

PARTIAL FRACTIONS

I. RATIONAL FRACTIONS AND PARTIAL FRACTIONS

Suppose we wish to express

$$\frac{2}{x+1} + \frac{3x+6}{x^2+2}$$

as a single fraction. The Lowest Common Denominator (LCM) is $(x+1) \cdot (x^2+2)$ and so

$$\begin{aligned} \frac{2}{x+1} + \frac{3x+6}{x^2+2} &= \frac{2(x^2+2) + (3x+6)(x+1)}{(x+1)(x^2+2)} \\ &= \frac{2x^2 + 4 + 3x^2 + 3x + 6x + 6}{(x+1)(x^2+2)} \\ &= \frac{5x^2 + 9x + 10}{(x+1)(x^2+2)} \end{aligned}$$

It is often necessary to reverse the above procedure and express the rational fraction

$$\frac{5x^2 + 9x + 10}{(x+1)(x^2+2)}$$

as

$$\frac{2}{x+1} + \frac{3x+6}{x^2+2}$$

which are called the partial fractions.

Partial fractions are useful for integrating rational fractions.

Reminder: a rational fraction is a fraction whose numerator and denominator are both polynomials.

Reminder: A polynomial is a mathematical expression involving a sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable with constant coefficients is given by: $a_n x^n + \dots + a_2 x^2 + a_1 x^1 + a_0$. A practical example of a polynomial is:

$$5x^2 + 9x + 10$$

II. HOW TO FIND PARTIAL FRACTIONS

1. Numerator is of smaller degree than the denominator:

Deg Num < Deg Denom

➤ Example a:

Express the following rational fraction in terms of partial fractions:

$$\frac{x+2}{(1+x)(2-x)}$$

The numerator $x+2$ is of the first degree while the denominator $(1+x)(2-x) = 2+x-x^2$ is of the second degree and thus we shall be able to find suitable partial fractions.

Now only experience will enable you to decide the form of the partial fractions, but we must remember that in each case the numerator must be of smaller degree than the denominator. If A and B are constant (i.e. numbers), then let:

$$\frac{x+2}{(1+x)(2-x)} = \frac{A}{1+x} + \frac{B}{2-x}$$

$$\Rightarrow \frac{x+2}{(1+x)(2-x)} = \frac{A(2-x) + B(1+x)}{(1+x)(2-x)}$$

$$\Rightarrow x+2 = A(2-x) + B(1+x)$$

Now **by choosing suitable values of x** it is possible to find the values of the constants A and B .

- When $x = 2$:

$$2+2 = A(2-2) + B(1+2)$$

$$\Rightarrow 4 = A \times 0 + B \times 3$$

$$\Rightarrow B = \frac{4}{3}$$

- When $x = -1$:

$$-1+2 = A(2-[-1]) + B(1+[-1])$$

$$\Rightarrow 1 = A \times 3 + B \times 0$$

$$\Rightarrow A = \frac{1}{3}$$

Thus:

$$\frac{x+2}{(1+x)(2-x)} = \frac{1}{3 \cdot (1+x)} + \frac{4}{3 \cdot (2-x)}$$

➤ **Example b:**

Express the following rational fraction in terms of partial fractions:

$$\frac{2x^2 + 7x - 17}{(x-1)(x-2)(x+3)}$$

This is possible since the numerator is of the second degree, which is one less than the degree of the denominator. Let:

$$\frac{2x^2 + 7x - 17}{(x-1)(x-2)(x+3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+3}$$

$$\Rightarrow \frac{2x^2 + 7x - 17}{(x-1)(x-2)(x+3)} = \frac{A(x-2)(x+3) + B(x-1)(x+3) + C(x-1)(x-2)}{(x-1)(x-2)(x+3)}$$

$$\Rightarrow 2x^2 + 7x - 17 = A(x-2)(x+3) + B(x-1)(x+3) + C(x-1)(x-2)$$

• When $x = 1$:

$$2(1)^2 + 7(1) - 17 = A(1-2)(1+3) + B(1-1)(1+3) + C(1-1)(1-2)$$

$$\Rightarrow 2 + 7 - 17 = A(-1)(4) + B(0)(4) + C(0)(-1)$$

$$\Rightarrow -8 = A \times (-4)$$

$$\Rightarrow A = 2$$

• When $x = 2$:

