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THE DEVELOPMENT OF YOUNG CHILDREN'S
EARLY NUMBER AND OPERATION SENSE
AND ITS IMPLICATIONS FOR EARLY
CHILDHOOD EDUCATION¹

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Survey responses such as “I don’t do mathematics” and “Young children shouldn’t do mathematics . . . it’s not appropriate” indicate that many early childhood teachers feel uncomfortable about teaching mathematics (Copley, 2004a, p. 403). Indeed, early childhood educators have long viewed young children and mathematics education like water and oil, as things that do not mix. As a result, they have tended to focus on literacy and to overlook numeracy. Balfanz (1999) observed that this has not always been the case. Indeed, “some of the founding figures of early childhood education, like Fredrick Froebel and Maria Montessori, . . . advanced the notion that young children are capable of complex mathematical thought and enjoy using mathematics to explore and understand the world around them” (p. 3). Based on their careful observations of children

in natural settings, both Froebel and Montessori incorporated rich mathematical experiences into their early childhood programs. However, between 1820 and 1920, the influence of such pedagogues was contested by social theorists, including leading psychologists and educators of the era, and was then eclipsed by them (see Balfanz, 1999, for a detailed account). In recent years, though, attitudes toward teaching mathematics to young children, including preschoolers, again has changed dramatically. For example, the National Council of Teachers of Mathematics (NCTM, 2000) concluded that “the foundation for children’s mathematical development is established in the earliest years” (p. 73), and, along with the National Association for the Education of Young Children (NAEYC), now takes the position that “high-quality, developmentally

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appropriate mathematics education for children 3-6 is vital” (Clements, Copple, & Hyson, 2002, p. 1; see also <http://www.naeyc.org/resources/position>).

In Part I of this chapter, we discuss key reasons for this renewed and intense national interest in early childhood mathematics education. This includes summarizing the findings of recent research that have helped propel mathematics education of young children into the spotlight. In Part II, we discuss what recent research on number and operation sense suggests about the nature of early childhood mathematics education. Number and operation sense are an intuitive feel for how numbers are related and behave. It stems from well-interconnected knowledge about numbers and how they operate or interact.

A CASE FOR EARLY CHILDHOOD
MATHEMATICS EDUCATION

In our increasingly technology- and information-based society, *mathematical proficiency* has become as important a gatekeeper as literacy. Despite the often-heated debate about reforming mathematics instruction, sometimes dubbed the “math wars” (see Baroody, 2004b, and Ralston, 2004, for analyses), there is considerable agreement that the aim of mathematics instruction from preschool to college should be mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001). Such proficiency entails *conceptual understanding*, *computational fluency*, *strategic mathematical thinking*, and a *productive disposition*. Conceptual understanding can be thought of as a web of connections (e.g., Ginsburg, 1977; Hiebert & Carpenter, 1992). Unlike rote knowledge that can be applied only to familiar tasks (*routine expertise*), meaningful knowledge (*adaptive expertise*) can be flexibly adapted and appropriately applied to learning new concepts or skills or to solving novel problems (Hatano, 1988, 2003; cf. Wertheimer, 1945). In a real sense, then, conceptual understanding is a key basis for the other aspects of mathematical proficiency (Baroody, 2003). Computational fluency involves using basic skills appropriately and flexibly, as well as efficiently (i.e., quickly and accurately). Conceptual understanding can aid appropriate and efficient use of skills and is probably critical to their flexible use. Strategic mathematical thinking includes creative problem solving and adaptive reasoning, both of which are facilitated by a rich web of interconnected knowledge.² A productive disposition includes interest and the confidence to solve challenging problems, both of which are more likely if mathematics makes sense to a child. Furthermore, there is considerable agreement these aspects of mathematical proficiency should be fostered in an *intertwined manner* (Kilpatrick et al., 2001).

However, it has long been clear that our public schools, particularly failing (many inner-city and rural) schools, do not adequately foster mathematical proficiency, especially among children at risk for school failure (e.g., those from low-income families and those with learning difficulties), and that serious reform is necessary (e.g., Carnegie Forum on Education and the Economy, 1986; McNight et al., 1987; National Commission on Excellence in Education, 1983). Indeed, international studies indicate U.S. students’ mathematical achievement lags significantly behind as early as first grade (e.g., Stigler, Lee, & Stevenson, 1990). Concern about the quality of mathematics instruction, economic competitiveness, and equity (e.g., equal opportunity to develop mathematical proficiency and to obtain good jobs) has led, for instance, the NCTM (1989, 2000) and the National Research Council (Kilpatrick et al., 2001) to advocate fostering the mathematical proficiency of *all* children (e.g., Baroody, 2004d; Fuson, 2004; Thornton & Bley, 1994).

How, though, are concerns about mathematical proficiency and equity related to early childhood education? Consider three important and interrelated findings from recent research (for detailed reviews, see, e.g., Clements, Sarama, & DiBiase, 2004; Ginsburg, Klein, & Starkey, 1998):

1. Preschoolers can develop a wealth of *informal* mathematical knowledge. (Informal knowledge is gleaned from everyday activities in what are not normally considered instructional settings—such as home, playground, grocery store, shopping mall, family car, or park—[Gelman & Massey, 1987; Ginsburg, 1977; Ginsburg, Cannon, Eisenband, & Pappas, 2006; Pound, 1999]. Such knowledge is usually represented nonverbally or verbally and often learned incidentally. In contrast, formal mathematical knowledge is school taught, largely represented in written form, and frequently the result of deliberate efforts by teachers and students. The distinction between informal and formal knowledge should be thought of as a continuum and, thus, in some cases can be quite fuzzy. Nevertheless, there is now widespread use of the term *informal knowledge* in the U.S. and elsewhere. For instance, within the Directorate for Education and Human Resources, the National Science Foundation has a unit called the *Elementary, Secondary, and Informal Sciences Division*, which includes in its mission promoting the infrastructure and resources needed to improve pre-K science, technology, engineering, and mathematics education.)

2. This informal knowledge is a critical basis for understanding formal mathematics, successfully mastering and retaining basic skills, devising and applying effective problem-solving and reasoning strategies, and developing a productive disposition.

²*Strategic mathematical thinking* encompasses two aspects of mathematical proficiency identified by Kilpatrick et al. (2001), namely strategic competence and adaptive reasoning. I will not distinguish between these two competencies and will use the more general term *strategic mathematical thinking* for two reasons. One is that the two components are intricately interrelated. For example, logical reasoning is a key strategy for solving problems. Another reason is that mathematical thinking involves an exploratory side as well as the logical side (Clements & Battista, 1992). In *Adding It Up* (Kilpatrick et al., 2001), the importance of deductive reasoning is correctly emphasized. Indeed, note that adaptive reasoning is defined in terms of this type of reasoning. However, little mention is made of other key types of mathematical reasoning—those that comprise the exploratory side of mathematics, namely intuitive and inductive reasoning. I wish to avoid painting a one-sided portrait of mathematical reasoning, mathematicians, and the science of mathematics.

3. Within as well as across areas of knowledge, preschoolers' informal knowledge can differ significantly among and within social classes, medical conditions, and other categories.

These findings and their implications for early childhood mathematics instruction are discussed in turn.

Finding 1: Preschoolers' Surprising Informal Knowledge

Over the course of the twentieth century, psychologists came to dramatically different conclusions about the nature of young children's mathematical competence and its basis. After reviewing these two changes in the conventional wisdom regarding children's mathematical knowledge, we draw some theoretical conclusions and discuss their educational implications.

Changes in the conventional wisdom regarding the nature of preschoolers' informal knowledge. Like a pendulum, the conventional wisdom about young children's number and arithmetic competence has swung from extremely pessimistic to extremely optimistic and then back toward the middle.

Early Pessimistic Views. For most of the century, psychologists held a pessimistic view and focused on what children *can't do*. This view reinforced the rationale for narrow and limited mathematics instruction in early childhood. For example, William James (1890) described an infant's perception of the world as a "great, blooming, buzzing confusion." Geary (1996) facetiously observed that such a description better characterizes the parents of newborns than the newborns themselves.

Edward L. Thorndike (1922) further concluded that young children were so mathematically inept that "little is gained by [doing] arithmetic before grade 2, though there are many arithmetic facts that can [be memorized by rote] in grade 1" (p. 198). He epitomized the social theorists' assumption that "young children started school with no prior mathematical knowledge or experience and that limited instruction" on the first 10 numbers, simple addition and subtraction, and recognition of basic shapes was sufficient for the early grades (Balfanz, 1999, p. 8). Given this perspective, it made little sense to emphasize mathematics in kindergarten and even less sense to do so in the preschool years. A minimalist mathematics curriculum is still evident and even predominant today in preschools and kindergartens (Balfanz, 1999; see also Copley, 2004a; Ginsburg, Inoue, & Seo, 1999; Hunting & Kamii, 2002).

Thorndike's (1922) work influenced early childhood mathematics instruction in another powerful way. Advocates of his associative-learning theory essentially concluded that early number and arithmetic knowledge should be learned by rote. Direct

instruction and repetitive practice continue to this day to be the instructional mainstay of many parents and early childhood teachers (Ginsburg et al., 1998).

Interestingly, Thorndike's (1922) views also contributed indirectly to narrow and limited mathematics instruction in early childhood settings. As the quotes from Copley (2004a) at the beginning of the chapter illustrate, many social theorists and early childhood educators believed that organized mathematics instruction in early childhood was not only unnecessary but inappropriate or even harmful. That is, many feared that the direct instruction and drill advocated by Thorndike as necessary to learn basic number and arithmetic competencies would, for example, stifle creativity and interest in learning in preschoolers (and older children). Thus, social theorists recommended postponing mathematics instruction until after the beginning of formal schooling. That is, they provided early childhood educators, many of whom felt uncomfortable about mathematics (Coates & Thompson, 1999; Patton & Kokoski, 1996), a noble excuse for avoiding the topic either for the most part or altogether.

Piaget's (e.g., 1965) work had several different effects. On the one hand, it helped focus attention on young children's informal knowledge. His theory underscored the key point that mathematical thinking and knowledge do not simply blossom in the school age children as a result of formal instruction (see, e.g., Ginsburg, 1977). Their genesis begins in the preschool years, indeed in infancy, with informal experiences. Piaget's (1964) mutually dependent principles of assimilation and accommodation imply that cognitive development is a gradual building or constructive process and that preschool experiences are an essential basis for understanding school taught mathematics.³ Furthermore, the primitive reasoning of young children paves the way for more sophisticated reasoning that emerges in late childhood and adolescence.

At the height of Piaget's influence in the 1960s and 1970s, his focus on preschoolers' informal knowledge created interest in preschool mathematics education. These educational efforts focused on fostering children's logical reasoning ability, as this was thought to be the basis for constructing an understanding of number and arithmetic (see, e.g., Furth & Wachs, 1974; Gibb, & Castaneda, 1975; Maffel & Buckley, 1980; Sharp, 1969).

On the other hand, Piaget's work had the effect of limiting expectations about what young children can learn and be taught. His views (Piaget, 1965) about the abilities and capacities of young children were relatively pessimistic. For instance, he believed that preschoolers are *preoperational* thinkers, incapable of logical and systematic thinking or constructing abstract concepts (e.g., a true concept of number or understanding of arithmetic). He ridiculed attempts to accelerate the development of logical operations or concepts by means of training as the "American question" (e.g., Elkind, 1988; Hall, 1970; Philips, 1969). Piagetians argued that teaching number and arithmetic concepts before the concrete operational stage did not make

³Assimilation is the process of interpreting new information in terms of existing knowledge. Piaget (1964) considered it, not associative learning, the primary fact of mental life. For example, prior existing knowledge is probably the best predictor of whether a child will understand new content. The process of assimilation necessarily entails its complementary process accommodation. Because new information does not exactly fit existing schemata, assimilation requires these existing mental structures to change to accommodate the new information and reality. The process of accommodation enables children to construct more accurate knowledge of the world as opposed to an idiosyncratic view of it.

TABLE 11.1. Principles Underlying (Non-verbal or Verbal) Object Counting (Gelman & Gallistel, 1978)

Stable-Order Principle: The tags (nonverbal symbols or number words) must always be used in the same order when enumerating collections.
One-to-One Principle: One and only one tag must be assigned each item in a collection.
Abstraction Principle: For enumeration purposes, a collection can be composed of unlike objects.
Cardinality Principle: The last tag used in the enumeration process has special significance because, it not only labels the last item, it represents the total.
Order-irrelevance Principle: The order in which the items of a collection are enumeration does not matter because it does not affect the cardinal value of the collection.

sense and that premature instruction was worse than no teaching at all (e.g., Hall; May & Kundert, 1997). To this day, some constructivists argue that an abstract or genuine understanding of number does not develop until children are at least 4 years of age or even older (Bermejo, 1996; Copeland, 1979; Kamii, 1985; Munn, 1998; Steffe, Cobb, & von Glasersfeld, 1988).

The Shift to a Highly Optimistic View. In the last quarter of the 20th century, psychologists adopted a highly optimistic view and focused on what children *can do* (Gelman, 1979). After establishing that posttoddlers develop a variety of mathematical competencies (see, e.g., Gelman & Gallistel, 1978; Ginsburg, 1977; Hughes, 1986; Schaeffer, Eggleston, & Scott, 1974; Wagner & Walters, 1982), researchers turned their attention to even younger preschoolers. For example, Wynn (1998) noted that “findings over the past 20 years have shown infants are sensitive to number” (p. 5). Specifically, she argued they are born with an ability to recognize and distinguish among *oneness*, *twoness*, and *threeness* and can even reason about or operate on very small numbers (e.g., recognize that one object added to another makes two and that two objects minus one is one)—all before they develop verbal-based counting competencies. Indeed, Gelman (e.g., Gelman & Meck, 1992) argued that children are innately endowed with counting principles (see Table 11.1)—principles that allow infants to nonverbally count (using nonverbal tags or representations) and toddlers

to quickly learn number words and how to use them to count collections.

The Recent Shift to Middle Ground Views. Some research over the last 10 years indicates that nativists such as Wynn (1992a, 1998) may be too optimistic and that a more balanced view of children’s informal mathematical knowledge is needed (Baroody, Benson, & Lai, 2003; Haith & Benson, 1998; Leslie, 1999; Mix, Huttenlocher, & Levine, 2002a, b; Wakeley, Rivera, & Langer, 2000a, b; Simon, 1997).

Consider, for example, the work of Janellen Huttenlocher and colleagues (e.g., Huttenlocher, Jordan, & Levine, 1994) regarding preschoolers’ nonverbal addition and subtraction ability (see Figure 11.1). These researchers found that:

- Children between 2.5 and 3 years of age were successful in mentally determining sums and differences with the “intuitive numbers” (collections of one to three or possibly four items) only about 25% of the time.
- A majority of children did not succeed on even the simplest trials (1 + 1 and 2 + 1) of the nonverbal addition and subtraction task until 3 years of age.

Huttenlocher et al. concluded that, if 1.5 to 3 year olds typically cannot perform simple arithmetic operations, it is not likely infants can do so.

