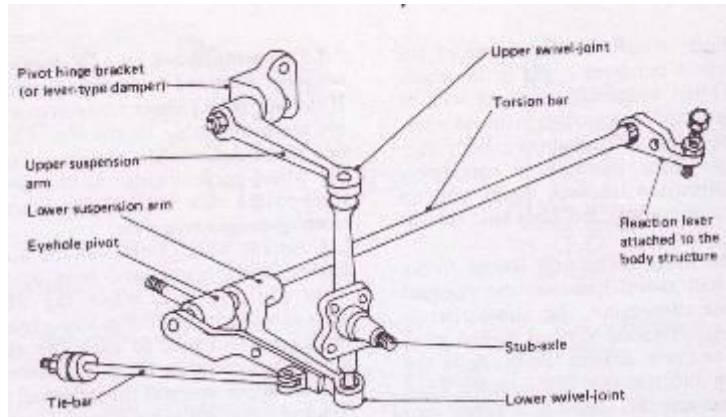


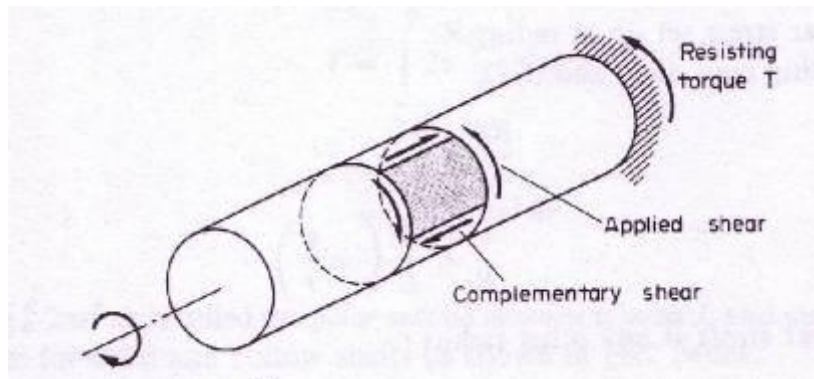
Torsion of Shafts



The above shows a Torsion-bar Transverse double wishbone suspension system. This uses a torsion bar instead of a helical spring resulting in a simple and compact suspension system.

Simple Torsion Theory

When a uniform circular shaft is subjected to a torque it can be shown that it is in a state of pure shear, the moment of resistance being equal and opposite to the applied torque as shown below;

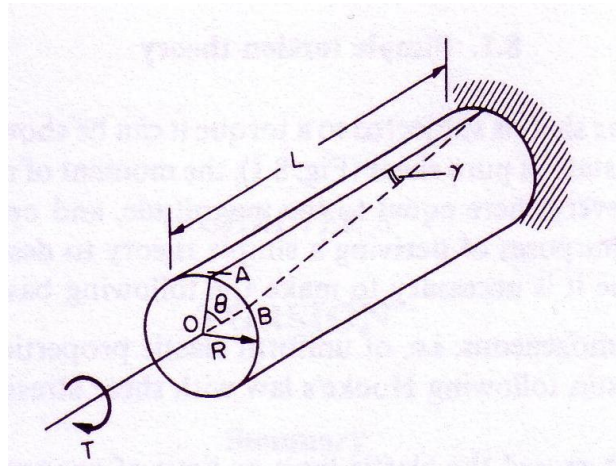


To derive a simple theory to describe the behaviour of shafts subjected to torque we make the following assumptions;

- The material is homogeneous – i.e. it has uniform elastic properties throughout.
- The material remains elastic
- Circular sections remain circular
- Cross sections remain plane
- Cross sections rotate as if rigid, i.e. every diameter rotates through the same angle.

Using these assumptions the theory of pure torsion can be derived which gives excellent correlation with experimental results.

Consider the below shaft that is subject to a torque T at one end whilst the other is fixed;



If this shaft twists through an angle θ , point A moves to point B and AB subtends an angle γ at the fixed end. This is the distortion of the shaft, i.e. the **shear strain**.

