

Arithmetical algorithms in Möbius number systems

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Iterative systems

X compact metric space, A finite alphabet

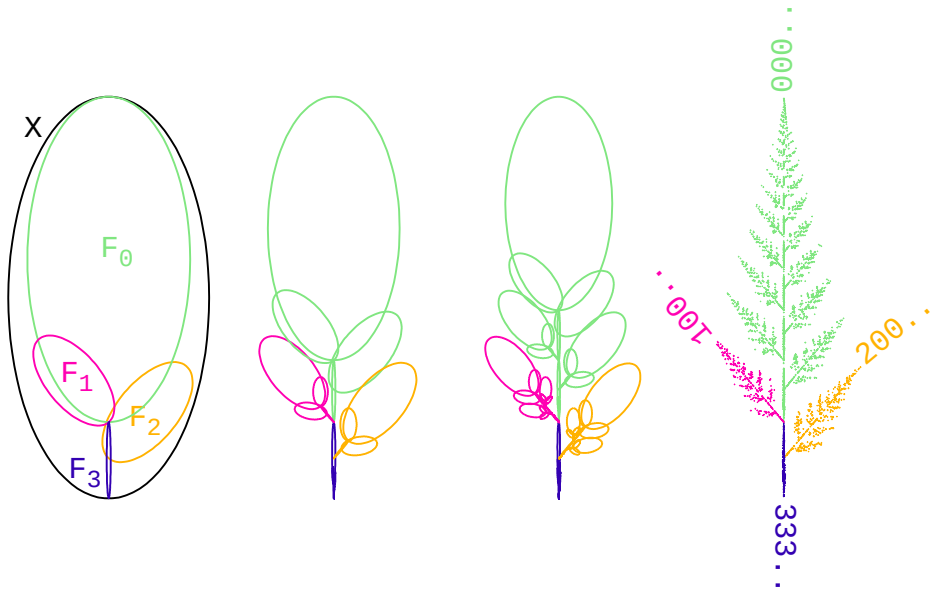
$(F_a : X \rightarrow X)_{a \in A}$ continuous.

$(F_u : X \rightarrow X)_{u \in A^*}$, $F_{uv} = F_u \circ F_v$, $F_\lambda = \text{Id}$

Theorem(Barnsley) If $(F_a : X \rightarrow X)_{a \in A}$ are contractions, then there exists a unique attractor $Y \subseteq X$ with $Y = \bigcup_{a \in A} F_a(Y)$, and a continuous surjective symbolic mapping $\Phi : A^{\mathbb{N}} \rightarrow Y$

$$\{\Phi(u)\} = \bigcap_{n \geq 0} F_{u_{[0,n)}}(X), \quad u \in A^{\mathbb{N}}$$

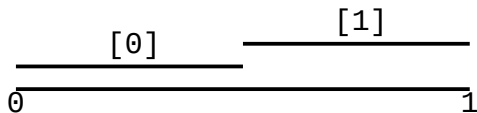
Fractal attractor $A = \{0, 1, 2, 3\}$



Binary system $A = \{0, 1\}$, $\Phi_2 : A^{\mathbb{N}} \rightarrow [0, 1]$

$$F_0(x) = \frac{x}{2}, \quad F_1(x) = \frac{x+1}{2}$$

$$\Phi_2(u) = \sum_{i \geq 0} u_i \cdot 2^{-i-1}, \quad u \in A^{\mathbb{N}}$$

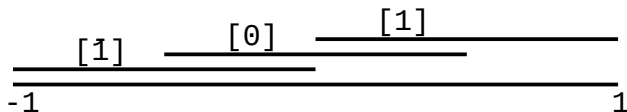


Binary signed system

$$A = \{\bar{1}, 0, 1\}, \quad \Phi_3 : A^{\mathbb{N}} \rightarrow [-1, 1]$$

$$F_{\bar{1}}(x) = \frac{x-1}{2}, \quad F_0(x) = \frac{x}{2}, \quad F_1(x) = \frac{x+1}{2}$$

$$\Phi_3(u) = \sum_{i \geq 0} u_i \cdot 2^{-i-1}, \quad u \in A^{\mathbb{N}}$$



In the standard decadic system, the addition is not algorithmic

$$\begin{array}{r} 0.333333333333333333333333333333... \\ 0.666666666666666666666666666666... \\ \hline ? \end{array}$$

$$\begin{array}{r} 0.33333333332 \qquad 0.33333333334 \\ 0.66666666664 \qquad 0.66666666668 \\ \hline 0.9999999999 \qquad 1.0000000000 \end{array}$$

The binary system Φ_2 is not redundant.
The binary signed system Φ_3 is redundant.