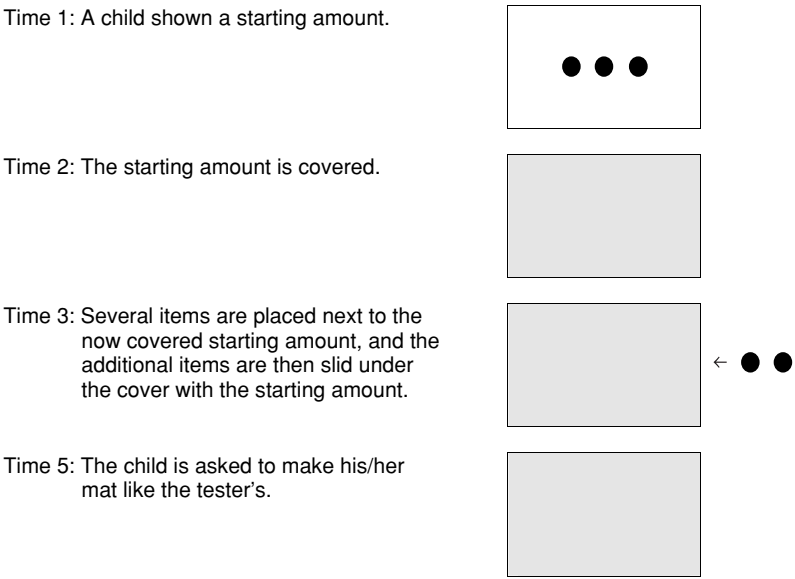


FIGURE 11.1. Nonverbal addition (and subtraction) task.

Two influential middle ground models are the *mental models view* (e.g., [Huttenlocher et al., 1994](#); [Mix et al., 2002a](#)) and the *progressive abstraction view* ([Resnick, 1992, 1994](#)). Each of these models focuses on a key but different aspect of number and arithmetic development.

According to the mental-models view, *how* children represent number evolves.

- **Pretransition 1: Inexact Perception-Based Representation.** Children initially represent all quantities *nonverbally* and *inexactly* using one or more perceptual cues such as area, contour length (total perimeter), density, and length ([Mix et al., 2002a, b](#)). For example, infants may represent two squares, each with sides 2 in long and a total perimeter of 16 in, as different from three squares with sides of 2 in each and a total perimeter of 24 in or two squares each with sides of 4 in and a total perimeter of 32 in.
- **Transition 1: Exact Nonverbal Representation.** Children next develop the ability to mentally represent the intuitive numbers nonverbally and *exactly* (Transition 1). Three factors may account for this transition. First, the evolution of object individuation (perceiving objects as discrete, permanent, and distinct) provides a basis for precursors to construct an understanding of one-to-one correspondence, which, in turn, provides a basis for identifying and representing collections and the groundwork for an informal understanding of numerical equivalence and number (e.g., [Mix et al., 2002a](#)). (Note that this view is consistent with [Piaget’s, 1965](#), hypothesis that one-to-one correspondence is a foundational concept of number, but unlike Piaget, an understanding of one-to-one—at least with small numbers—is seen as evolving well before 7 years of age, the onset of concrete operational thought.) Second, between 2.5 and 3.5 years of age, children begin developing the ability for symbolic representation (e.g., [Huttenlocher, Jordan, & Levine, 1994](#)). Third, another important step toward understanding number is recognizing that—like color ([Sandhofer & Smith, 1999](#)), size, and weight—it is an important basis for categorizing and, thus, identifying and comparing things ([Mix, et al. 2002a](#)).
- **Transition 2: Number-Word Representation.** Transition 1 and the development of counting permit the *exact verbally based* representation of any number ([Mix et al., 2002a](#)).

Because children can represent and operate on exact but nonverbal representations of number (Transition 1) before they can do so with exact and verbal representations, they are able to solve nonverbal addition and subtraction problems (again see Figure 11.1) before they can solve verbal versions of the same problems (word problems) or correctly respond to verbal presented expressions such as, “How much is two and two?” ([Levine, Jordan, & Huttenlocher, 1992](#)).

According to the progressive-abstraction view ([Resnick, 1992, 1994](#)), *what* children represent also evolves. In this view, concepts and reasoning moves from concrete (context-specific) to abstract (general; cf. [Baroody, Wilkins, & Tiilikainen, 2003](#); [Benson & Baroody, 2002](#)).

- *Protoquantitative (context-specific, qualitative) reasoning.* Initially, children reason about non-quantified amounts in a

global way. For example, a child knows that adding an item to a collection “makes it larger.”

- *Context-specific quantitative reasoning.* With the acquisition of counting (or other processes for representing collections exactly), children can reason about specific quantities in a particular and meaningful context. For instance, given two cookies and then offered one more, a child might count his new total and discover that one cookie added to two cookies always makes three cookies altogether.
- *Numerical reasoning.* This involves reasoning about (verbal) numbers without reference to a particular and meaningful context. In effect, numbers are now treated as nouns instead of as adjectives as in the previous level. For example, a child might discover or reason that two and one more is the next number after two in the counting sequence, namely *three*.
- *Abstract reasoning.* Children eventually recognize general principles that apply to any context or number. For instance, preschoolers typically discover the following rule for comparing numbers: The sum of any number (*n*) and 1 more is the number after *n* in the counting sequence.

[Resnick’s](#) last three levels are essentially a restatement of [Colburn’s \(1828\)](#) view:

The idea of number is first acquired by observing sensible objects. Having observed that this quality is common to all things with which we are acquainted, we obtain an abstract idea of number. We first make calculations about sensible objects; and we soon discover the same calculations will apply to things very dissimilar; and finally that they can be made without reference to any particular thing. Hence from particulars we establish general principles, which serve as the basis of our reasonings, and enable us to proceed, step by step, from the most simple to the most complex operations. (p. 4)

Changes in the conventional wisdom regarding the basis of preschoolers’ informal knowledge of number and arithmetic. Like a pendulum, the conventional wisdom about the role of language in number development has also shifted back and forth over the last 100 years or so (see [Mix, Sandhofer, & Baroody, in press](#), for a detailed discussion).

The First Shift. [Dewey \(1898\)](#) and [Thorndike \(1922\)](#) concluded that children’s initial training in mathematics should focus on counting. [Russell \(1917\)](#) set the tone for the rest of the twentieth century by denouncing this informal approach and arguing that *initial* mathematical training should focus on developing children’s logical thinking about classes. Subsequently, nativists, Piagetians, and the previously mentioned middle-ground theorists ([Huttenlocher](#) and [Resnick](#)) all agreed on one point, namely, that language development does not play a role (directly or otherwise) in children’s initial construction of an inexact number concept. All but perhaps [Resnick \(1992\)](#) would further agree that it is not essential for the *initial* construction of exact representations also.

[Piaget \(1965\)](#), for example, dismissed verbal and object counting as skills learned by rote, skills that had no impact on constructing a number concept. He argued that the construction of a number concept depended on the development and synthesis of the logical thinking abilities necessary for classifying

and ordering (e.g., Gibb & Castaneda, 1975). More recently, early verbal counting (specifically the numbers from 1 to 12 or so in English) was considered merely a singsong (Ginsburg, 1977) or a meaningless string of sounds (e.g., Fuson, 1988). Clearly, children’s initial uses of number words are non-functional (Baroody, 1987; Mix et al., 2002a). However, this perspective has obscured the roles even nonfunctional number words may play in constructing a number concept.

The Second Shift. Although nativists (e.g., Gelman, 1991; Wynn, 1998), Piagetians (e.g., Beilin, 1975; Beilin & Kagan, 1969), and proponents of the mental models (e.g., Huttenlocher et al., 1994; Mix et al., 2002a) would agree that language is important in transcending the limits of early number or quasi-number representation and extending a concept of number, these theorists did not feature language playing a role in the initial formation of a concept of cardinal number (Benson & Baroody, 2003). Since the turn of the 21st century, some researchers have proposed that learning the number words is key to constructing such a concept (Baroody, Benson, & Lai, 2003; Benson & Baroody, 2002; Mix & Sandhofer, in press; Sandhofer & Mix, 2003; Spelke 2003a, 2003b; Van de Walle, Carey, & Prevor, 2000). For example, Spelke has argued that this learning is what prompts the transition from a relatively concrete (perception-based) quasi-numerical representation of the intuitive numbers to a relatively abstract (numerical) representation and the transition from an inexact to an exact representation of larger numbers. That is, number-word learning helps explain why (“how”) children’s representation of number changes and “what” changes.

More specifically, in her *core knowledge view*, Spelke (2003a, b) proposed that infants are born with specialized, task-specific cognitive systems, which form the core of advanced cognitive abilities. She further proposed that infants use two distinct and initially uncoordinated core systems to represent and operate on (e.g., compare, add, subtract) the intuitive numbers and larger numbers, namely object individuation and number sense, respectively (cf. Spelke & Tsivkin, 2001).

According to Spelke (2003a, b), infants use an object-individuation system to represent exactly the intuitive numbers but treat the items of such collections as distinct or unconnected entities. This representation enables infants to recognize that adding an item to another results in an item and another item, which makes it appear they understand “one and one is two,” or removing an item from an item and another item results in a single item, which makes it appear they understand “two take away one is one.” However, it does not permit them to view collections of one to about three items as a set—as a group whose cardinal (numerical) value can be compared to the cardinal value of other sets. That is, contrary to the nativist’s view, it does not guarantee that preverbal children recognize that single entities share the commonality of *oneness*, that pairs of items share the commonality of *twoness*, and that triplets share the commonality of *threeness*. In brief, basic perception-based processes may enable preverbal children to behave in ways that create the *appearance* of understanding number and arithmetic but without real understanding of these concepts (see also Van de Walle et al., 2000).

Spelke (2003a, b) further suggested that infants use a number-sense system to represent inexactly (approximately) larger collection. This enables them to treat larger collections as an approximate set and make numerical comparisons across sets. However, because infants fail to represent these collections as persisting, distinct items, they are incapable of adding or subtracting one with such collections.

According to the core knowledge view (Spelke, 2003a, b), then, infants represent small collections as individual items but not as sets and large numbers as sets but not as individual items. A number concept, though, requires representing collections as *sets of individuals*—simultaneously recognizing the whole and its constitute units or parts (see also Piaget, 2001; von Glasersfeld, 1982).

What then allows children to integrate the object individuation and number sense core systems and thus, treat collections of any size as sets of individuals? According to Spelke, the learning of the first few number words prompts the formation of a numerical concept. Likewise, Van de Walle et al. (2000) have suggested that number words provide a vehicle for abstracting the intuitive numbers. The word “two,” for instance, may enable children to construct a notion of “twoness” that applies to various pairs of items despite apparent or physical differences (see also Baroody, Benson, & Lai, 2003; Benson & Baroody, 2002, 2003; Mix et al., 2002a). Support for the proposition that language plays a role in constructing a number concept comes from the growing evidence that words facilitate the formulation of other function-based or kind concepts by infants and toddlers (Balaban & Waxman, 1997; Dueker & Needham, 2003; Fulkerson & Mansfield-Koren, 2003; Sandhofer & Smith, 1999; Waxman & Markow, 1995; Xu, 1998, cited in Van de Walle et al., 2000).

Theoretical conclusions. The middle-ground views previously described (mental models, progressive abstraction, and core knowledge views) have not as yet been thoroughly tested, are not entirely consistent with each other and the existing data, and are by themselves incomplete descriptions of early number arithmetic development.

According to the integrated view (Baroody, Benson, & Lai, 2003; Baroody et al., 2004), infants may indeed form inexact and then exact nonverbal representations of one or two things and even discriminate among small collections under some circumstances, but they probably do not have a concept of oneness or twoness. That is, they can apprehend the equivalence of such collections without representing their cardinal values (Leslie, 1999). Number symbols (usually number words but possibly written numbers) serve as the catalyst for a number concept, which includes understanding equivalence based on cardinal values.

Figure 11.2 illustrates a hypothetical developmental trajectory for key aspects of early number and arithmetic development suggested by the integrated view. Note that the foundation for the depicted developmental hierarchy is (usually) verbal number recognition. In contrast to the highly optimistic view that a concept of number and the counting principles summarized earlier in Table 11.1 are innate, verbal number recognition provides a basis for constructing cardinality concepts (see

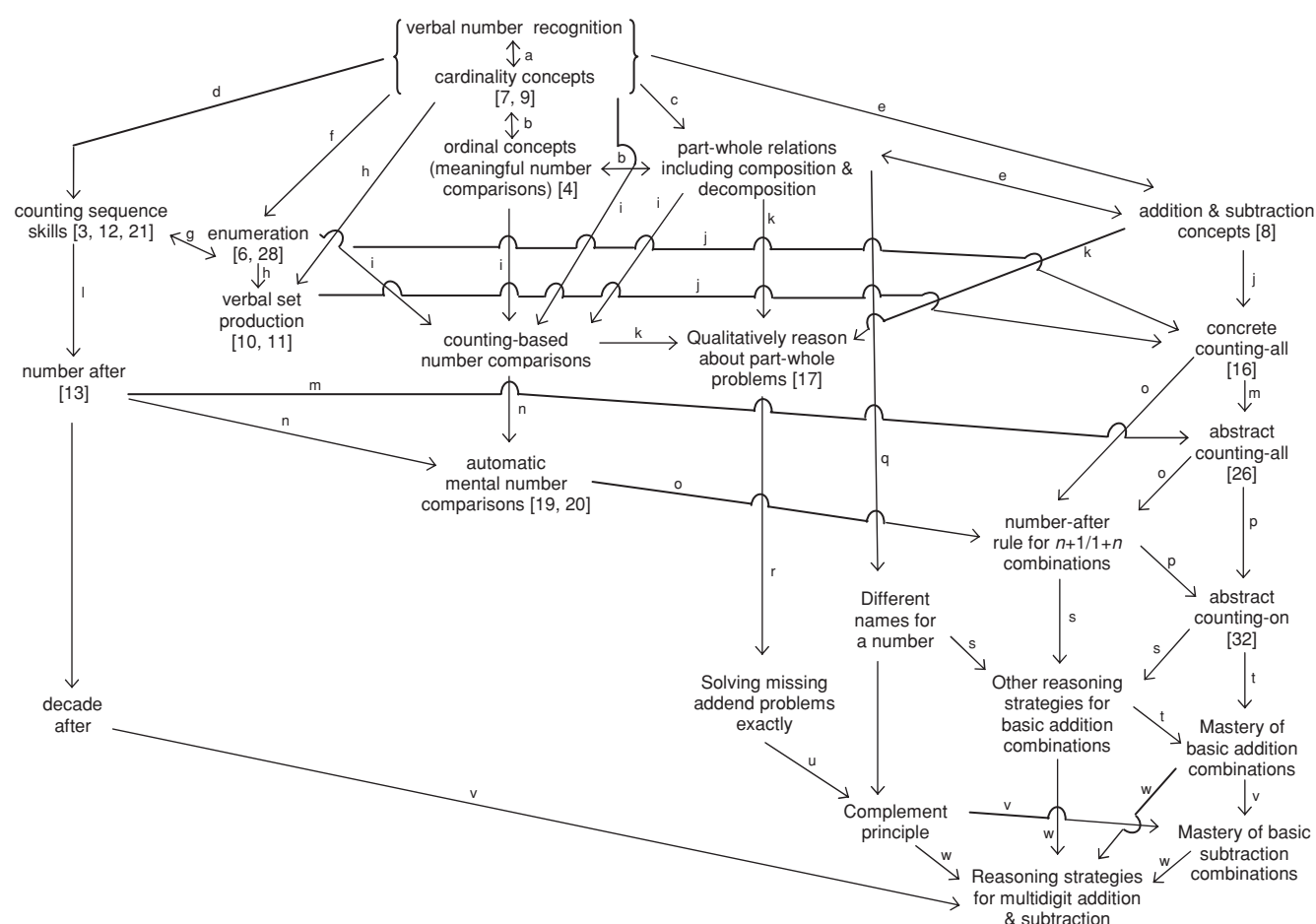


FIGURE 11.2. A hypothetical learning trajectory for some key number and arithmetic skills.*

Note. The arrows in the figure above and footnotes below indicate developmental relations. The numbers within the brackets indicate the TEMA-3 (Ginsburg & Baroody, 2003) test items that gauge a particular competence. In conjunction with this test, this figure can be used as a diagnostic tool to identify a deficiency in informal or formal knowledge and its developmental prerequisites, which can then be assessed and remedied if necessary.

