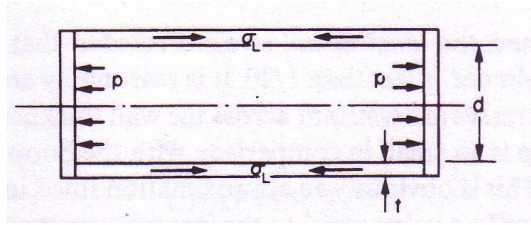
$$\therefore 2\sigma_H Lt = pdL$$

Thus circumferential stress is defined as;

$$\sigma_H = \frac{pd}{2t}$$

Longitudinal Stress

Now consider an enclosed cylinder as shown below;



The total force on the end of the cylinder due to the internal pressure = $p \times \text{area} = \frac{p\pi d^2}{4}$

The area of metal resisting this force $\sim$ circumference $\times$ $t = \pi dt$

Since, $\text{stress} = \frac{\text{force}}{\text{area}}$, longitudinal stress must therefore be defined as;

$$\therefore \sigma_L = \frac{pd}{4t}$$

Notice that the **Hoop stress is twice the longitudinal stress**

Change in Length

The change in length of the cylinder can be calculated from the longitudinal strain (neglecting the radial stress)

$$\text{Longitudinal strain} = \frac{1}{E}(\sigma_L - \nu\sigma_H)$$

Thus, change in length = longitudinal strain $\times$ original length

$$\text{Change in length} = \frac{pd}{4tE}(1 - 2\nu)L$$

Change in Diameter

The change in diameter of the cylinder can be calculated from the diametral strain;

$$\text{Diametral strain} = \frac{\text{change in diameter}}{\text{original diameter}}$$

This can be found by considering the circumferential change. The stress acting around a circumference is the hoop or circumferential stress σ_H giving rise to the circumferential strain ϵ_H

$$\text{Change in circumference} = \text{strain} \times \text{original circumference} = \epsilon_H \times \pi d$$

$$\text{Therefore the new circumference} = \pi d + \pi \epsilon_H d = \pi d(1 + \epsilon_H)$$

But this is the circumference of a circle of diameter $d(1 + \epsilon_H)$

$$\therefore \text{New diameter} = d(1 + \epsilon_H)$$

$$\therefore \text{Change in diameter} = d\epsilon_H$$

$$\therefore \text{Diametral strain, } \epsilon_D = \frac{d\epsilon_H}{d} = \epsilon_H$$

i.e. **the diametral strain = the hoop strain**

Thus,

$$\text{The change in diameter} = d\epsilon_H = \frac{d}{E}(\sigma_H - \nu\sigma_L)$$

$$\text{The change in diameter} = \frac{pd^2}{4tE}(2 - \nu)$$

Change in internal volume

The volumetric strain can be shown to be the sum of three mutually perpendicular direct strains, i.e.

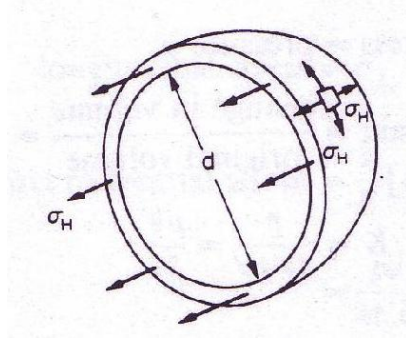
$$\begin{aligned} \text{Volumetric strain} &= \epsilon_L + 2\epsilon_D = \epsilon_L + 2\epsilon_H \\ &= \frac{1}{E}(\sigma_L - \nu\sigma_H) + \frac{2}{E}(\sigma_H - \nu\sigma_L) \\ &= \frac{pd}{4tE}(5 - 4\nu) \end{aligned}$$

Thus change in volume = volumetric strain x original volume

$$\text{Change in volume} = \frac{pd}{4tE}(5 - 4\nu)V$$

Thin spherical shell under internal pressure

Due to the symmetry of a sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoop stresses of equal value and a radial stress. Once again if we assume that the thickness to diameter ratio is less than 1:20 the radial stress can be assumed to be negligible in comparison to the hoop stress. Hence the stress system is one of biaxial hoop stress. Consider the following;



As before if we equate the pressure on one half of the sphere with the force from the hoop stress;

$$p \times \frac{\pi d^2}{4} = \sigma_H \times \pi d t$$

$$\therefore \sigma_H = \frac{pd}{4t}$$

Change in internal volume

As for the cylinder

change in volume = original volume x volumetric strain

$$\begin{aligned} \text{Volumetric strain} &= \text{sum of three mutually perpendicular strains} \\ &= 3\varepsilon_D = 3\varepsilon_H \\ &= \frac{3}{E}(\sigma_H - \nu\sigma_H) \\ &= \frac{3pd}{4tE}(1-\nu) \end{aligned}$$

$$\therefore \text{Change in internal volume} = \frac{3pd}{4tE}(1-\nu)V$$

Effects of end plates and joints

So far we have assumed uniform material properties throughout the cylinders and spheres that we have considered, neglecting the effects of endplates and joints. In general, the strength of the components will be reduced by rivets for example. This is taken into account by introducing the **joint efficiency factor** η (expressed as a decimal) into the equations, hence our equations become;

Cylinders;

$$\sigma_H = \frac{pd}{2t\eta_L}$$

Where η_L is the efficiency of the longitudinal joints

$$\sigma_L = \frac{pd}{4t\eta_C}$$

Where η_C is the efficiency of the circumferential joints

Spheres;

$$\sigma_H = \frac{pd}{4t\eta}$$

Examples

- 1) A cylinder is 300 mm mean diameter with a wall 2 mm thick. Calculate the maximum pressure difference allowed between the inside and outside if the stress in the wall must not exceed 150 MPa.

[2 Mpa]

- 2) A thin cylinder 75 mm internal diameter, 250 mm long with walls 2.5 mm thick is subjected to an internal pressure of 7 MN/m^2 . Determine the change in internal diameter and the change in length.

If, in addition to the internal pressure, the cylinder is subjected to a torque of 200 N m, find the magnitude and nature of the principal stresses set up in the cylinder. $E = 200 \text{ GN/m}^2$. $\nu = 0.3$.

[33.4E-6m, 26.2E-6m, 51.1MPa, 106.5MPa]

- 3) A cylinder has an internal diameter of 230 mm, has walls 5 mm thick and is 1 m long. It is found to change in internal volume by $12.0 \times 10^{-6} \text{ m}^3$ when filled with a liquid at a pressure p . If $E = 200 \text{ GN/m}^2$ and $\nu = 0.25$, and assuming rigid end plates, determine:

- (a) the values of hoop and longitudinal stresses;
- (b) the modifications to these values if joint efficiencies of 45 % (hoop) and 85 % (longitudinal) are assumed;
- (c) the necessary change in pressure p to produce a further increase in internal volume of 15 %. The liquid may be assumed incompressible.

[28.8MPa, 14.4MPa, 33.9MPa, 32MPa, 0.188MPa]

Notes adapted from/further reading;

Mechanics of Materials 1, E.J. Hearn