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Real orientation-preserving Möbius transformations

$$M_{a,b,c,d}(x) = \frac{ax + b}{cx + d}, \quad ad - bc > 0$$

act on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

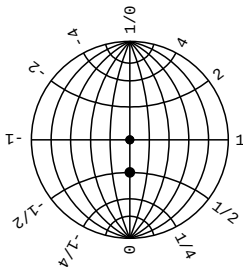
and on $\mathbb{U} = \{z \in \mathbb{C} : \Im(z) > 0\}$

Stereographic projection $\mathbf{d}(z) = \frac{iz+1}{z+i}$

$\mathbf{d} : \overline{\mathbb{R}} \rightarrow \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ unit circle

$\mathbf{d} : \mathbb{U} \rightarrow \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ unit disc

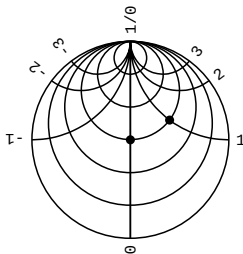
Disc Möbius transformations $\hat{M} = d \circ M \circ d^{-1}$



$$\hat{F}_0(z) = \frac{3z-i}{iz+3}$$

$$F_0(x) = x/2$$

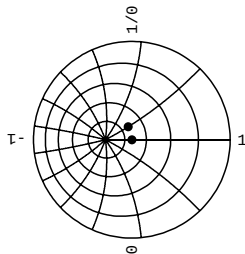
hyperbolic



$$\hat{F}_1(z) = \frac{(2+i)z+1}{z+2-i}$$

$$F_1(x) = x + 1$$

parabolic



$$\hat{F}_2(z) = \frac{(7+2i)z+i}{-iz+(7-2i)}$$

$$F_2(x) = \frac{4x+1}{3-x}$$

elliptic

Möbius number system(MNS) (F, Σ)

$(F_a : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}})_{a \in A}$ Möbius iterative system

$$F_u = F_{u_0} F_{u_1} \cdots F_{u_{n-1}}, \quad u \in A^n$$

convergence space:

$$\mathbb{X}_F = \{u \in A^{\mathbb{N}} : \lim_{n \rightarrow \infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}\}$$

symbolic map $\Phi : \mathbb{X}_F \rightarrow \overline{\mathbb{R}}$

$$\Phi(u) = \lim_{n \rightarrow \infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}, \quad u \in \mathbb{X}_F$$

$\Sigma \subseteq \mathbb{X}_F$ is a subshift such that $\Phi : \Sigma \rightarrow \overline{\mathbb{R}}$ is continuous and surjective.

Interval almost-cover $\mathcal{W} = \{W_a : a \in A\}$

W_a open intervals with $\bigcup_{a \in A} \overline{W_a} = \overline{\mathbb{R}}$

Expansion graph: $x \xrightarrow{a} F_a^{-1}(x)$ if $x \in W_a$

$$x \xrightarrow{u_0} F_{u_0}^{-1}(x) \xrightarrow{u_1} F_{u_0 u_1}^{-1}(x) \xrightarrow{u_2} \dots$$