^a By seeing different examples of a number labeled with a unique number word and nonexamples labeled with other number words, children can achieve reliable verbal number recognition (VNR) and the construction of cardinality concepts. That is, they can abstract oneness, twoness, and, in time, threeness and fourness—view collections of items as both parts and as whole (Rosu & Baroody, 2004) and as the same amount despite the physical differences of the items within or across collections (Baroody, Benson, & Lai, 2003; Benson & Baroody, 2002).

^b Meaningful VNR enables a child to directly see that a collection of two is more than one and three is more than two. This is aided by the ability to see two as one and one and three as two and one. VNR, then, can underlie children's recognition that the number words represent ordinal relations (e.g., a collection of three is not merely different than one involving two items but is [one] more than it).

^cConceptually based VNR enables a child to see (decompose) collections of two (a whole) as one and one (into its parts) and likewise collections of three as two and one or even one, one, and one (Rosu & Baroody, 2004).

⁴By using VNR, for example, to recognize a collection of one as “one,” adding an item and using VNR to recognize the new collection as “two,” adding yet another item and using VNR to recognize this even larger collection as “three,” children could effectively re-discover the conventional order of the first few number words (“one, two, three”) that start the counting sequence (Rosu & Baroody, 2004). That is, VNR applied to successively larger collections may help children discover what [Gelman & Gallistel \(1978\)](#) called the *stable order principle*.

^eExperiences composing and decomposing small, easily subitized collections may be the basis for constructing an informal concept of addition (and subtraction). For example, meaningful VNR enables children to directly see that one item added to another results in two and that one item removed from two leaves one. Once children can reliably recognize three, they can do the same for two and one or three take away two. This composing process fosters an understanding of part-whole relations and vice versa.

Figure 2. Continued.

^f VNR linked to cardinality concepts help children understand the whys and the hows of enumeration. By observing others enumerate small collections they can recognize, children can discover that enumeration is another process by which to label a collection with a number word. Similarly, they can construct an understanding of the how-to-count principles:

- By observing adults enumerate small collections they can recognize, children may be more likely to abstract the one-to-one principle—to understand why an adult labels each item with a single number word (otherwise the outcome is different than the cardinal value as determined by verbal number recognition).
- By observing others model counting with collections they visually recognize, children can discover the cardinality principle—that the last number word used represents the total (Baroody et al., 2004). With a collection of three, for example, a child who can recognize the collection as “three” can understand why others emphasize or repeat the last number word used in the enumeration process.
- By observing adults enumerate heterogeneous collection and labeling it with the cardinal value that can be “seen” (determined by VNR), verbal number recognition may also underlie children’s discovery of the abstraction principle.
- By observing adults or themselves enumerate small numerically identifiable collections in different directions or arrangements, they may also induce the order-irrelevance principle.

Once children understand the purpose of enumeration, VNR can serve to check and correct enumeration efforts with small collections (e.g., it can focus attention on labeling each item with one and only one number word so that the outcome matches the results of VNR). Meaningful enumeration fuels the drive to master enumeration subskills (e.g., learning how to better keep track of counted and uncounted items).

^g Meaningful enumeration can also fuel the drive to master more of the counting sequence. In turn, this mastery is a prerequisite for enumerating larger and larger collections.

^h Meaningful VNR provides a basis for verbally producing small collections (e.g., responding to a request “Give me two blocks” by putting out two blocks from a pile of blocks). This, in turn, may help a child understand how counting may be useful in verbally producing collections. Specifically, by putting out three items in response to a give-me-*n* request, and then counting the collection to check their effort, children may recognize the cardinal-count concept—that the requested *n* the last number word when counting out a collection and that the counting-out process should stop. Alternatively, VNR may—in a similar manner—help children make sense of adult demonstrations of the counting-out procedure.

ⁱ Meaningful VNR of small collections, particularly in order of size, plus the insight that numbers have an ordinal meaning and part-whole relations can lead to the discovery that the number word further along in the counting sequence represents the larger collection (e.g., Schaeffer et al., 1974).

^j A concept of addition as increasing the size of a collection (and indirectly part-whole relations including composition and decomposition and ordinal concepts) provide the conceptual basis for inventing counting-based addition strategies. Because the most basic of these strategies (concrete counting-all) entails representing each addend with items and then counting all the items put out to determine the total, verbal set production and enumeration skills are necessary prerequisites for this strategy.

^k An understanding of part-whole relations (e.g., composition and decomposition) and addition and subtraction and skill at mentally comparing numbers can enable children to reason qualitatively about missing-addend problems. Consider, for example, the problem: *Susan had three cookies, then her mom gave her some more, now she has five cookies. How many cookies did mom give her?* Effective qualitative reasoning would entail recognizing that the answer had to be a number smaller than five (Sophian & Vong, 1995; Sophian & McGorray, 1994).

^l In time children’s increasing familiarity with the counting sequence enables them to specify the number after another without counting from one.

^m Children, at some point, realize that they can count or represent *both* addends at the same time they count to determine the sum (Fuson, 1988, 1992). This insight might be prompted by children’s use of concrete counting all to compute the sums of *n* + 1 problems or expressions and the discovery that the answer to such items is the number-after after *n*. For instance, while solving a problem involving “three and one more,” a child might represent each addend with a finger pattern, count all the fingers extended (“1, 2, 3, 4”), and realize that the sum *four* is just one count beyond *three*. This discovery could lead children to eschew counting out objects to represent each addend of a *n* + 1 problem or expression in favor of verbally counting up to the cardinal value of *n* and then verbally counting once more to determine the sum. Children might then further reason that problems or expressions involving *n* + 2, *n* + 3, and so forth could, in principle, be solved in a similar manner—via an abstract counting-all strategy.

ⁿ Once children recognize that the counting sequence can be used to compare numbers and they master the number-after skill, they can efficiently compare neighboring numbers as well as those further apart in the counting sequence.

Figure 2. Continued.

^oChildren may first memorize by rote a few $n + 1$ combinations. However, once they recognize that such combinations are related to their existing counting knowledge—specifically their (already efficient) number-after knowledge—they do not have to repeatedly practice the remaining $n + 1$ combination to produce them. That is, they discover the *number-after rule* for such combinations: “the sum of $n + 1$ is the number n in the counting sequence.” This reasoning process can be applied efficiently to any $n + 1$ combination for which a child knows the counting sequence, even those a child has not previously practiced (Baroody, 1988, 1992a). Regarding link v, this can include large combinations such as $1,000,128 + 1$. (Note that the application of the number-after rule with multi-digit numbers builds on previously learned and possibly automatic *rules* for generating the counting sequence.) In time, the number-after rule for $n + 1$ combinations becomes automatic and can be applied quickly, efficiently, and without thought.

^pResearch has found that both typically developing children and those with learning difficulties discovered the “number-after rule for $n + 1$ combinations” (the sum of $n + 1$ is the number after n in the counting sequence) just before inventing the general counting-on strategy. This insight into the connection between adding one and extant counting knowledge appears to have served as a scaffold for generalized counting-on (e.g., if $7 + 1$ is the number after 7, then $7 + 2$ must be two numbers after 7: “8, 9”; Baroody, 1995, 1996b; Bråten, 1996). Experiences concretely and mentally composing and decomposing increasingly large numbers may facilitate this discovery.

^qA part-whole concept and experience with composition and decomposition may underlie an understanding of “number families” or the different-names-for-a-number concept (a number can be represented in various ways because a whole can be composed or decomposed in various ways) and is one key link between number and arithmetic. The number represented by 5 can also be represented by, for example, $0 + 5$, $1 + 4$, $2 + 3$, $3 + 2$, $4 + 1$, $5 + 0$, $6 - 1$, $7 - 2$ (as well as |||||, ||||, V, fi ve, and •••••).

^rIn time children use their conceptual understanding of part-whole relations, including composition and composition and qualitative reasoning about such situations, to construct strategies for determining missing addends exactly.

^sMeaningfully learning the basic addition combinations entails constructing concepts and discovering patterns or relations. For example, children who construct the “big ideas” of composition and decomposition are more likely to exhibit adaptive expertise, to translate addition combinations (particularly larger ones) into easier or known expressions, and invent such reasoning strategies as the “doubles plus or minus one” (e.g., $7 + 8 = 7 + [7 + 1] = [7 + 7] + 1 = 14 + 1$) or “make a ten” (e.g., $9 + 7 = 9 + [1 + 6] = [9 + 1] + 6 = 10 + 6 = 16$).

^tResearch further indicates that fluency with basic number combinations may involve flexible use of a variety of efficient strategies (LeFevre, Smith-Chant, Hiscock, Daley, & Morris, 2003). With practice, many of the reasoning strategies children devise may become semiautomatic or automatic. Recent evidence further indicates that the mental representation of basic number-combination knowledge of mental-arithmetic experts is a web of richly interconnected ideas. For example, evidence indicates that understanding of commutativity not only enables children to learn all basic multiplication combinations by practicing only half of them (Baroody, 1999c) but also enables them to store both combinations as a single representation in long-term memory (Butterworth, Marschesini, & Girelli, 2003; Rickard & Bourne, 1996; Rickard, Healey, & Bourne, 1994; Sokol, McCloskey, Cohen, & Aliminosa, 1991). This view is further supported by the observation that the calculation prowess of arithmetic savants does not stem from a rich store of isolated facts but from a rich number sense (Heavey, 2003). In brief, the counting and reasoning phases may be essential for laying the conceptual groundwork and providing at least some the strategies for fluency with the basic combinations (the mastery phase).

^uChildren who construct the “big ideas” of composition and decomposition are also more likely to notice that addition and subtraction are related (Whole – Part 1 = Part 2 $\longleftrightarrow$ Part 1 + Part 2 = Whole)—a relation that is otherwise not salient to young children—and to invent the “missing-addend” reasoning strategy for subtraction (e.g., for $9 - 4 = ?$: $4 + ? = 9$, $4 + 5 = 9$; $? = 5$; Baroody, 1985, 1999a; Baroody, Ginsburg, & Waxman, 1983; Canobi, 2004; Fuson, 1988, 1992; Putnam, deBettencourt, & Leinhardt, 1990).

^vAs children practice the “missing-addend” reasoning strategy for subtraction, this reasoning strategy can become increasingly automatic.

^wAs with the single-digit arithmetic, children invent increasingly sophisticated mental-computation procedures (Cobb & Wheatley, 1988; Kamii, 1989) and fluency entails flexibly drawing on a variety of strategies. As with single-digit addition and subtraction, they discover patterns and relations that allow them to devise rules or reasoning strategies, which can become automatic. For example, children discover that adding single-digit numbers and 10 results in an analogous teen (e.g., $7 + 10$ or $10 + 7 \rightarrow$ seven + teen). This may become embodied as the general (algebraic) “teen rule”: $n + 10$ to $n +$ teen. Children also discover relations that allow them to exploit their existing knowledge. Some discover, for instance, that adding $40 + 30$ can be thought of as four 10s and three 10s and related to $4 + 3 = 7$ for a sum of seven 10s or 70. These discoveries are far more likely if children are accustomed to thinking in terms of the big ideas of composition and decomposition.

TABLE 11.2. Three Views of the Developmental Relations Among Number Recognition, Number Words, a Number Concept, and Object Counting	
Immediate number recognition, sometimes called <i>subitizing</i>, entails quickly recognizing the number of items in a collection without counting. Although sometimes viewed as a single process, it probably involves a number of skills, including nonverbal number recognition (e.g., equating $\bullet\bullet\bullet$ with $\square\square$ and verbal number recognition (e.g., immediately recognizing or labeling $\bullet\bullet\bullet$ or $\bullet\bullet\bullet$ as “three”). The latter encompasses recognizing regular patterns (e.g., equating a triangular array such as $\begin{smallmatrix} \bullet \\ \bullet\bullet \end{smallmatrix}$ with “three”) or decomposing collections into smaller recognizable collections and either using addition or multiplication to determine the total (e.g., viewing $\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix}$ as 2 and $3 = 5$ or viewing $\begin{smallmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{smallmatrix}$ as 3 groups of 4 or 12). Whether nonverbal number recognition, verbal number recognition, or object counting (enumeration) indicate a conceptual understanding of number is matter of considerable debate. Proponents of the Piagetian or extremely pessimistic view imply a skills-before-concepts (skills-first) view, whereas nativist or proponents of the extremely optimistic view advocate a concepts-first view. The Integrated View is consistent with intermediate perspectives, namely the iterative view and the simultaneous view.	
Skills-First View In the skills-first view, number (and arithmetic) skills are learned by rote memorization through imitation, practice, and reinforcement. The result is that skill learning is piecemeal. Through applying their skills, children discover number (and arithmetic) regularities view and concepts. Specifically, von Glasersfeld (1982) argued that the early reciting of number words is a meaningless skill. He also hypothesized that the skill of verbal number recognition is initially a perceptual skill that does not imply an understanding of number, because children view collections as perceptual configurations—as a set or whole, not as a set of individual items or units. He further argued that enumeration is also initially a skill learned by rote. Although the counting process forces children to treat counted objects as individual items (units), they do not conceptually understand that counting is a tool for determining the total number of units (the cardinal value of the collection or the whole). Thus, the initial use of number words, verbal number recognition, and enumeration do not entail viewing collection in terms of a true number concept—as <i>a set of individual items</i>. All three skills do not start to become meaningful until children use them in conjunction. By enumerating a collection they also can immediately recognize and verbally label, children realize that the former results in the total and that the perceptual configuration of latter is comprised of units. In brief, the simultaneous application of meaningless verbal-number-recognition and object-counting skills leads to a number concept and the meaningful use of these skills.	
Concepts-First View In the concepts-first view, children’s conceptual understanding enables them to devise meaningful procedures or skills. As indicated earlier, according to nativists accounts (e.g., Gelman & Gallistel, 1978), children have an innate understanding of number and counting principles and this prior understanding underlies infants’ ability to nonverbally recognize the number of items in small collections and toddlers’ capacity to rapidly learn number words and enumeration procedures.	
Iterative or Simultaneous View: Meaningful Subitizing and Number Word Use Underlies the Meaningful Learning of Object Counting (Enumeration Procedure) According to the iterative view, conceptual knowledge can lead to the invention of procedural knowledge, the application of which can lead to a conceptual advance, which in turn can lead to more sophisticated procedural knowledge, and so forth (Baroody, 1992b; Baroody & Ginsburg, 1986; Fuson, 1988; Rittle-Johnson & Siegler, 1998). Alternatively, a skill can be learned by rote and its application can lead to the discovery of a concept. This understanding can lead to a procedural advance and reflection of its application can lead to a deeper understanding and so forth. In some cases, concept and skill coevolve (the simultaneous view; Rittle-Johnson & Siegler). Children’s initial use of the first few number words may well be meaningless as von Glasersfeld (1982) suggested. However, the use of these words in conjunction with seeing examples and nonexamples of each can imbue them with meaning. As children hone the skill of verbal number recognition, they construct a cardinal concept of one, two, three, and four—abstracting oneness, twoness, threeness, and fourness (Baroody, Benson, & Lai, 2003; Benson & Baroody, 2002). In this view, then, children’s initial procedural knowledge (the verbal number recognition skill) and conceptual knowledge (cardinal concepts of one to about three or four) develop simultaneously. Moreover, this coevolution of number skill and concept may also enable children to see two as one and one and three as two and one or as one and one and one—as a collections composed of units or a whole composed of individual parts (Freeman, 1912). For example, in the case of Alice (Rosu & Baroody, 2005), a 2-year-old saw a collection of one and labeled it “one.” After seeing an item added to the initial collection, she commented, “Another one” and used visual number recognition to label the collection “two.” (The child had not yet learned how to enumerate even small collections and, indeed, did not consistently generate the first few number words in the standard order.) After yet another item was added, the girl commented, “Another one” and again used verbal number recognition to label the collection “three.” Apparently, she treated the three objects as a collection of units—as a whole composed of individual parts. The coevolution of verbal number recognition skill and conceptual knowledge of the intuitive numbers’ cardinality can also serve as the foundation for meaningful enumeration. Once children can reliably and meaningful recognize and label small collections, the use of this conceptually based verbal number recognition skill in conjunction with seeing enumeration modeled by others can enable children to understand the why and the how of the latter. Specifically, meaningful verbal number recognition can enable children to discern the purpose of enumeration (its another way of determining the total number of items of a collection or its cardinal value) and the rationale for enumeration procedures (e.g., why others emphasize or repeat the last number word used in the enumeration process because it represents the total number of items or cardinal value of the collection; see footnote f in Figure 11.2 for further discussion).	