Therefore,

$$\gamma = \frac{R\theta}{L}$$

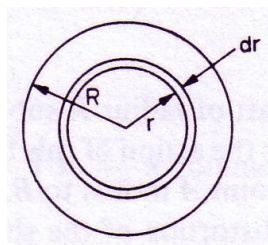
By definition, the shear modulus G is given by shear stress/shear strain;

$$G = \frac{\tau}{\gamma}$$

So if we combine the above equations we get;

$$\boxed{\frac{\tau}{R} = \frac{G\theta}{L}}$$

Now let's consider the stresses in the shaft. If we look at a cross section and divide it up into elements of radius r and thickness dr, each element would be subject to a shear stress τ' ;



The shear force on each element;

$$\begin{aligned} &= \text{stress} \times \text{area} \\ &= \tau' \times 2\pi dr \end{aligned}$$

This force will produce a torque about the centre axis of the shaft;

$$\begin{aligned} &= \tau' \times 2\pi dr \times r \\ &= 2\pi \tau' r^2 dr \end{aligned}$$

Therefore the total torque will be given by;

$$T = \int_0^R 2\pi \tau' r^2 dr$$

Since τ' is a function of the radius r , we must replace it before integrating;

$$T = \int_0^R 2\pi \frac{G\theta r^3}{L} dr = \frac{G\theta}{L} \int_0^R 2\pi r^3 dr$$

$\int_0^R 2\pi r^3 dr$ is termed the **polar second moment of area, J**, giving;

$$\boxed{\frac{T}{J} = \frac{G\theta}{L}}$$

Combining this with our earlier equation when considering the shear strain, we get the **simple theory of torsion**;

$$\boxed{\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}}$$

Polar Second Moment Of Area

As stated above the polar second moment of area (J) is given by $\int_0^R 2\pi r^3 dr$, thus we can calculate the following standard values for J;

Solid shaft;

$$\boxed{J = \frac{2\pi R^4}{4} = \frac{\pi D^4}{32}}$$

Hollow shaft;

$$\boxed{J = \frac{\pi}{2} (R^4 - r^4) = \frac{\pi}{32} (D^4 - d^4)}$$

Shear Stress and Shear Strain in Shafts

We have already shown that the shear stress at the outside radius R is given by;

$$\tau = \frac{G\theta}{L} R$$

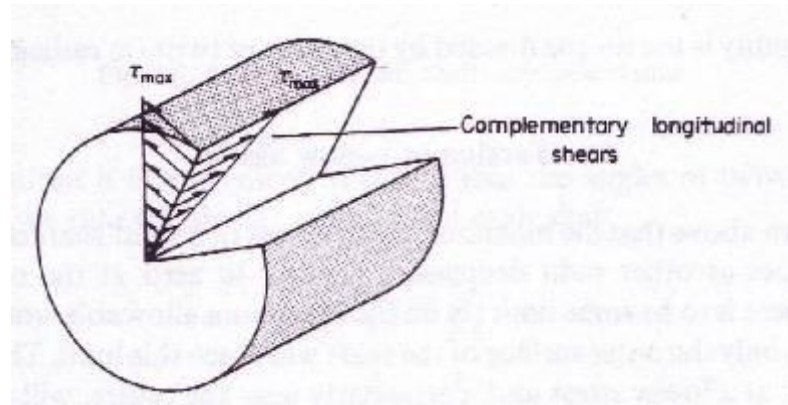
Or expressed in terms of shear strain γ ;

$$\tau = G\gamma$$

It follows that if we were to consider an internal radius r , we would get the following;

$$\boxed{\tau' = \frac{G\theta}{L} r = G\gamma}$$

This indicates that the shear stress and shear strain vary linearly with radius and are a maximum at the outside radius. The applied shear stress in the plane of the cross section are accompanied by **complementary stresses** of equal value on longitudinal planes as shown below;



Section Modulus

Part of the torsion theory formula $\frac{T}{J} = \frac{\tau}{R}$ is often written in terms of the maximum shear stress that occurs on the outside of the shaft when R is a maximum;

$$\tau_{max} = \frac{TR}{J}$$

J/R is termed the **polar section modulus, Z**, i.e.

$$\tau_{max} = \frac{T}{Z}$$

thus we can also calculate the following standard values for Z;

Solid shaft;

$$Z = \frac{\pi D^3}{16}$$

Hollow shaft;

$$Z = \frac{\pi (D^4 - d^4)}{16D}$$

Torsional Rigidity

The angle of twist per unit length of shafts is given by the torsion theory as;

$$\frac{T}{GJ} = \frac{\theta}{L}$$

The term GJ is the **torsional rigidity** of the shaft, i.e.

$$GJ = \frac{T}{\theta/L}$$

i.e. the torsional rigidity is the torque divided by the angle of twist (in radians) per unit length.