$$x \in W_{u_0}, F_{u_0}^{-1}(x) \in W_{u_1}, F_{u_0 u_1}^{-1}(x) \in W_{u_2}$$

$$W_u := W_{u_0} \cap F_{u_0}(W_{u_1}) \cap \dots \cap F_{u_{[0,n)}}(W_{u_n})$$

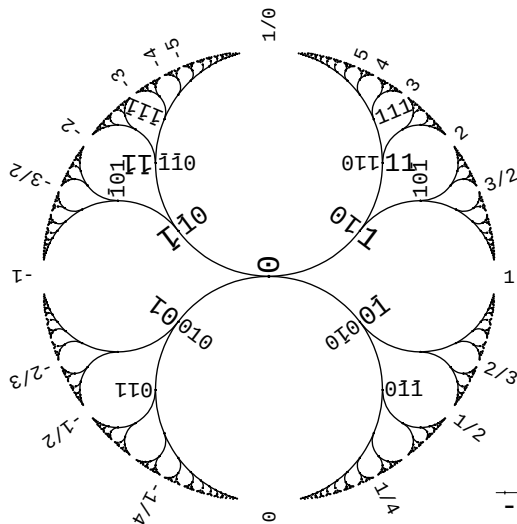
$x \in W_u$ iff u is the label of a path with source x :

Expansion subshift:

$$\mathcal{S}_{\mathcal{W}} := \{u \in A^{\mathbb{N}} : \forall n, W_{u_{[0,n)}} \neq \emptyset\}$$

If $\mathcal{W}$ is a cover then arithmetic algorithms exist.

Continued fractions $a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \dots}} = F_1^{a_0} F_0 F_1^{a_1} F_0 \dots$



$$F_{\bar{1}}(x) = x - 1$$

$$F_0(x) = -1/x$$

$$F_1(x) = 1 + x$$

forbidden words:

00, 1 $\bar{1}$, $\bar{1}1$, 101, $\bar{1}0\bar{1}$

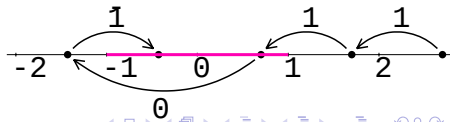
Interval almost-cover:

$$W_{\bar{1}} = (\infty, -1)$$

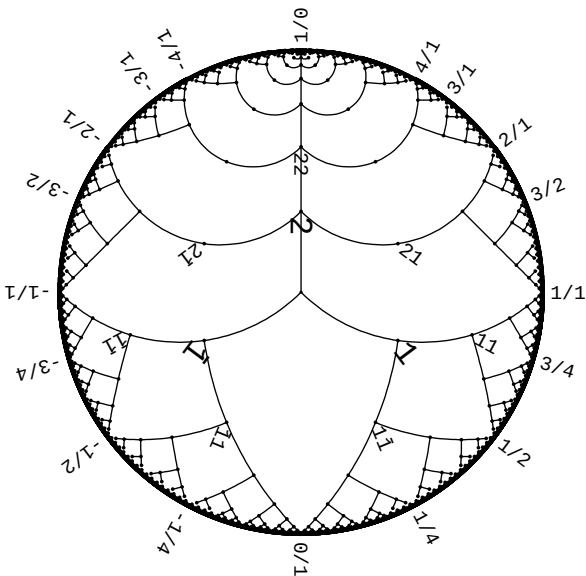
$$W_0 = (-1, 1)$$

$$W_1 = (1, \infty)$$

$$x \xrightarrow{a} F_a^{-1}(x) \text{ if } x \in \overline{W_a}$$



Binary signed system, $A = \{\bar{1}, 1, 2\}$



$$F_{\bar{1}}(x) = (x - 1)/2$$

$$F_1(x) = (x + 1)/2$$

$$F_2(x) = 2x$$

$$W_{\bar{1}} = (-1, 0)$$

$$W_1 = (0, 1)$$

$$W_2 = (1, -1)$$

$\mathcal{S}_{\mathcal{W}}$ is a SFT.