footnote a in Figure 11.2) and discovering all or at least most of the counting principles (see footnote in Figure 11.2). It also differs from the highly pessimistic (early Piagetian) view that number development occurs independently of language and that verbal, or even object counting, is learned merely as a rote skill and, thus, not integral to the construction of a number concept. The developmental trajectory suggested by the Integrated View, then, represents a moderate position in the concepts-before-skills versus skills-before-concepts debate (Baroody, 2004c; see Table 11.2 for details). Note that verbal number recognition is hypothesized to be the developmental basis for other key number and arithmetic concepts as well (again see Figure 11.2, particularly footnotes b, d, and e).

Educational implications of research finding 1

Mathematical Learning Begins Early, Very Early. Because of the earlier conventional wisdom about mathematical development (e.g., the pessimistic views of James, 1890; Thorndike, 1922; Piaget, 1965), early childhood educators have, since the beginning of the twentieth century, generally not appreciated preschoolers' significant potential for informal mathematical knowledge and thinking. As a result, instruction in this area has typically been overlooked. Number and arithmetic instruction, if implemented at all, often focused on learning by rote the counting sequence, reading and writing numerals, and perhaps, single-digit number combinations such as "two and two makes four."

Although children may not be born with innate knowledge of number and arithmetic, as nativists contend, a concept of number and arithmetic begins to emerge in toddlers and is more robust than earlier scholars (e.g., James, 1890; Thorndike, 1922; and Piaget, 1965) suggested. Two key implications for childhood educators follow from existing research:

- Number and arithmetic concepts may not be naturally given but require nurturing.
- This nurturing can begin early, indeed, as early as toddlerhood (Baroody, 2000; Baroody & Benson, 2001; Clements et al., 2002). For example, toddlers and somewhat older children can be engaged in such one-to-one correspondence activities such as completing shape or knob puzzles or retrieving a cookie for each of several children, classification activities such as sorting objects into egg cartons by rows, equivalence activities such as determining whether there are the same number of spoons as bowls or whether two people have fair shares of candies, ordering or ordinal number activities such as gauging whether a 2-year-old or a 3-year-old is older, and addition or subtraction activities such as gauging a child's age at their next birthday or how many candies are will be left if one is eaten.

Learning Number Words May Help Toddlers Construct an Understanding of Number. Learning number words may help toddlers to abstract oneness, twoness, and threeness.

- Toddlers may benefit from seeing a variety of both examples and *nonexamples* of the intuitive numbers. For instance, by

seeing ••, ΔΔ, and □□ (examples of pairs) all labeled "two," young children may recognize that the appearance of the items in the collections is not important (e.g., shape and color are irrelevant to number). Furthermore, seeing •,•••, Δ, ΔΔΔ, □□, and □□□ (nonexamples of pairs) labeled as "not two" or as a number word other than "two" may help them define the boundaries of the concept of "two" and more accurately or selectively apply this word.

- Learning to recognize regular number patterns (e.g., immediately seeing that the ••• array on a die, a triangular array such as •••, and letter-shaped arrays such as •• and ••• as "three") may speed the process of abstracting the intuitive numbers. This stands in contrast to Piaget's (1965) dismissal of number-pattern recognition as a skill learned by rote and unimportant to number-concept development. (Note, though, that pattern recognition should be done as a component of the previous point. Otherwise, children may equate a number word with a particular pattern and not the general numerosity it represents.)
- Although helping children to discern other conceptual categories such as the colors can lay the foundation for discovering that numbers are a useful means of categorizing or classifying things, there is little evidence that logical training is necessary for developing a concept of number or at least certain types of number concepts (e.g., Baroody, 1987; Clements & Callahan, 1983).
- There is considerable evidence that counting experiences are related to the richness of a number sense and play a key role in extending children's understanding of number beyond the intuitive numbers (e.g., Baroody, 1987, 1992b; Fuson, 1988, 1992; Gelman & Gallistel, 1978). Indeed, research suggests that practice on meaningful counting activities leads to improved performance on logical tasks as well as number tasks (Payne & Huinker, 1993).

Finding 2: Informal Knowledge as a Critical Basis for Success With Formal Mathematics

Figure 11.2 illustrates the cumulative nature of mathematical development—how the earliest informal knowledge provides the basis for other informal competencies, which in turn ultimately provide the bases for formal competencies.

Research

Informal Knowledge as a Foundation for School Learning. A now large body of research indicates that the quality and quantity of children's informal experiences and knowledge is a key foundation for mathematical achievement in school (see reviews by, e.g., Dowker, 1998; Ginsburg et al., 1998; Kilpatrick et al., 2001). Blevins-Knabe and Musun-Miller (1996), for instance, found that the frequency with which parents engaged their children in number-related activities in the home was correlated to numerical knowledge as measured by an achievement test, namely, the *Test of Early Mathematics, Second Edition (TEMA-2*, Ginsburg & Baroody, 1990). Longitudinal

research indicates that preschool mathematical performance or achievement is predicative of mathematical achievement in school (e.g., Jimerson, Egeland, & Teo, 1999; Shaw, Nelsen, & Shen, 2001; Stevenson & Newman, 1986; Young-Loveridge, 1989). Furthermore, the quality of child care is at least modestly predictive of long-term mathematical achievement (Broberg, Wessels, Lamb, & Hwang, 1997; Peisner-Feinberg et al., 2001; Roth, Carter, Ariet, Resnick, & Crans, 2000).

However, citing Clements (1984) as an exception, Arnold, Fisher, Doctoroff, and Dobbs (2002) concluded that there is little experimental evidence documenting a causal connection between early (preschool) knowledge and mathematical performance in school. Moreover, the vast majority of existing empirical work focuses on post-Transition 2 developments, such as the relation between informal counting skills (e.g., verbal- or object-counting proficiency) and informal arithmetic skills (e.g., devising counting- or reasoning-based strategies to add or subtract) or formal number skills (e.g., recognizing numerals) or arithmetic competencies (e.g., mastery of basic number combinations). (For a detailed discussion, see, e.g., Baroody, 1987, 2004a; Baroody & Tiilikainen, 2003; Bideaud, Meljac, & Fischer, 1992; Clements & Sarama, in press; Cowan, 2003; Fuson, 1988, 1992; Ginsburg, 1977; Ginsburg & Baroody, 2003; Hughes, 1986; Kilpatrick et al., 2001; Mix et al., 2002a; Nunes & Bryant, 1996; Pound, 1999; Verschaeffel, Greer, & DeCorte, in press). Put differently, although there is substantial evidence for relations such as g to i and k to w in Figure 11.2, there is currently little or no evidence for the relations between nonverbal skills or verbal number recognition and other informal concepts or skills (e.g., relations a to f in Figure 11.2) and their long-term impact on school achievement.⁴ Although intuitively it makes sense that non-verbal skills or verbal number recognition would be indirectly or directly associated or predictive of later school success, clearly research is needed to determine whether this is so.

Knowledge Gaps: A Key Source of Learning Difficulties. Ginsburg (1977) noted that a key source of learning difficulty is a gap between a child’s existing knowledge and instruction. For primary-level children, in particular, existing knowledge consists largely of their informal mathematical knowledge. Gaps can occur for essentially two reasons:

1. *Unconnected Formal Instruction.* A gap can occur because formal instruction does not build on a child’s informal mathematical knowledge or strengths. This makes it difficult, if not impossible, to understand school instruction. The consequences can be routine expertise (Hatano, 1988, 2003) at best and learning difficulties and math anxiety at worse. Not surprisingly, mathematical learning difficulties and underachievement are often the result of an increasing gap between children’s informal knowledge and the symbolic (written) procedures and formulas formally taught in school (Ginsburg, 1977; Griffin, Case, & Capodilupo, 1995).

2. *Spotty or Inadequate Informal Knowledge.* A gap can also occur when a child does not have well-developed informal knowledge. Inadequate informal knowledge can greatly delay or hamper the learning of school mathematics in at least two ways (e.g., Allardice & Ginsburg, 1983; Baroody, 1987, 1996a, 1999b; Baroody with Coslick, 1998; Donlan, 2003; Ginsburg, 1977; Ginsburg & Baroody, 2003; Jordan, Hanich, & Urberti, 2003; New, 1998): (a) If a teacher attempts to relate formal instruction to common everyday experiences, a child without such experiences will not have a basis for meaningfully assimilating the school instruction. (b) Inadequate informal knowledge can also prevent children from inventing informal problem-solving strategies or interfere with their efficient execution or effective application. For instance, Donlan (2003) observed it is widely assumed that the development of arithmetic skills and the learning of number combinations, in particular, depend primarily on counting skill[s] (e.g., Fuson, 1988; Ginsburg, 1977)” (p. 342).

Case study evidence (e.g., Baroody, 1987; Dowker, 1998; Ginsburg, 1977) and a growing amount of longitudinal research (Oakes, 1990b; Roth et al., 2000; Shaw et al., 2001) indicate that inadequate informal mathematical knowledge is detrimental to school mathematics achievement. For example, Baroody (1984) found that improficiency with counting backward, which depends on learning the forward sequence and number-before relations, can seriously interfere with children’s ability to informally determine differences or solve subtraction problems. This is because they often use a counting-down procedure to model subtraction situations, a complex procedure that requires counting backward efficiently (e.g., for 9–7, “*Eight* is one taken away, *seven* is two, *six* is three, *five* is four, *four* is five, *three* is six, and *two* is seven taken away—so the answer is two”). Inefficient, particularly inaccurate, informal (counting-based) arithmetic procedures, in turn, make it less likely that children will discover the patterns and relations necessary to invent more sophisticated reasoning strategies (e.g., “9–7 = ? can be thought of as 7 + ? = 9” or 10–7 is 3 and 9 is one less than 10, so 9–7 is 2”; Baroody, in press; Dowker, 1998) or to memorize arithmetic facts accurately (e.g., Donlan, 2003).

Analogously, Donlan (2003), citing his own and Fazio’s (1994, 1996, 1999) longitudinal research, noted that children with specific language impairments often “show substantial deficits in counting procedures” (p. 355). These and short-term memory deficits “may combine to limit strategy development in simple arithmetic” (p. 355), the cumulative effect of which can seriously restrict conceptual development.

In this same vein, Mazzocco and Myers (2003) found that, although no single criterion or combination of criteria in kindergarten consistently predicted which children would be categorized as having a mathematical learning disability in Grades 2 or 3, a score of less than the tenth percentile on the *TEMA-2* (Ginsburg & Baroody, 1990) was the most consistent

⁴Research in this area has just begun, and little available is often ambiguous. Hannula and Lehtinea (2004, in press), for instance, found remarkable individual differences among 3- to 7-year-olds in their use of enumeration and possibly verbal number recognition in the service of solving a nonverbal problem (imitating a tester who put out, e.g., 2, 2, and 1 items successively). They equated performance on this task with a tendency to attend to numerical information (SFON for Spontaneous Focusing On Numerosity) and concluded conceptual and procedural advances in enumeration facilitated SFON, which in turn prompted a greater tendency to apply and thus practice enumeration, further promoting SFON, and so on. Although this conclusion makes intuitive sense, unfortunately the correlational analysis used did not permit a causal analysis.

criteria. They attributed this result to the fact that the *TEMA-2* assessed a wide range of informal, as well as formal, concepts and skills. Mazzocco, Myers, and Thompson (2003) subsequently found that a battery of four tests administered to preschoolers was predictive of which children would be designated “learning disabled” in second grade. Interestingly, a composite of three informal items (cardinality, comparisons of one-digit numbers, and mentally adding one-digit numbers) and one formal item (reading numerals) from the *TEMA-2* were nearly as predictive as the entire battery (Mazzocco & Thompson, in press).

Educational implications of research finding 2. There have been numerous efforts to survey the specific number and arithmetic competencies of children entering first grade (Brownell, 1941; Buckingham & MacLatchy, 1930; Callahan & Clements, 1984; Ginsburg & Baroody, 1983, 1990, 2003) and kindergarten (e.g., Bjoneruds, 1960; Reaney & West, 2003; Rea & Reys, 1971; Reys & Rea, 1970; see Kraner, 1977, for a review). There have even been a number of efforts, particularly in recent years, to survey such knowledge among preschoolers (Ginsburg & Baroody; McLaughlin, 1935; again see Kraner for a review). Perhaps the most notable is the national effort by the National Center for Education Statistics to track development from birth through first grade (<http://nces.ed.gov/ecls>).

Despite these survey efforts and the extensive research literature on preschoolers’ number and arithmetic development (see, e.g., Baroody, 1987, 2004a; Baroody & Wilkins, 1999; Bideaud, Maljac, & Fischer, 1992; Clements et al., 2004; Frye, Braisby, Lowe, Maroudas, & Nicholls, J. 1989; Fuson, 1988, 1992; Gelman, 1998; Gelman & Gallistel, 1978; Ginsburg, 1977; Ginsburg, Inoue, & Seo, 1999; Ginsburg et al., 1998; Hall, 1891; Mix et al., 2002a; Rittle-Johnson, & Siegler, 1998; Sophian, 1995, 1998; Sophian, Harley, & Martin, 1995; Wynn, 1990, 1992b), preschool and elementary-level educators have historically overlooked the importance of informal mathematical knowledge for success with formal mathematical instruction. Typically, the result has been that informal strengths and weakness have not been assessed and deficiencies in informal knowledge have not been remedied (Baroody & Ginsburg, 1991; Ginsburg & Baroody, 2003).