Torsion of hollow shafts

It has been shown that the maximum shear stress on a shaft carrying torque happens at the outer surface and decreases to zero at the centre. Therefore in applications where weight reduction is important (as in the aerospace industry), it is often advisable to use hollow shafts. As an example of the increased torque-weight ratio possible with a hollow shaft, a shaft with an inside diameter half the outside diameter will see an increase in maximum stress of 6% over a solid shaft of the same diameter. However, the weight reduction achieved would be approximately 25%.

Examples

1)

(a) A solid shaft, 100 mm diameter, transmits 75 kW at 150 rev/min. Determine the value of the maximum shear stress set up in the shaft and the angle of twist per metre of the shaft length if $G = 80 \text{ GN/m}^2$.

(b) If the shaft were now bored in order to reduce weight to produce a tube of 100 mm outside diameter and 60 mm inside diameter, what torque could be carried if the same maximum shear stress is not to be exceeded? What is the percentage increase in power/weight ratio effected by this modification?

[24.3MN/m², 0.348 degrees/m, 4.15kNm, 36%]

2)

Determine the dimensions of a hollow shaft with a diameter ratio of 3:4 which is to transmit 60 kW at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MN/m^2 and the angle of twist to 3.8° in a length of 4 m.

For the shaft material $G = 80 \text{ GN/m}^2$.

[75.3mm, 56.5mm]

3)

A shaft 50 mm diameter and 0.7 m long is subjected to a torque of 1200 Nm. Calculate the shear stress and the angle of twist. Take $G = 90 \text{ GPa}$.

[48.89MPa, 0.871°]

4)

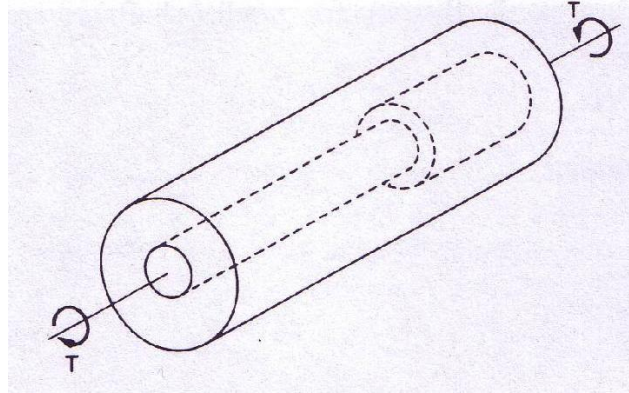
Repeat the previous problem but this time the shaft is hollow with an internal diameter of 30 mm.

Comment on the results. What is the weight save?

[Shear stress and angle increase by 15% with a 36% weight save]

Composite Shafts – Series Connection

If two or more shafts of different material, diameter or basic form are connected such that they **carry the same torque**, they are said to be connected in series. In this case each shaft may be considered separately, apply simple torsion theory to each in turn, and the composite shaft will therefore break at its weakest component. Consider the below shaft that could be considered to be two hollow shafts connected in series. Each shaft carries the same torque, T ;



From simple torsion theory;

$$T = \frac{GJ\theta}{L}$$

Since both shafts carry the same torque then;

$$\frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2}$$

In some applications it is useful to ensure that the angles of each shaft are the same ($\theta_1 = \theta_2$), so that for similar materials ($G_1 = G_2$), then;

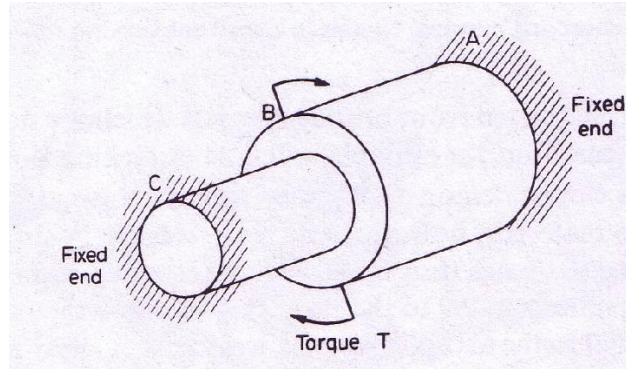
$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

Or;

$$\frac{L_1}{L_2} = \frac{J_1}{J_2}$$

Composite Shafts – Parallel Connection

If two or more shafts of different material, diameter or basic form are connected such that the **torque is shared between them**, they are said to be connected in parallel. Consider the below example;



Total Torque, $T = T_1 + T_2$

In this case the angle of twist of shaft 1 must equal shaft 2, therefore;