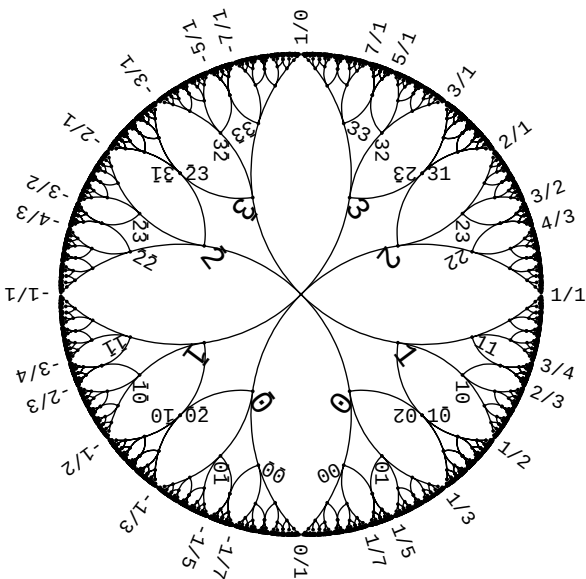
forbidden words:

$\bar{1}2, 12, 2\bar{1}1, 21\bar{1}$

$$\mathcal{S}_{\mathcal{W}} = \{2^n u : u \in \{\bar{1}, 1\}^{\mathbb{N}}\}$$

$$\Phi(2^n u) = \sum_{i=0}^{\infty} u_i \cdot 2^{n-i}$$

The bimodular octanic system $A = \{\bar{3}, \bar{2}, \bar{1}, \bar{0}, 0, 1, 2, 3\}$



$$F_{\bar{3}}(x) = 2x - 1$$

$$F_{\bar{2}}(x) = 2x/(1 - x)$$

$$F_{\bar{1}}(x) = (x - 1)/2$$

$$F_{\bar{0}}(x) = x/(2 - x)$$

$$F_0(x) = x/(x + 2)$$

$$F_1(x) = (x + 1)/2$$

$$F_2(x) = 2x/(x + 1)$$

$$F_3(x) = 2x + 1$$

$$W_{\bar{3}} = (\infty, -2)$$

$$W_{\bar{2}} = (-3, -1)$$

$$W_{\bar{1}} = (-1, -\frac{1}{3})$$

$$W_{\bar{0}} = (-\frac{1}{2}, 0)$$

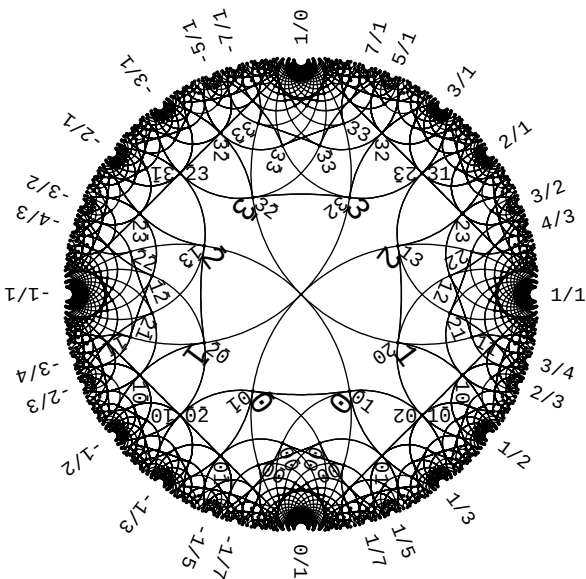
$$W_0 = (0, \frac{1}{2})$$

$$W_1 = (\frac{1}{3}, 1)$$

$$W_2 = (1, 3)$$

$$W_3 = (2, \infty)$$

The redundant bimodular octanic system



$$F_3(x) = 2x - 1$$

$$F_2(x) = 2x/(1 - x)$$

$$F_1(x) = (x - 1)/2$$

$$F_0(x) = x/(2 - x)$$

$$F_0(x) = x/(x + 2)$$

$$F_1(x) = (x + 1)/2$$

$$F_2(x) = 2x/(x + 1)$$

$$F_3(x) = 2x + 1$$

$$W_3 = (3, -2)$$

$$W_2 = (-3, -\frac{1}{2})$$

$$W_1 = (-2, -\frac{1}{3})$$

$$W_0 = (-\frac{1}{2}, \frac{1}{3})$$

$$W_0 = (-\frac{1}{3}, \frac{1}{2})$$

$$W_1 = (\frac{1}{3}, 2)$$

$$W_2 = (\frac{1}{2}, 3)$$

$$W_3 = (2, -3)$$