With the trend toward reform-based or Standards-based instruction, ensuring that students have solid informal knowledge is more important than ever. This is because such instruction—unlike traditional instruction—focuses on achieving all aspects of mathematical proficiency, and—unlike meaningless skill learning—conceptual understanding, appropriate and flexible application of skills, and autonomous strategic mathematical thinking depend on accessing ample prior knowledge. This may help to explain why evaluation research (e.g., Boaler, 1998; Fuson, Carroll, & Drueck, 2000; Woodward & Baxter, 1997) often finds that Standards-based curricula help to promote the conceptual understanding, mastery of basic skills, and problem-solving ability of high and average achievers but not low achievers, who typically have serious gaps in their informal knowledge. In brief, two key implications follow:

1. Formal mathematical instruction, including the mastery of basic skills (e.g., reading and writing of numerals and fluency

with single-digit number combinations), should build on children’s informal mathematical knowledge.

2. Gaps in informal knowledge need to be identified and filled early (during the preschool years or the first years of school).

If school instruction does not build on children’s informal knowledge, it will probably not make sense and thus, will make learning mathematics unduly difficult and the achievement of all aspects of mathematical proficiency unlikely (Baroody with Coslick, 1998). This is particularly true if gaps in informal knowledge are not filled early because the result all too often is that children become victims of a spiral of frustration, failure, and despair (Ginsburg & Baroody, 2003).

Finding 3: Individual Differences in Informal Knowledge

The research on individual difference is summarized first (see also Dowker, 1998). The debate about the source of most of these individual differences is then addressed.

Research on individual differences. Significant individual differences in verbal and written number and arithmetic skills are already evident when children enter kindergarten or first grade (Baroody, 1987; Klein & Starkey, 2004; Starkey & Klein, 2000; Wright, 1991). For example, Canobi (2004) found individual differences in conceptual knowledge of part-whole relations and problem-solving ability among 6- and 7-year-olds and that differences in the former had a significant impact on the latter. Moreover, because differences in conceptual knowledge were independent of grade, Canobi concluded that these differences were due to variation in conceptual development during the preschool years.

Furthermore, children with low skills at the onset of school tend to remain relatively low achievers throughout their formal education, and many develop math anxiety, do not pursue more advanced courses in mathematics, or otherwise disengage from mathematics (Wright, 1994).

Recently, research has focused on individual differences in nonverbal number and arithmetic skills. The results regarding individual differences among preschool children in this area have not been entirely consistent. On the one hand, Jordan and colleagues (Jordan, Huttenlocher, & Levine, 1992, 1994; Jordan, Levine, & Huttenlocher, 1994; see Jordan, Hanich, & Uberti, 2003, for a detailed discussion), for example, have found that:

- preschool children are more successful on *nonverbal* arithmetic tasks than on *verbal* (word-problem) arithmetic tasks; and
- children from low-income families were as successful at nonverbal arithmetic but not as successful at verbal arithmetic as were children from middle-income families.

Jordan et al. (2003) concluded that social class differences are not apparent with nonverbal mathematical competencies because such skills do not require social support to develop but differences do appear with verbal number and arithmetic skills because these competencies do require social mediation.

In effect, the former could be considered akin to cognitive universals. It would follow, then, that individual differences would be significantly less evident for nonverbal than for verbal number and arithmetic skills, even among children from low-income families.

On the other hand, other researchers (e.g., Baroody, Benson, & Lai, 2003; Klein & Starkey, 2004; Starkey & Klein, 2000) have found individual differences among 2- to 4-year-olds on nonverbal or precounting number tasks. For instance, Starkey and Cooper (1995) found that 2-year-olds differed significantly in their ability to recognize the number of items in small collections of one to four items. Some evidence suggests that nonverbal number skills with larger intuitive numbers (three and four) at least depend on verbal number recognition (reliably and discriminately recognizing and verbally labeling a collection; Benson & Baroody, 2003). Analogously, Mix (1999) found success on all but the simplest equivalence tasks (e.g., simultaneous homogeneous collections of the same kind of item) involving intuitive numbers and concluded that a verbal represented aided or was critical to success on more advanced tasks (e.g., simultaneous homogeneous collections of dissimilar objects such as four blocks and four balls, simultaneous heterogeneous collections, or collections presented successively). These results suggest that facility with nonverbal number skills, with larger intuitive numbers at least, may depend on verbal number skills and, thus, social mediation.

In brief, important individual differences may occur with verbal number and arithmetic competencies and at least some nonverbal math skills that are socially mediated. Further research is needed to determine what role number words play in numerical and arithmetic competencies, particularly with the intuitive numbers and whether delays or difficulties with achieving proficiency in verbal number recognition of these have a domino effect that ultimately has an adverse impact on formal achievement.

The primary cause of individual differences in nonverbal and verbal mathematical competencies among preschoolers. Undoubtedly, some individual differences are due to genetic or acquired organic impairment. See, for example, Delazer (2003) and Macaruso and Sokol (1998) for a review of the effects of brain injuries on mathematical competencies, Baroody (1999b) for the effects of mental retardation, Donlan (1998, 2003) for a review of the effects of specific language deficits, and Nunes and Moreno (1998) for reviews of the effects of hearing impairments. However, most individual differences are probably due to the lack of opportunity. Support for this conjecture comes from two sources of group differences, research on the effects of (a) social class and (b) culture.

The Effects of Poverty. Although research does not consistently find social class differences, even within the same study some trends are evident. Overall, research indicates that socioeconomic status (SES) is a key factor in school success. For example, middle-class children outperform children from low-income families on at least some number and arithmetic tasks (e.g., Denton & West, 2002; Griffin & Case, 1997; Griffin, Case, & Siegler, 1994; Kirk, Hunt, & Volkmar, 1975; Starkey & Klein, 2000; [Starkey, Klein, & Wakeley, 2004](#); but cf. [Seo & Ginsburg, 2004](#); see [Arnold & Doctoroff, 2003](#), [Kilpatrick et al., 2001](#), and [Secada, 1992](#), for reviews). “Students reporting higher levels of parental education tend to have higher average scores on [National Assessment of Educational Progress] assessments” ([Kilpatrick et al., 2001](#), p. 143).

A spiral of failure or “poor math trajectories in low-SES children begins very early” ([Arnold & Doctoroff, 2003](#)). Indeed, poverty is associated with cognitive, achievement, and behavior differences among preschoolers (e.g., [Duncan & Brooks-Gunn, 2000](#)). Low social or economic status can adversely affect early informal knowledge ([Case, 1975](#); [Entwisle & Alexander, 1990, 1996](#); [Shaw et al., 2001](#)). Students identified as having lower SES enter school with lower achievement levels in mathematics than middle- or high-SES students (see reviews by [Arnold & Doctoroff, 2003](#), [Kilpatrick et al., 2001](#), and [Secada, 1992](#)). More specifically, many children from low income families have relatively little productive informal experiences and, as a result, enter kindergarten with significantly less number and arithmetic knowledge ([Denton & West, 2002](#); [Griffin, Case, & Siegler, 1994](#); [Hughes, 1986](#); [Jordan, Huttenlocher, & Levine, 1994](#), [Klein & Starkey, 1995](#); [Saxe, Guberman, & Gearhart, 1987](#)). For example, [Ginsburg and Russell \(1981\)](#) found that although children from low-income families had a number of informal strengths, they were significantly less likely to informally solve arithmetic word problems.

Poverty itself, of course, does not cause learning difficulties. A chief candidate for the intervening or linking variable between them is family process ([Brooks-Gunn, 2003](#)). For example, often because of their own academic difficulties, parents in low-income families provide less support for academic learning in the form of monitoring, scaffolding, and expectations ([Brooks-Gunn, Klebanov, & Liaw, 1995](#); [Campbell & Mandel, 1990](#); [Entwisle & Alexander, 1996](#); [Halle, Kurtz-Costes, & Mahoney, 1997](#); [Jimerson et al., 1999](#); [Klein & Starkey, 1995](#)). For example, middle-class parents tend to provide a broader range of activities and more complex tasks and did so more often than working-class parents ([Saxe et al., 1987](#); [Starkey et al., 1999](#)).⁵ In brief, the behavior and beliefs of many low SES parents make it less likely their children will achieve the four aspects of mathematical proficiency. This is not to say there are not important ex-

⁵There continues to be considerable debate about the relative contributions of genetics and environment. Some have argued that innate endowment significantly influences the level of intelligence, and the latter, in turn, affects SES and the inadequate or otherwise ineffective parent stimulation for learning informal mathematics. Although there may be an element of truth in this argument, particularly for extreme cases, it seems at least plausible that most social class variation in the quantity and quality of parental stimulation stems from environmental factors. For example, [Benigno and Ellis \(2004\)](#) found that middle-class parents were more likely to use games as an opportunity to teach number skills if they were playing with only a preschooler than if they were playing with a preschooler and an older sibling. It is not clear why this is the case, but many people who have raised children might attribute at least some of the drop off in attentiveness to fatigue. This sibling-fatigue factor may be even more pronounced in families with low income because such families often have more children than in middle-class families.

ceptions. For instance, Moll, Amanti, Neff, and Gonzalez (1992) discussed the funds of knowledge available in low SES Latino homes.

Another chief candidate for a linking variable between social class and low achievement is that the quality of schools serving low SES children is generally inferior to that of schools serving higher SES children. Poverty is significantly associated with academic failure in first grade and becomes more so with time (Griffin et al., 1995; McLoyd, 1998; Pagani, Boulrice, & Tremblay, 1997), a result that suggests public education is not helping or even failing disadvantaged children (Brooks-Gunn, 2003).

Indeed, the curricula of elementary schools serving high-poverty communities tend to focus on routine expertise (mastery of basic facts and other skills), whereas those serving affluent are more likely to focus on adaptive expertise and strategic mathematic thinking (Anyon, 1980; Matti & Weiss, 1999; Oakes, 1990a). This may be due in part because fostering all aspects of mathematical power requires highly qualified teachers and schools serving minority and low-income children are less likely to have such a faculty than schools serving more affluent children (Darling-Hammond, 2000; Ferguson, 1991, 1998; Greenwald, Hedges, & Laine, 1996; Wright, Horn, & Sanders, 1997). This, in turn, is due at least in part to inferior working conditions (e.g., larger class size and less teacher autonomy), conditions that work against recruiting and retaining highly qualified teachers (Darling-Hammond, 1997; Gilford & Tenebaum, 1995; Ingersoll, 1999). Another reason for inferior curricula and instruction is that schools serving low-income or minority students have fewer technology resources than more affluent schools (Educational Testing Service Policy Information Report, 1997).

Cross-Cultural Evidence. The superior mathematical performance of Asian schoolchildren has been well documented (see, e.g., McKnight et al., 1987; Miura & Okamoto, 2003; Song & Ginsburg, 1987; Stevenson, Lee, & Stigler, 1986; Stevenson & Stigler, 1992; Stigler, Lee, Lucker, & Stevenson, 1982; Takeuchi & Scott, 1992; Towse & Saxton, 1998). As Ginsburg, Choi, Lopez, Netley, and Chao-Yuan (1997) noted, one hypothesis about the source of this discrepancy “is that Asian children enter school with superior informal mathematical abilities in such areas as counting, concrete and mental arithmetic, concepts of more and less, and the like. This informal mathematical knowledge might provide Asian children with a superior foundation for learning what is later taught . . . in school” (p. 166; cf. Starkey et al., 1999).

However, the relatively little cross-cultural research on preschoolers’ informal mathematical knowledge has yielded mixed results. Consistent with the hypothesis just discussed, Geary, Bow-Thomas, Fan, and Siegler (1993) found that even before formal instruction, Chinese children outperformed American children in mental addition. Inconsistent with the hypothesis, Baroody and Lai (2003) did not find a significant difference between U.S. and Taiwanese 4- to 6-year-olds’ informal understanding of the addition-subtraction inverse principle (e.g., adding two items to a collection can be undone by taking away two other items).

Similarly, Song and Ginsburg (1987) found that, on a variety of competencies measured by a standard test of early

mathematical ability (Ginsburg & Baroody, 1983), U.S. 4- and 5-year-olds significantly outperformed their Korean counterparts. However, these results are not conclusive because the test was developed for U.S. children and may not have been entirely appropriate for Korean children (Ginsburg et al., 1997). Using a more sensitive instrument, Ginsburg et al. found that Chinese, Japanese, and Korean preschoolers exhibited a small but consistent advantage over U.S. preschoolers.

Both Miura (1987; Miura, Kim, Chang, & Okamoto, 1988; Miura & Okamoto, 1989) and Miller (1996; Miller, Smith, Zhu, & Zhang, 1995; Miller & Stigler, 1987) have noted that the highly regular nature of Asian (Chinese, Japanese, and Korean) counting sequences facilitates the learning of this sequence and grouping-by-ten concepts and that the somewhat irregular nature of the English counting sequence interferes with mastering these competencies (see Miura & Okamoto, 2003, for a review). However, Towse and Saxton (1998) observed that an association between differences in language and differences in performance on a conceptual (grouping-by-ten) task does not rule out alternative explanations or demonstrate a direct causal relation. They argued that differences in the nature or amount of parent-child interactions or preschool experience may be more important than differences in the orderliness of counting sequences. For example, parents in Asian countries value education and mathematics more than U.S. parents, and this may translate into more informal instruction in the home. (Tudge and Doucet, 2004, found that U.S. parents engaged their 3-year-olds in mathematically related activities infrequently, regardless of ethnicity or social class.) Asian countries have well-organized and extensive preschools (Towse & Saxton, 1998). In both informal home settings and more formal preschool settings, Asian cultures may put more emphasis on key concepts such as composition (composing a whole from its parts, perhaps in different ways) and decomposition (decomposing a whole into its constituent parts, perhaps in different ways) by, for instance, making available an abacus, which can highlight that $1 + 4$, $2 + 3$, $3 + 2$, or $4 + 1$ are all ways of making five.

Interestingly, though, Miller (e.g., 1996) found the advantage of a regular counting sequence did not carry over to understanding counting principles or applying object-counting skills, such as understanding the cardinality principle. Case’s (Case, Griffin, McKeough, & Okamoto, 1991; Case & Okamoto, 1996; Griffin & Case, 1995b) results were analogous to Miller’s. They tested 4-6- and 10-year-old children from high, medium, and low social classes from the United States, Canada, Japan, and China over a 6-year span. In the domain of number, Case and Okamoto (1996) found, for instance, large and consistent differences favoring the Japanese in particular numerical skills and concepts (e.g., number knowledge and computational skill) but not in central numerical structures (general concepts). They concluded that high parental expectations and involvement and language structure may help Japanese children perform better in the particular numerical skills. However, although specific instructional opportunities differ widely across the two groups, more general educational opportunities do not. The everyday lives of children in both cultures provide ample opportunity to encounter quantitative problems in such domains as number

and arithmetic and to construct central structures or general concepts in these domains. Whether this is true of only a few (those assessed by extant research), many, most, or all central concepts (e.g., composition and decomposition), however, is currently unclear.

Educational implications. In the past, early childhood and primary-level educators largely underestimated either the serious individual differences among children’s informal knowledge or the need for targeted opportunities to close gaps in this knowledge. These oversights are embodied in the traditional policies or practices for enrolling children at risk or with a developmental delay in kindergarten. One practice is to provide no intervention before kindergarten and, if children fail to prosper in kindergarten, have them repeat this grade (i.e., provide the same experiences again a year later). The second practice is to delay the start of kindergarten by a year. In effect, both traditional practices are based on the assumption that some children just take longer to develop and simply postponing the start of formal mathematics (and reading) instruction (entry into first grade) will be sufficient.