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2}$$

In most applications for parallel shafts the lengths of the shafts are the same (i.e. $L_1 = L_2$);

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2}$$

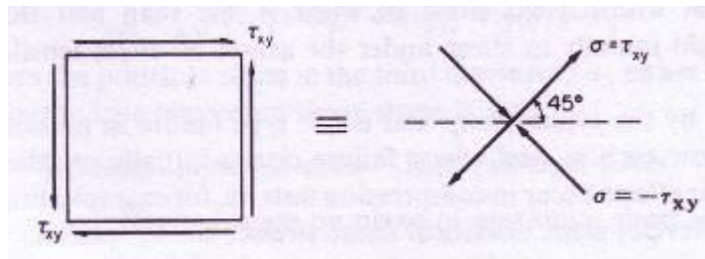
Principal Stresses

When considering an element subject to pure shear the following relationships are true;

$$\sigma_\theta = \tau_{xy} \sin 2\theta$$

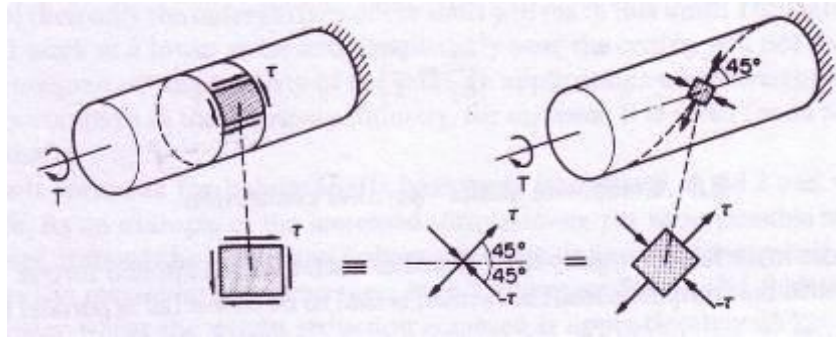
$$\tau_\theta = -\tau_{xy} \cos 2\theta$$

Thus if a system is subjected to pure shear an equivalent direct stress system is produced with one set compressive and one tensile at 45° , e.g;

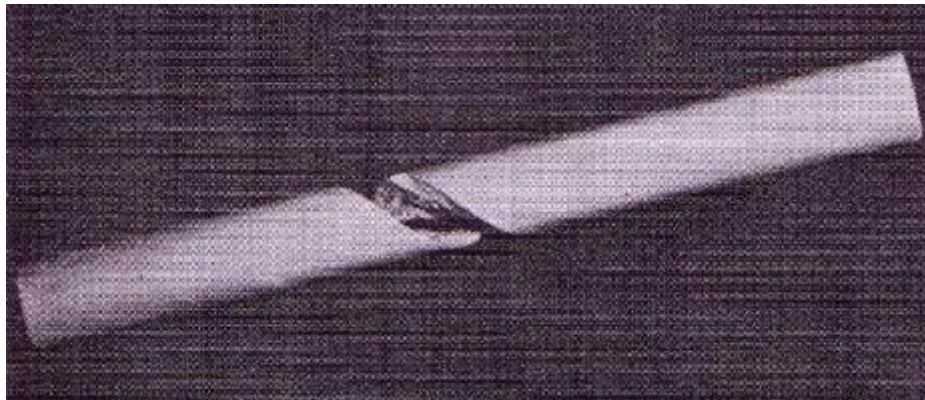


This has great significance in the measurement of shear stresses or torques on shafts using strain gauges where the gauges are arranged to record the direct strains at 45° to the shaft axis.

A shaft in pure torsion places the shaft into pure shear so considering the above diagram we can consider the shear and principal stresses in a shaft subjected to torsion;

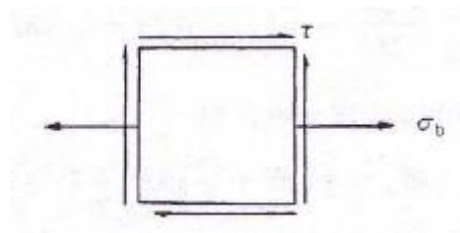


Thus shafts that are constructed from brittle materials which are notably weaker under direct stress than in shear (e.g. cast iron) will fail by cracking along a helix inclined at 45° to the shaft axis. This is demonstrated by twisting a piece of chalk to failure as can be seen below. Ductile materials, however, which are weaker in shear, fail on the shear planes at right angles.