However, research indicates that “time alone may not serve to improve academic achievement” of children at risk or developmentally delayed (Reaney & West, 2003). Put differently, the historical practices of delaying the start of the first year of kindergarten, repeating kindergarten, or transition classes are not effective (May & Kundert, 1997). Other research on individual differences and related factors have created enormous interest in early childhood mathematics instruction for a number of interrelated reasons:

- The evidence that opportunity is a key factor in individual differences underscores the need and even the civil right for equitable education (Miller, 1995; Moses, 2001; Oakes, 1990a; Schoenfeld, 2002).
- Children from low-income families or minority groups are particularly likely to lack the opportunities to develop robust informal mathematical knowledge, a disadvantage that is further compounded by poor formal instruction. Therefore, such children are especially prone to be caught in a spiral of failure and, thus, are more often excluded from professional and technology-related jobs, jobs that require a high degree of mathematical proficiency.
- An increasing number of families, both parents work (Hoffman, 1989) and, thus, an increasing number of preschoolers are enrolled in preschools or other early care (U.S. Department of Education, 2000).
- In part because of the research and reform effort discussed earlier, the mathematics goals and curricula for preschool, kindergarten, and later primary grades have become more demanding (Karweit, 1994).
- Gaps in mathematical achievement stem from structural factors such as poverty and other inequities (Oakes, Joseph, & Muir, 2004). Furthermore, the number of preschoolers living in poverty—including ESL children—and, thus, at risk for school failure is increasing (McLoyd, 1998).

- Current preschool learning environments typically do “not provide focused, systematic support for young children’s mathematics development” (Starkey et al., 2004, p. 101). This is particularly critical for children from low-income families or otherwise at risk (see bullets 2 to 4, above).
- Although not conclusive (Karweit, 1994), some evidence indicates that early childhood intervention is an important investment for later school success (e.g., Campbell, Pungello, Miller-Johnson, Burchinal, & Ramey, 2001; Frank Porter Graham Child Development Center, 1999; National Institute of Child Health and Human Development, 1998; Reynolds & Temple, 1998; Schweinhart, Barnes, & Weikart, 1993; Weikart & Schweinhart, 1992; Weikart, 1998; Weikart & Schweinhart, 1997; Zigler & Styfco, 1998; but cf. Gray, Ramsey, & Klaus, 1983).

Ensuring that all students have solid foundational knowledge, then, seems to be an essential first step for achieving *equity* (e.g., NCTM, 1989, 2000; Kilpatrick et al., 2001). Early intervention is now viewed as one key step toward ensuring a level playing field. This is particularly true when intervention builds on the informal strengths that at-risk children do have and target specific gaps in their informal knowledge (Clements & Sarama, 2003; Ginsburg & Baroody, 1990, 2003).

Unfortunately, relatively few intervention studies have to date focused on mathematical knowledge. Of those that have, fewer yet have focused on the effects of long-term intervention or the long-term effects of intervention of any kind. For example, Arnold et al. (2002) examined the effects of intervention lasting 6 weeks on participants immediately after the end of intervention. Several long-term early interventions have had a positive impact on the children at risk for academic failure at least in the short term (Kim, Charmaraman, Klein, & Starkey, 2000; Klein & Starkey, 2004; Sarama, 2004; Sophian, 2004; Starkey & Klein, 2000).

Another limitation of existing research is that the dependent measures involved global assessment such as standard mathematics achievement scores (e.g., Arnold et al., 2002). For example, Dev, Doyle, and Valente (2002) found that after 2 years of intervention, children at risk for reading and mathematics difficulties showed improvement on the WRAT-III and were no longer in need of special education services at the end of second grade. A number of recent efforts (e.g., Starkey et al., 2004) have entailed examining mathematical performance in some detail (i.e., include training and testing of, e.g., matching and comparing, sorting, patterning, sequencing, and numbers). However, even this level of specificity is relatively general and not sufficient to detail conceptual and procedural learning trajectories. For instance, number entails a variety of specific concepts and skills, such as enumeration (counting a collection to determine its cardinal value) and verbal production (counting out a requested number of items as in “Give me three blocks”).

One notable early intervention effort parallels the *Reading Recovery* program. The *Math Recovery* program (Wright, Martland, & Stafford, 2000; Wright, Martland, Stafford, & Stanger, 2002) builds on constructivist theory (e.g., Cobb & Wheatley, 1988; Steffe & Cobb, 1988) and involves individual instruction

by specially trained teachers for 30 min a day, 4 days a week for 12 to 20 weeks. Informal number, counting, and arithmetic competencies are evaluated and intervention entails challenging children to use increasingly sophisticated strategies in these areas. The aim of the program is to help children at risk match the informal competencies of their more advantaged peers. Efforts to evaluate this program have only recently begun.

THE NATURE OF EARLY CHILDHOOD MATHEMATICS EDUCATION

Early in the 20th century, John Dewey experimented with reforming instruction. His “Progressive Education Movement” was largely a reaction to traditional instruction and, thus, in many ways a direct opposite of it. In place of direct instruction by a teacher, for example, children were encouraged to explore their world unencumbered by adult intervention. Disappointed by the results of his early experiment with educational reform, Dewey (1963) summarized the lessons learned in *Experience and Education*. These lessons are still valuable today as early childhood educators consider how to improve preschool intervention and revise primary-level instruction. In effect, Dewey laid out three criteria for successfully changing instruction.

1. **Methods: *How to teach.*** New teaching methods cannot be substituted for traditional methods merely because they are different from the latter. *New teaching approaches, methods, or tools must have their own theoretical, empirical, and practical justification.*
2. **Goals: *What to Teach.*** Instruction cannot simply consist of a hodgepodge of activities without clear educational purposes. *Teachers must strive to provide educative experiences* (experiences that lead to worthwhile learning or a basis for later learning; see also Sophian, 2004), not *mis-educative experiences* (activities for the sake of activity and that may actually impede development). In a similar vein, Lillian Katz cautioned that “just because [children] can does not mean they *should!*” For example, “little children can learn to do formal sums but that does not mean they should—or that it would be in their long-term interests to do so. . . . They may be learning that mathematics is boring” (Pound, 1999, p. 35).
3. **Timing: *When to Teach.*** Educative experiences result “from an interaction of external factors, such as the nature of the subject matter and teaching practices, and internal factors, such as a child’s [developmental readiness] and interests” (Baroody, 1987, p. 37). Teachers, then, must ensure that external and internal factors mesh. Dewey’s principle of interaction bears on the issues of how, what, and when to teach. In regard to the latter, teachers need to consider the developmental readiness when deciding to introduce a topic, using, for instance, children’s questions as a guide.

In this part of the chapter, we review some of the relevant research to each of the three issues discussed above and discuss its educational implications.

Research Implications Regarding How to Teach Mathematics

In general, we have found that the younger the child, the less willing he or she was to answer thoughtfully, or at all, as a task moved from one initiated by the participant to one imposed by an adult (Baroody, 2001, 2002; Benson & Baroody, 2003). Consider the case of Nathaniel (2.8 years); Baroody, Benson, and Lai (2003). Adult-initiated testing indicated he had little nonverbal or verbal number knowledge. He did not, for example, respond correctly to a matching or a nonverbal production task involving one to four items. (The tasks require creating a collection equivalent to a visible collection and a previously seen but hidden collection, respectively.) He also refused to play a game that entailed determining the cardinal value of small collections. However, as the tester was cleaning up, Nathaniel asked for a soldier, and she allowed him to keep one until the next session. As tester was leaving, he became upset and said, “I want two soldiers.” He went into her bag and grabbed another soldier and said, “Aha! I got two!” He apparently recognized that the one soldier in his hand was not enough (not “two”); he also appeared to know that he needed another soldier to have two soldiers. In brief, the adult-initiated testing did not accurately reflect his knowledge, which was exhibited in self-initiated activities (see also Mix, 2002). (Parenthetically, it seems plausible that low SES children may be even less accepting of adult-imposed tasks and that traditional research on the effects of social class underestimate the mathematical—and other intellectual—competencies of these children; H. P. Ginsburg, personal communication, March 1, 2004.)

How can such results be reconciled with Dewey’s (1963) guidelines? Listed below is a continuum of types of teaching or testing tasks and approaches to teaching or testing (Baroody, 2001):

- Traditional direct instruction and drill/Adult-imposed task—Directed response (e.g., a direct question or work sheet)
- Guided discovery learning/ Adult-initiated task—Invited response (e.g., a project or math game)
- Flexible guided discovery learning/Child-initiated task—Guided response (e.g., building on a child’s question)
- Unguided discovery learning/Child-initiated task—Unguided response (e.g., free play)

Our research indicates that an overly direct approach risks not engaging children and may not be effective, particularly with younger children. Dewey’s experience indicated that unguided discovery approach risks not providing educative experiences. Early childhood educators face a balancing act—that is, an approach that is neither too direct nor too hands off. The two middle courses listed here, then, seem to be the most promising. See, for example, Clements et al. (2004), Ginsburg et al. (in press), and Pound (1999), for general instructional guidelines and examples of developmentally appropriate preschool curricula. See Zur and Gelman (2004) for an excellent example of how careful observations by teachers can

lead to a developmentally appropriate and interesting educational (computational estimation) game. See Benigno and Ellis (2004) for an interesting analysis of how parents can take advantage of a game as an opportunity to teach basic number skills.

Consider, for example, the role of manipulatives. On the one hand, research suggests that direct instruction with manipulatives is not likely to provide an educative experience for young children. That is, simply demonstrating concrete models for children and then requiring them to imitate such manipulative-based procedures may not promote conceptual learning (e.g., Baroody 1989c; Clements & McMillen, 1996; Fuson & Burghardt, 2003; Miura & Okamoto, 2003; Resnick, 1982; Seo & Ginsburg, 2003; cf. Sophian, 2004). It is difficult, if not impossible, to impose understanding. On the other hand, the presence of manipulatives alone in a free play context does not guarantee an educative experience either. That is, simply providing manipulatives without a purpose, direction, guidance, or feedback may not promote conceptual understanding, computational fluency, or strategic mathematical thinking. Manipulatives may be most useful when children have a purpose of their own or an adult creates one and when children reflect on their use or peers or adults cause them to do so. Meaningful use of manipulatives also entails children relating (assimilating) this experience to their existing knowledge. This may be more likely to happen if students are asked to use what they know to devise their own solutions (guided or flexibly guided discovery) than if they are shown how to use manipulatives (direct instruction). (See Balfanz, 1999, for a thoughtful discussion of how Froebel and Montessori artfully used manipulatives to guide discovery of mathematical ideas.)

Our conclusions are consistent with those of the NAEYC and the NCTM. Clements et al. (2002), for example, noted that it is doubtful that incidental experiences or learning will promote educative experiences. They concluded that existing evidence indicates young children are ready for organized, sequenced experiences embedded in specific activities (e.g., reading children’s literature, music or art activities), play (e.g., math games or physical activities), projects, or a careful combination of approaches.

Not coincidentally, a number of scholars (e.g., Bredekamp & Copple, 1997) now consider the Vygotskian, rather than the Piagetian, perspective ‘the central theoretical grounding within the field of early childhood education’ (Winsler, 2003, p. 254).⁶ Vygotsky (1962) differed with Piaget (e.g., 1965) in key ways, including:

- a focus on sociocultural factors as well as cognitive factors;
- a central role for teachers as well as for peers and individual reflection;
- a key role for language in shaping learning and thought.

These key interrelated differences parallel three important Vygotskian concepts now commonly cited in developmental and early childhood literature:

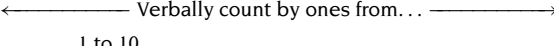
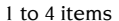
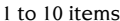
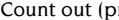
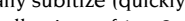
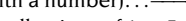
- The *zone of proximal development* is the domain-specific—or even activity-specific—competence a child can exhibit with careful and minimal assistance. This zone exists between the zone of competence (what a child can do independently) and the zone of incompetence (what a child cannot learn even with help). See Ginsburg et al. (in press) for an interesting and informative discussion of teaching experiments that use problem-solving situations “under adult guidance or in collaboration with more capable peers” (Vygotsky, 1986, p. 86) to take advantage of young children’s zone of potential development.
- *Scaffolding* refers to the (minimal or indirect) support a teacher, parent, or others can provide (e.g., asking the child a question or creating cognitive conflict, instead of telling him or her what to do). It entails using an engaging activity to assess a child’s zone of proximal development and, ideally, carefully decreasing guidance to keep the child in this zone.
- Gradual *internalization* of cultural tools. For instance, thought is viewed as self-directing language that has become subvocal. (See Chapter 4 of Vygotsky, 1978, 1987, for a detailed discussion of the transformations that occur, the interaction between rational and cultural lines of development, and how his views on the topic differ from the behaviorist views of Watson, 1930).

The key implication of Vygotsky’s (1962, 1978) theory for the topic under discussion is his central thesis that “early number understanding emerges first in social interaction with more capable partners and only gradually becomes internalized” (Benigno & Ellis, 2004, p. 6). Put differently, whereas a Piagetian approach is generally consistent with an unguided discovery learning approach, a Vygotskian approach is more consistent with a guided discovery or flexible guided discovery learning approach. Winsler (2003) noted there has been an explosion of Vygotskian-based educational research in recent years. In the area of mathematics, he cited, for example, Berenson (1997); Bussi and Bartolini (1998); De Abreu (2000); Jones and Thornton (1993); Lerman (1996, 2001); Saxe, Dawson, Fall, and Howard (1996); Saxe and Guberman (1998); and Waschesio (1998). For a comprehensive survey of Vygotsky’s thinking regarding developmental and educational psychology, see Langford (2004).

Note that effectively implementing guided discovery learning has important implications for class size. Such instruction is difficult to implement with 15 or more students. Even preschool teachers with small class sizes need to consider how to involve curriculum supervisors/specialists, parents, preservice teachers, older children, peers, and volunteers trained to perform

⁶The influence of Vygotsky on educational practice is a worldwide phenomenon. For example, educators from both mainland China and Taiwan have concluded that traditional curricula are not effective in promoting all aspects of mathematical proficiency (e.g., Chang, 1999; Huang, 1988) and that reform based on Vygotsky’s views (e.g., child-centered but adult guided instruction) is needed (Chung, 2000). In China, the reform movement began in the early 1990s under the name of “Quality Education” (Zhou, 1995). In Taiwan, the reform movement likewise began in the early 1990s (e.g., Government Information Office, 2003).

TABLE 11.3. Four Levels of Number Goals (Based on a Portion of Table 1.1 in Clements, 2004)

Topic	2–4 years	4–5 years
	Counting	
	Counting can be used to find out <i>how many</i> in a collection.	
<i>b. Another key element of object-counting readiness is learning standard sequences of number words.</i>		
<i>c. Object counting involves creating a one-to-one correspondence between a number word in a verbal counting sequence and each item of a collection, using some action indicating each action as you say a number word.</i>		
	Count the items in a collection and know the last counting word tells "how many"	
<i>d. Number patterns can facilitate determining the number of items in a collection or representing it.</i>		
	Count out (produce) a collection of a specified size.	
		
	Verbally subitize (quickly "see" and label with a number)...	

specific instructional tasks (e.g., reading a story to one group of children as another small group undertakes an investigation). They need to consider other ways of multiplying effort such as engaging children with educative computer programs, audio-visual work station (e.g., an instructional station with a story, questions, instruction, etc. on a video- or audiotape, CD, or DVD), educational television programs, and educative activities, projects, or play stations.

Research Implications Regarding What to Teach

Our discussion of what to teach will focus on (a) levels of developmentally appropriate goals, (b) the need for integrated instruction (instruction that encompasses several or many goals), and (c) the need for teachers to focus on big ideas (major goals), rather than specific facts or procedures.

Topic 1: Levels of developmentally appropriate goals.

Early childhood educators need clear-cut goals ranging from global aims (“really big ideas”) to specific objectives (Baroody, 2004d). For example, see in Clements (2004) the Developmental Guidelines for Number and Operations (his Table 1.1) and in Baroody (2004a), A Sample of Possible Early Childhood Number and Operations Goals (his Table 7.1). A portion of the former, which illustrates three levels of goals, is shown in Table 11.3.

It makes sense for pre- and in-service teacher education to begin with the most general goals. These really big ideas can help teachers focus on what is most important to accomplish, are less overwhelming to learn than are more detailed goals, can help them see connections between diverse examples or topics, and can bolster confidence that they can be effective mathematics teachers. Once really big ideas are grasped, teachers may feel a need and thus welcome increasingly specific goals and information about how to achieve them. In effect, levels of goals are consistent with the view that worthwhile learning is an active and gradual building process, which can be facilitated by big

ideas that enable the learner to see connections or relation with a minimum of guidance (Baroody et al., 2004).

Topic 2: Integrated instruction. The detailed list of specific competencies in Table 11.1 is not intended as a laundry list of skills that a teacher focuses on one at a time. It is intended to guide curriculum developers, curriculum coordinators, and teachers for what might be developmentally appropriate for a particular age range. The activities used to engage children in learning or practicing even specific competencies should provide *rich, integrated* experiences (e.g., projects, games, real problems) that promote the development of several competencies simultaneously (see, e.g., Figure 11.3).

Topic 3: Focusing on big ideas. Teachers should focus on helping students discover and understand *big ideas*—overarching concepts that connect numerous topics and applications (Baroody et al., 2004). For example, the really big idea of equally dividing up a whole (*equal partitioning*) and its informal analogy of fair sharing underlies such diverse concepts as division, fractions, measurement, averages (the mean), and the unit principle (e.g., $5 = 1 + 1 + 1 + 1 + 1$). For instance, $12 \div 3$ can be informally viewed as sharing 12 items among three people and $2/3$ can be done so as the size of each person's share of two cookies are shared fairly among three people (see Baroody, 2004a, for a detailed discussion).

As Paulos (1991) advised, “*stress a few basic principles and [leave] most of the details to [the student]*” (p. 7). If students understand the big ideas, most will be able to rediscover or reinvent the principles, properties, and procedures central to elementary arithmetic, including the commutative principle of addition, reasoning strategies using known number combinations to logically determine unknown sums or differences, and renaming procedures. Understanding big ideas can help students understand the rationale for specific methods (e.g., procedures and formulas), adapt them to meet the challenge of new problems or tasks (“adaptive expertise”), and see how various concepts and procedures are related. This can help students

Animal Spots (Wynroth, 1986)

Animal Spots requires players to throw a die with 0 to 5 dots to determine how many pegs ("spots") they can take for their leopard or giraffe (an animal figure cut out of wood with holes drilled for pegs). (The game also can be played with 0- to 10-dot die or cards.) The first child to fill his or her animal with spots is the winner. Playing *Animal Spots* entails oral counting, enumeration, the cardinality principle, and production of sets. To ensure accurate set production, have children count their pegs into a dish before placing them in their animal. By learning the rules of a game, a child learns a key mathematical concept such as equivalence ("same number as"). Thus, games are used to teach concepts as well as practice skills.

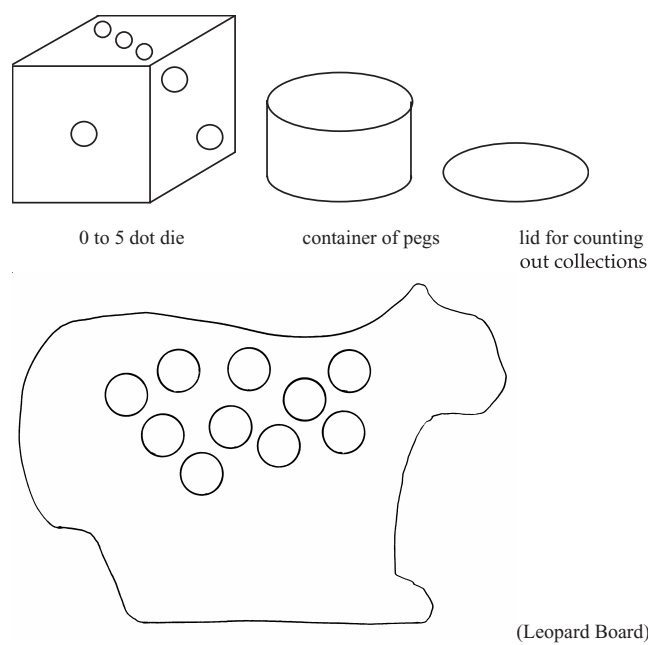


FIGURE 11.3. Animal spots (Wynroth, 1986).

see that mathematics is a system of knowledge. This in turn can make learning diverse ideas and procedures much easier. An example of a really big idea (composition and decomposition) is described in Table 11.4 and instructional games useful in fostering reflection on this big idea are illustrated in Figure 11.4.

Research Implications for When to Teach

Our discussion of when to teach will focus on (a) hypothetical learning trajectories, which provide a guide for constructing hierarchies of developmentally appropriate instruction, and (b) the issue of developmental readiness.

Topic 1: Hypothetical learning trajectories (HTLs). Efforts to define instructional sequences or potential developmental paths are not new, but recent efforts to detail HTLs (originally proposed by Simon, 1995) represent a significant improvement in such efforts (Baroody et al., 2004; Clements & Sarama, 2004; Gravemeijer, 2004). In the past, instructional sequences or learning hierarchies focused on skill hierarchies (e.g., Gagné

& Briggs, 1974) or relatively simple ladder-like sequences of concepts and skills (e.g., Baroody, 1989a; Griffin et al., 1994; Ginsburg & Baroody, 1983, 1990, 2003). Unlike many previous efforts to define learning sequences, HLTs entail a combination of all of the following: (a) goals for meaningful student learning, (b) tasks geared to achieve this learning, and (c) hypotheses about the process of student learning (local learning theories). Specifically, HLTs are based on constructivist research of how children actively build conceptual understanding; entail systematic development of curriculum, instructional activities, and assessment tools; and are based on domain- or even task-specific theory that can entail multiple hypothetical developmental paths or web-like mental structures (Battista, 2004; Lesh & Yoon, 2004).

HLT then can serve as a uniting force for developing and implementing standards (goals), curriculum or instructional activities, teaching strategies, assessment, and teacher training (Clements et al., 2002). Furthermore, HLTs serve to focus educators' attention on teaching students rather than merely implementing a curriculum. That is, successful teachers focus on student learning—children's progress through a conceptually

TABLE 11.4. An Example of a Big Idea: Composition and Decomposition

A fundamental idea underlying various aspects or topics of mathematics is that a <i>quantity or whole can consist of parts and be broken apart (decomposed) into them and that the parts can be combined (composed) to form the whole:</i> • Other names for a number (e.g., different parts can compose to make the same whole, such as $1 + 7$, $2 + 6$, $3 + 5$, and $4 + 4 = 8$...and a whole can be decomposed in constitute parts in different ways, such as $8 = 1 + 7$, $2 + 6$, $3 + 5$, $4 + 4$...) • invention of reasoning strategies and mastery of larger addition combinations (e.g., $7 + 8 = 7 + [7 + 1] = [7 + 7] + 1 = 14 + 1 = 15$ or $7 + 8 = [7 - -2] + [8 + 2] = 5 + 10 = 15$) • part-whole number relations (e.g., one of two or more parts is smaller than its whole and conversely a whole is larger than any one of its multiple parts) • missing-addend (part) problems (e.g., in the problem below or in the equation $4 + ? = 6$, the missing part must be smaller than the whole 6 and when added to 4 equals 6) Georgia had 4 dresses. Her mother bought her some more. Georgia found 6 dresses hanging in closet. How many new dresses did Georgia's mom buy? • invention of reasoning strategies and mastery of subtraction combinations (e.g., for $6 - 4 = ?$, the whole 6 minus the part 4 is equal to the other part, which when added to 4 makes 6; i.e., $6 - 4 = ?$ is related to the missing-addend expression $4 + ? = 6$ and, because $4 + 2 = 6$, the unknown part is 2) • renaming (carrying and borrowing) procedures (e.g., for $37 + 28$, a child must be able to recognize that the sum of the ones digits 15 can be decompose into a 10 and a 5 and that this 10 must be added to the three 10s3 and two 10s shown in the 10s place) • geometry (e.g., a square can be decompose into two [right, isosceles] triangles and vice versa).

based trajectory—not on progressing through a curriculum (Fuson et al., 2000).⁷

When recommendations for instructional activities and local learning theories (e.g., hypotheses about common misconceptions or other learning difficulties, how students might make connections or construct understanding, and what questions might naturally arise) are coupled with the developmental trajectories outlined in Figure 11.2, this hierarchical sequence can serve as a HLT. For sources of instructional guidelines, activities, and local learning theories regarding the components of this hierarchical sequence, see Baroody (1987, 1989a), Baroody with Coslick (1998), Clements et al. (2004), Copley (1999, 2000, 2004b), Fromboluti and Rinck (1999), and Ginsburg and Baroody (2003; particularly Table 1.1, Chapter 6 in the *Examiner's Manual*,and the *Assessment Probes and Instructional Activities Manual*).

As an example of a research-based HLT, consider initial number instruction. Mothers reported that the most common number activity in which they engaged their preschoolers, particularly 3.5- to 4.5-year-olds, was verbal and object counting (Blevins-Knabe & Musun-Miller, 1996). Our own case study work indicates that many parents begin number instruction with counting objects (Baroody et al., 2004). This appears to

be confusing to children—a miseducative experience—because number words are used in two different ways:

- During the counting process, they are used as *ordinal* terms (to specify the order of the items).
- At the end of the counting process, the last number word used also has a *cardinal* meaning, namely, it specifies the cardinal value of the collection (the total number of items in the collection).

Understanding the second point above (the *cardinality principle*) is fundamental to meaningful object counting or functional enumeration. Children's confusion about this switch in number meaning (the absence of a cardinality principle) can be manifested by, for example, recounting a collection when asked, "How many?"(cf. Fluck, 1995; Fluck & Henderson, 1996). Our observations indicate parents' efforts to impose counting on children often leads to learning the object counting process by rote and its nonfunctional, mechanical, or error-prone application (Baroody et al., 2004).

A more optimal instructional route suggested by research is to first encourage *verbal number recognition* of small collections (Baroody et al., 2004). For example, by initially

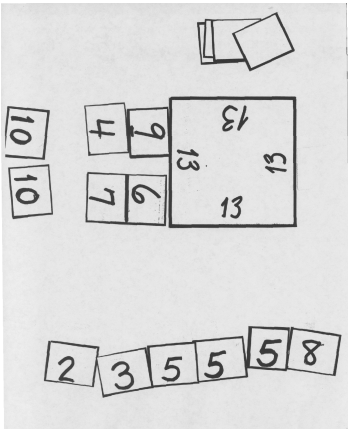
⁷An argument against using HLTs is that such an approach draws heavily on imperfect research and does not take into account what Vygotsky (1978) called children's zone of proximal development. Therefore, basing trajectories on such research can be dangerously restrictive. Even among proponents of HLTs, there are serious disagreements (see, e.g., Clements & Sarama, 2004). For example, Lesh and Yoon (2004) worried that linear (ladderlike) sequences and even multiple-path (branching-tree) trajectories (a) suggest a one correct instructional-path approach, (b) imply stable stages of understanding across different but applicable tasks, and (c) do not adequately represent rich (weblike) learning situations.

However, proponents of HLT recognize that the research basis for a HLT is imperfect (and will always be incomplete to some degree). HLTs are tentative local theories or conjectures, which is precisely why they are called *hypothetical* learning trajectories. A key aspect of HLT approach is its cyclical nature: Construct an HLT based on the best theory and research available, mathematical history, mathematical needs, and so forth. Develop instructional activities to promote each component of the HLT. Test the HLT by trying out the instructional activities and adjust the HLT and instructional activities accordingly and so on. Note that this try and adjust process would take into account children's zone of proximal development. Furthermore, Lesh and Yoon's (2004) concerns are not an entirely fair of most HLT efforts (see Baroody et al., 2004, for a detailed discussion).

The Number Goal Game

Two to six children can play this game. A large center card (square) is placed in the middle with a number such as 13 printed on it. Each player draws six small squares numbered 1 to 10 from a pile of squares all facing down. The players turn up their squares. On his or her turn, a player may compose two or more squares to a sum equal to the number in the center.

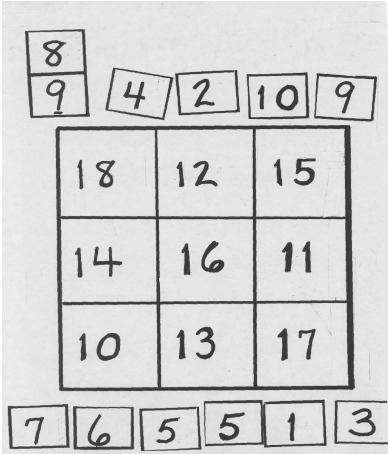
If a player had squares 2, 3, 5, 5, 5 and 8, she could combine 5 and 8 and 3, 5, and 5 to make 13. As each solution would be worth 1 point, the player would get two points for the round. If the player had chosen to combine 2 + 3 + 8, no other possible combinations of 13 would be left, and the player would have scored only 1 point for the round.



An alternative way of playing (scoring) the game is to award points for both the number of parts used to compose the target number (e.g., the play 3 + 5 + 5 and 5 + 8 would be scored as 5 points, whereas the play 2 + 3 + 8 would be scored as 3 points).

Number Goal—Tic Tac Toe (or Three in a Row)

This game is similar to the *Number Goal Game*. Two children can play this game. Each player draws six small squares numbered 1 to 10 from a pile of squares all facing down. The players turn up their squares. On his or her turn, a player may compose two or more squares to create a sum equal to one of the numbers in the 3 x 3 grid. If a player can do this, s/he places her/his marker on the sum in the 3 x 3 grid, discards the squares used, and draws replacement squares. The goal is the same as *Tic-Tac-Toe*—that is, to get three in a row.



Using Egyptian Hieroglyphics to Underscore Decomposition and Base Ten/Place-Value Ideas (Based on Investigation 6.1 in Baroody with Coslick, 1998)

= 1  = 10  = 100

Challenge student to find how many ways the digits 1, 2, and 5 can be combined to form a three-digit numeral. Ask them if 153, 215, 251, 512, and 521 represent the same number as 125 (Whitin & Wilde, 1992). Repeat the challenge with Egyptian hieroglyphics. Discuss the similarities and differences between our Arabic numerals and Egyptian hieroglyphics.)