Combined stress systems – combined bending and torsion

In most applications shafts that transmit torque are also subjected to bending, if only by virtue of self-weight of the gears that they carry. In the case of shafts, bending gives rise to tensile stress on one surface and compressive stress on the opposite surface, whilst torsion gives rise to pure shear throughout the shaft. An element on the tensile surface will thus be subjected to the stress system shown below;



To find the principal stresses we could use Mohr's Circle or use the following equation;

$$\sigma_{1,2} = \frac{1}{2} \left[\sigma_x + \sigma_y \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \right] \quad [1]$$

Alternatively we could consider the shaft subjected to an equivalent torque or equivalent bending moment.

Combined bending and torsion – Equivalent bending moment

From simple bending theory it can be shown that;

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

Or

$$\frac{My}{I} = \sigma$$

Applying this to shafts we get

$$\frac{MD}{2I} = \sigma \quad [2]$$

Simple torsion theory states that;

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$$

Or

$$\frac{TR}{J} = \tau$$

Again, putting this in terms of the diameter gives us;

$$\frac{TD}{2J} = \tau$$

For a circular shaft $J=2I$, therefore;

$$\frac{TD}{4I} = \tau \quad [3]$$

If we put equations [2] and [3] into [1] we get the following relationship;

$$\sigma_1 = \frac{1}{2} \left(\frac{D}{2I} \right) \left[M + \sqrt{M^2 + T^2} \right]$$

Thus if we want to find an equivalent bending moment M_e , that would provide the same stress as the combined bending and torsion system then;

$$\frac{M_e D}{2I} = \frac{1}{2} \left(\frac{D}{2I} \right) \left[M + \sqrt{M^2 + T^2} \right]$$

i.e. the equivalent bending moment is given by;

$$M_e = \frac{1}{2} \left[M + \sqrt{M^2 + T^2} \right]$$

Combined bending and torsion – Equivalent torque

From the previous section we saw that

$$\sigma_1 = \frac{1}{2} \left(\frac{D}{2I} \right) \left[M + \sqrt{M^2 + T^2} \right]$$

And since for $J=2I$

$$\sigma_1 = \frac{1}{2} \left(\frac{D}{J} \right) \left[M + \sqrt{M^2 + T^2} \right]$$

Similarly it can be shown that

$$\sigma_2 = \frac{1}{2} \left(\frac{D}{J} \right) \left[M - \sqrt{M^2 + T^2} \right]$$

Again, referring to complex stress theory we have shown that;

$$\tau_{\max} = \frac{1}{2} (\sigma_1 - \sigma_2)$$

Therefore;

$$\tau_{\max} = \frac{1}{2} \left[\left\{ \frac{1}{2} \left(\frac{D}{J} \right) \left[M + \sqrt{M^2 + T^2} \right] \right\} - \left\{ \frac{1}{2} \left(\frac{D}{J} \right) \left[M - \sqrt{M^2 + T^2} \right] \right\} \right]$$

$$\tau_{\max} = \frac{1}{2} \left(\frac{D}{J} \right) \sqrt{M^2 + T^2}$$

From simple torsion theory;

$$\frac{TD}{2J} = \tau$$

Thus to get an equivalent torque T_e , that would give the same stress as the combined bending and torsion system then;

$$\frac{T_e D}{2J} = \frac{1}{2} \left(\frac{D}{J} \right) \sqrt{M^2 + T^2}$$

i.e. the equivalent torque is given by;

$$T_e = \sqrt{M^2 + T^2}$$

Examples

- 1) (a) A steel transmission shaft is 510 mm long and 50 mm external diameter. For part of its length it is bored to a diameter of 25 mm and for the rest to 38 mm diameter. Find the maximum power that may be transmitted at a speed of 210 rev/min if the shear stress is not to exceed 70 MN/m^2 .
- (b) If the angle of twist in the length of 25 mm bore is equal to that in the length of 38 mm bore, find the length bored to the latter diameter.

[25.2kW, 210mm]

- 2) A circular bar ABC , 3 m long, is rigidly fixed at its ends A and C . The portion AB is 1.8 m long and of 50 mm diameter and BC is 1.2 m long and of 25 mm diameter. If a twisting moment of 680 N m is applied at B , determine the values of the resisting moments at A and C and the maximum stress in each section of the shaft. What will be the angle of twist of each portion?
- For the material of the shaft $G = 80 \text{ GN/m}^2$.

[58.3Nm, 621.7Nm, $25.33 \times 10^6 \text{ N/m}^2$, $19.0 \times 10^6 \text{ N/m}^2$, 1.3°]