FIGURE 11.4. Examples of a composition-decomposition activity.

labeling various examples of single instances “one” and various examples of pairs “two” and larger collections (nonexamples) as “not one” or “not two,” children can abstract a concept of oneness and twoness. Once this is achieved, a child can in a similar way be helped to abstract threeness and then fourness.

With this basis, counting instruction should make more sense to young children. If a parent models counting a collection of three items, for example, it is far more likely a child will understand the switch in how the last number word is used. Specifically, when the parent models “One, two, three, see three,” or “One, two, t-h-r-e-e,” the child who has already used verbal number recognition to see that there are three things in the collection may better understand why their parent repeated or emphasized the word “three” and, thus, more likely to abstract the cardinality principle. Is this *the best* way to help children understand? Perhaps, perhaps not, but it seems to be a more effective path than what many parents and preschool teachers currently use.

Topic 2: Developmental readiness: Estimation and zones of comfort. Our discussion of developmental readiness will focus on the overlooked topic of estimation, a key component of number sense. When (and also how) estimation should be introduced remains unclear. Some researchers have argued that, given the relative difficulty of the topic, it makes sense to postpone its instruction until formal schooling begins (Brade, 2003), perhaps even until second grade (Fuson & Hall, 1983). Others have argued that such instruction needs to begin early before schooling creates the mistaken belief that only exact answers are desirable and to help children construct effective estimation strategies (Baroody with Coslick, 1998; Baroody & Gatzke, 1991). Because estimation is a key component of number sense, it is important to address whether such instruction is appropriate for preschoolers or kindergartners.

Another reason we focus on estimation is a key issue that arises from research on the topic, namely that there are zones of comfort and engagement ranging from interested involvement to disinterested disengagement. This issue is akin to Piaget’s moderate novelty principle and has broad implications for planning and implementing any preschool mathematics instruction.

Research Evidence. Both numerical estimation (gauging the approximate size of a collection) and computational estimation (gauging the approximate outcome of an operation) are, at heart, problem-solving activities requiring thoughtful analysis and flexibility (Markovits & Sowder, 1994; Siegel, Goldsmith, & Madson, 1982). Unfortunately, many school age students do not appreciate the value of estimation and respond inflexibly and poorly on estimation tasks (Carpenter, Coburn, Reys, & Wilson, 1976; Kouba, Carpenter, & Swafford, 1989).

How estimation competence and difficulties emerge remains unclear. Relatively little research that has been done on the topic of numerical and computational estimation (Carter, 1986; Fuson & Hall, 1983) and, with few exceptions (e.g., Baroody, 1989b, 1992a; Barth et al., in press; Condry, Gramzow, & Clayton, 2003; La Mont, Barth, & Spelke, 2003; Sowder & Wheeler, 1989), research on the latter has focused on children over 10 years of age (Brade, 2003; Dowker, 2003). Almost no research has examined

the emergence of numerical or computational estimation among children at risk for or with learning difficulties (but see, Baroody, 1988).

Zones of Comfort. Dowker (1997, 2003) identified the following three levels of mental-computation and estimation competence among elementary school children, which can also be applied to gauging number.

1. *The zone of competence* (familiarity and comfort) in which they can determine the exact answer by computation or recall—or, in the case a collection, can visually recognize and accurately specify (“subitize”) the cardinal value of the collection.
2. *The zone of proximal competence* (moderate familiarity and comfort) in which they can effectively use existing knowledge to estimate the answer—or, in the case a collection, can reasonably approximate the cardinal value of the collection.
3. *The zone of incompetence* (unfamiliarity and discomfort) in which they “meltdown” (e.g., refuse to respond or resort to a wild guess).

Research on Preschoolers’ Zones of Comfort. When designing the tasks for our ongoing work with young preschoolers, we purposely limit the trials to one to four items for two reasons:

1. We wanted to avoid overwhelming our young participants—that is, avoid possible meltdowns.
2. If children could reliably use the number words “one” and “two” or even just “one,” it seemed plausible that “three” and “four” would be within their zone of proximal competence. This would seem particularly true if nativists are correct that children so young can form a nonverbal but relatively exact cardinal representation of collections of one to about three items (e.g., Starkey & Cooper, 1995; Wynn, 1992a, 1998).

Surprisingly, though, across a variety of tasks but particularly when required to form and use a mental representation, many children seemed to perceive collections of four and even three items as overwhelmingly large numbers and melted down (Baroody, Benson, & Lai, 2003). The size of the collection that caused a meltdown not only varied among children but varied for a particular child depending on the task complexity or demands.

Consider, for example, the case of Sarah (almost 3 years old). In session 1 with a matching game, she correctly created equivalent collections for visible collections of one, two, and three. However, asked to do so for four items, Sarah put out three items, shook her head to indicate that they were not the same, but said that she couldn’t make them the same. In session 2 a week later, she again responded correctly to the matching trials involving one, two, or three items. With four items, Sarah became visibly upset, put out three items, but seemed to be uncertain of her answer. In the *Hiding Game* (nonverbal production of a single collection administered during session 2), she likewise responded correctly to trials involving one, two, or three items. When presented the trial with four items, Sarah started

crying. In sessions 3 and 4, 1 and 2 weeks later, respectively, she likewise cried when a task became too challenging on the *Fast Hiding Game* (nonverbal subitizing in which the model collection was displayed for <2s) trial involving four. These results indicated that one to three was in her zone of competence and that four was in her zone of incompetence—whether the model collection remained visible or not—whether forming a mental representation for a single collection was not or was required.

On more challenging tasks involving the mental representation of two collections, Sarah's zone of incompetence expanded. In session 1 with the *Double Hiding Game* (nonverbal production of two collections), she successfully created equivalent collections for two previously viewed but hidden collections of one item each (the 1 & 1 trial). However, when administered the 3 & 2 trial, the child said, "I can't do it." When subsequently given the 2 & 4, trial, she started crying. In session 2, Sarah again responded correctly to the 1 & 1 trial. With the 2 & 3 trial, the girl ran away and said she didn't want to play again. In sessions 3 and 4, Sarah became tearful on the *Super Hiding Game* (nonverbal addition and subtraction) trials involving 3 − 2, 1 + 2, and 3 + 1. Comparable to Leslie's (1999) results with 12-month-old infants, these results indicate that Sarah could simultaneously form a mental representation of two collections of a single item each (zone of competence) but not of two larger collections (zone of incompetence). Whether she would have responded correctly, melted down, or given an approximate answer (zone of partial competence) for 1 & 2, 2 & 1, 2 & 2 is not known. In brief, these results indicate that with cognitively more demanding tasks (i.e. tasks that required mentally representing two collections and, thus, a greater load on working memory than those involving a single collection), Sarah's zone of competence shrank even further below four.

Hadley, a 3-year-old, similarly was able to nonverbally produce two items but melted down with four. After viewing four items for 3 seconds and seeing them hidden, she responded, "I don't know what that number is, Can I count it?" Encouraged by the tester to put out what she thought, Hadley requested a second look so that she could count the collection. Again encouraged by the tester to put out what she thought, the girl simply grabbed all the available chips.

Research on Preschoolers' Estimation. Our pilot work with 3-year-olds indicates that more developmentally advanced children spontaneously invent decomposition strategies to gauge either exactly or inexact collections of three to six items. For example, Bruce decomposed a collection of three items into two and one and a collection of four into two and two and quickly determined the total of each. Four-year-olds in a preschool program for at-risk children were able to indicate the size of a collection within ±25% of its actual cardinality about 75% of the time with briefly viewed (<2 s) collections of three to five items (Baroody, Eiland, Lai, & Li in preparation). However, reasonable estimates for collections of 7 to 11 fell to about 45%.

A number of children in the previously mentioned estimation study tried to count collections before they were hidden. Although some then gave as their estimate the last number used

in the enumeration process, some gave a number one or more after the last counting number used. Whether they counted the visible collection or not, a few children verbally counted when asked for an estimate. Although this appeared to be a haphazard process for a few, some tailored the length of their count to the relative size of the collection.

Children 8 years old or younger sometimes use a moderately sophisticated benchmark strategy (i.e., compared a collection to a mental representation of a well known number such as 10; Fuson & Hall, 1983; Siegel et al., 1982). Baroody and Gatzke (1991) found that potentially gifted kindergartners seem to use such a strategy effectively with collections less than 10 (with a benchmark of 5), but most had difficulty with larger collections and benchmarks. Brade (2003) replicated these results with typical kindergartners and first graders. Moreover, she found that computer-based estimation instruction from the *Building Blocks* curriculum twice weekly over a period of about 4 weeks did not significantly improve her participants' estimation skill. One possible reason for this is that the intervention was inadequate (e.g., too brief) to help participants construct the big ideas of composition and decomposition or use them to construct more sophisticated estimation strategies, such as decomposing a collection into more or less equal size and subitizeable parts (Siegel et al., 1982). Clearly, much more research is needed to address these issues.

Pedagogical Implications. The research just discussed has six instructional implications. The first below is general to all mathematics instruction; the remaining implications are specific to estimation instruction.

- *Instruction (and testing) should be sensitive to a children's developmental level in general and their individual zones of competence, partial competence, or incompetence in particular.* In order to ensure that external factors mesh with internal ones so as to promote meaningful learning (Dewey, 1963), preschool teachers should be aware that what constitutes an overwhelmingly large number varies from child to child and that this key internal factor can involve very small numbers and may be narrow (exceeded quickly).
- *Estimation experiences probably can begin during the preschool years if done in a developmentally appropriate manner.*
- *Consistent with the first two points, estimation instruction should begin with numbers just beyond children's zone of competence.* For 3-year-olds, this may be collections as small as about four items. For 4- and 5-year-olds, this may involve collections in the range of 5 to 10 items.
- Instruction that underscores the purposes of estimation and the gradual development of big ideas that underlie strategies for gauging exactly or approximately the size of small or larger collections, such as composition and decomposition, may be helpful. Such big ideas and the sophisticated estimation strategies that build on them arise from number sense and cannot quickly be imposed.
- Children should be encouraged to devise and share informal strategies for quickly recognizing intuitive numbers (e.g.,

decomposing three into two and one) and encouraged to apply them to increasingly larger collections, whether done so exactly or inexactly. Informal estimation strategies, such as stating a number beyond the last number reached during an aborted enumeration attempt and tailoring a verbal count response to the relative size of a collection, can be encouraged by prompting children to share their estimation strategy and encouraging students to evaluate whether it makes sense. The same can also be done with nonproductive strategies (e.g., citing the last number word generated in an aborted enumeration effort or haphazardly verbal count response). Strategies, particularly nonproductive ones, could be modeled by a teacher using a puppet such as Count Count or Cookie Monster, and students could be encouraged to help it by evaluating its strategy.

- Help older preschoolers construct a benchmark (mental image) of five; help kindergartners construct a mental image of 10.

CONCLUSIONS

As a society, we can no longer afford to overlook early childhood mathematics education. Given the wealth of informal mathematical knowledge preschoolers seem able to construct and its probable importance for later success with formal mathematics, it is important that all children be given the opportunity to develop mathematical proficiency before formal schooling. Given the evidence of significant individual and group differences in informal knowledge, it is imperative that children at risk for academic failure (e.g., those from low-income, uneducated, non-English-speaking, or single-parent families) be provided early and high quality intervention. In most cases, such intervention alone will not be sufficient to ensure success later during the school years (e.g., Zigler, 2003), but it may be an initial necessary ingredient for achieving equity.

Efforts are already under way to develop effective preschool interventions. Service organizations (e.g., NYAEC and NCTM), state and federal governments (e.g., the National Science Foundation and National Institutes of Health), and private foundations and industry (ExxonMobil Foundation) are supporting the development of appropriate goals or standards (see Clements et al., 2004) and coherent, research-based, and effective curricula and instructional materials such as *Big Math for Little Kids* (Balfanz, Ginsburg, & Greenes, 2003; Ginsburg, Greenes, & Balfanz, 2003; Greenes, Ginsburg, & Balfanz, 2004), *Building Blocks* (Clements & Sarama, 2003, 2004; Sarama & Clements, 2002, 2003), *Everyday Mathematics* (Carroll & Issacs, 2003; SRA/McGraw Hill, 2003), and *Number Worlds* (Griffin, 2000, 2004; Griffin & Case, 1995a, 1996; Griffin et al., 1995), and *Round the Rug Math: Adventures in Problem Solving* (Casey, Kersh, & Young, 2004). The NCTM and NYAEC jointly or separately have published several resources for pre-school educators detailing units, lessons, or activities for promoting specific goals (Copley, 1999, 2000, 2004b). A number of resources detail mathematical games appropriate for early childhood instruction, including Baroody (1987, 1989a), *Early Childhood Games*

Kit (Everyday Mathematics, 2003), Kamii (1985), and Wynroth (1986). Tools for assessing specific informal concepts and skills, such as the *TEMA-3* (e.g., Ginsburg & Baroody, 2003) are also now available.

Given that kindergarten sets the stage for formal mathematics instruction and success, another positive step would be a consistent nationwide policy regarding kindergarten (Vecchiotti, 2003). Currently there are no general standards for mandatory kindergarten attendance, the age of entry, and appropriate curriculum and instructional methods or screening and assessment practices.

The success of early intervention efforts, however, ultimately depends on preschool and kindergarten teachers who are highly knowledgeable about mathematics (including the relevant big ideas), children’s learning of it, and techniques for effectively fostering such learning (Baroody, 2004d). The knowledgeable teacher can, for example, see teaching opportunities in everyday routines, unexpected events, children’s questions, and play time and take advantage of them to artfully guide learning (Baroody with Coslick, 1998; Bruner, Jolly, & Sylva, 1976; Guha, 2002; Seo & Ginsburg, 2004). However, this more sophisticated type of teaching will require seriously upgrading the pre- and in-service preparation of preschool teachers (Hyson, 2001; Kersaint & Chappell, 2001; Sarama & DiBiase, 2004). This preparation should include integrated mathematical content, developmental, and pedagogical instruction that focus on understanding and how to foster developmentally appropriate big ideas.

A general trend in the research on mathematical development provides an important lesson for the mathematics instruction of preschoolers or kindergartners pre- or in-service teachers, and all students in between. Many researchers have remarked on the stark contrast between preschoolers’ surprising mathematical competence and schoolchildren’s disturbing mathematical incompetence. Nativists have argued that this paradox is evidence for “privileged domains,” knowledge with an innate basis that unfolds readily in the preschool years, and nonprivileged domains, knowledge not supported by an innate endowment and that must be learned with relative difficulty in school. An alternative and perhaps more plausible explanation is that preschoolers’ surprising competence with number and simple arithmetic is due to their opportunities to gradually develop adaptive expertise in a supportive, rather than a directive, environment (Baroody & Tiilikainen, 2003). Put differently, the purposeful, meaningful, and inquiry-based of preschoolers’ informal learning environment enables them to construct conceptual understanding, meaningful computational procedures, informed strategic thinking, and a productive disposition (i.e., all aspects of mathematical proficiency). Schoolchildren’s disturbing incompetence is due to the lack of such opportunities. In other words, the approach and focus of formal schooling, namely highly directive instruction with the aim of fostering routine expertise, ensures the lack of mathematical proficiency. As we prepare preschoolers, kindergartners, and their teachers, it is well worth keeping in mind the results of this natural experiment—that formal instruction might better imitate children’s informal preschool learning environment than vice versa.

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