$$2(2)^2 + 7(2) - 17 = A(2-2)(2+3) + B(2-1)(2+3) + C(2-1)(2-2)$$

$$\Rightarrow 8 + 14 - 17 = A(0)(5) + B(1)(5) + C(1)(0)$$

$$\Rightarrow 5 = B \times 5$$

$$\Rightarrow B = 1$$

• When $x = -3$:

$$2(-3)^2 + 7(-3) - 17 = A(-3-2)(-3+3) + B(-3-1)(-3+3) + C(-3-1)(-3-2)$$

$$\Rightarrow 18 - 21 - 17 = A(-5)(0) + B(-4)(0) + C(-4)(-5)$$

$$\Rightarrow -20 = C \times 20$$

$$\Rightarrow C = -1$$

Thus:

$$\frac{2x^2 + 7x - 17}{(x-1)(x-2)(x+3)} = \frac{2}{x-1} + \frac{1}{x-2} + \frac{-1}{x+3}$$

➤ **Example c:**

Express the following rational fraction in terms of partial fractions:

$$\frac{8x - 28}{x^2 - 6x + 8}$$

This is possible since the numerator is of the first degree, which is one less than the degree of the denominator. Let:

$$\frac{8x - 28}{x^2 - 6x + 8} = \frac{8x - 28}{(x - 2)(x - 4)} = \frac{A}{x - 2} + \frac{B}{x - 4}$$

$$\Rightarrow \frac{8x - 28}{(x - 2)(x - 4)} = \frac{A(x - 4) + B(x - 2)}{(x - 2)(x - 4)}$$

$$\Rightarrow 8x - 28 = A(x - 4) + B(x - 2)$$

- When $x = 4$:

$$8 \times 4 - 28 = A(4 - 4) + B(4 - 2)$$

$$\Rightarrow 32 - 28 = A(0) + B(2)$$

$$\Rightarrow 4 = 2B$$

$$\Rightarrow B = 2$$

- When $x = 2$:

$$8 \times 2 - 28 = A(2 - 4) + B(2 - 2)$$

$$\Rightarrow 16 - 28 = A(-2) + B(0)$$

$$\Rightarrow -12 = -2A$$

$$\Rightarrow A = 6$$

Thus:

$$\frac{8x - 28}{x^2 - 6x + 8} = \frac{6}{x - 2} + \frac{2}{x - 4}$$

2. Numerator is NOT of smaller degree than the denominator:

Deg Num $\geq$ Deg Denom

➤ Example a:

$$\frac{x^2 + 3x - 10}{x^2 - 2x - 3}$$

The first consideration is:

Is the numerator of lower degree than the denominator?

No, it is not, so we have to divide out by long division:

$$\begin{array}{r} x^2 + 3x - 10 \\ -(x^2 - 2x - 3) \\ \hline 5x - 7 \end{array} \bigg| \frac{x^2 - 2x - 3}{1}$$

$$\Rightarrow \frac{x^2 + 3x - 10}{x^2 - 2x - 3} = 1 + \frac{5x - 7}{x^2 - 2x - 3}$$

$$\Rightarrow \frac{x^2 + 3x - 10}{x^2 - 2x - 3} = 1 + \frac{5x - 7}{(x + 1)(x - 3)}$$

$$\Rightarrow 1 + \frac{5x - 7}{(x + 1)(x - 3)} = 1 + \frac{A}{x + 1} + \frac{B}{x - 3}$$

$$\Rightarrow 1 + \frac{5x - 7}{(x + 1)(x - 3)} = 1 + \frac{A(x - 3) + B(x + 1)}{(x + 1)(x - 3)}$$

$$\Rightarrow 5x - 7 = A(x - 3) + B(x + 1)$$

• When $x = 3$:

$$5 \times 3 - 7 = A(3 - 3) + B(3 + 1)$$

$$\Rightarrow 15 - 7 = A(0) + B(4)$$

$$\Rightarrow 8 = 4B$$

$$\Rightarrow B = 2$$

• When $x = -1$:

$$5 \times (-1) - 7 = A(-1 - 3) + B(-1 + 1)$$

$$\Rightarrow -5 - 7 = A(-4) + B(0)$$

$$\Rightarrow -12 = -4A$$

$$\Rightarrow A = 3$$

Thus:

$$\frac{x^2 + 3x - 10}{x^2 - 2x - 3} = 1 + \frac{3}{x+1} + \frac{2}{x-3}$$

➤ **Example b:**

$$\frac{2x^2 + 18x + 31}{x^2 + 5x + 6}$$

The first consideration is:

Is the numerator of lower degree than the denominator?

No, it is not, so we have to divide out by long division:

$$\begin{array}{r} 2x^2 + 18x + 31 \\ -(2x^2 + 10x + 12) \\ \hline 8x + 19 \end{array} \Bigg| \frac{x^2 + 5x + 6}{2}$$

$$\Rightarrow \frac{2x^2 + 18x + 31}{x^2 + 5x + 6} = 2 + \frac{8x + 19}{x^2 + 5x + 6}$$

$$\Rightarrow \frac{2x^2 + 18x + 31}{x^2 + 5x + 6} = 2 + \frac{8x + 19}{(x+2)(x+3)}$$

$$\Rightarrow 2 + \frac{8x + 19}{(x+2)(x+3)} = 2 + \frac{A}{x+2} + \frac{B}{x+3}$$

$$\Rightarrow 2 + \frac{8x + 19}{(x+2)(x+3)} = 2 + \frac{A(x+3) + B(x+2)}{(x+2)(x+3)}$$

$$\Rightarrow 8x + 19 = A(x+3) + B(x+2)$$

• When $x = -3$:

$$8 \times (-3) + 19 = A(-3+3) + B(-3+2)$$

$$\Rightarrow -24 + 19 = A(0) + B(-1)$$

$$\Rightarrow -5 = -B$$

$$\Rightarrow B = 5$$

• When $x = -2$:

$$8 \times (-2) + 19 = A(-2+3) + B(-2+2)$$

$$\Rightarrow -16 + 19 = A(1) + B(0)$$

$$\Rightarrow 3 = A$$

$$\Rightarrow A = 3$$

Thus:

$$\frac{2x^2 + 18x + 31}{x^2 + 5x + 6} = 2 + \frac{3}{x+2} + \frac{5}{x+3}$$

➤ **Example c:**

$$\frac{3x^3 - x^2 - 13x - 13}{x^2 - x - 6}$$

The first consideration is:

Is the numerator of lower degree than the denominator?

No, it is not, so we have to divide out by long division:

$$\begin{array}{r} 3x^3 - x^2 - 13x - 13 \\ -(3x^2 - 3x^2 - 18x) \\ \hline 2x^2 + 5x - 13 \\ -(2x^2 - 2x - 12) \\ \hline 7x - 1 \end{array} \bigg| \frac{x^2 - x - 6}{3x + 2}$$

$$\Rightarrow \frac{3x^3 - x^2 - 13x - 13}{x^2 - x - 6} = 3x + 2 + \frac{7x - 1}{x^2 - x - 6}$$

$$\Rightarrow \frac{3x^3 - x^2 - 13x - 13}{x^2 - x - 6} = 3x + 2 + \frac{7x - 1}{(x-3)(x+2)}$$

$$\Rightarrow 3x + 2 + \frac{7x - 1}{(x-3)(x+2)} = 3x + 2 + \frac{A}{x-3} + \frac{B}{x+2}$$

$$\Rightarrow 3x + 2 + \frac{7x - 1}{(x-3)(x+2)} = 3x + 2 + \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$$

$$\Rightarrow 7x - 1 = A(x+2) + B(x-3)$$

• When $x = -2$:

$$7 \times (-2) - 1 = A(-2+2) + B(-2-3)$$

$$\Rightarrow -14 - 1 = A(0) + B(-5)$$

$$\Rightarrow -15 = -5B$$

$$\Rightarrow B = 3$$

- When $x = 3$:

$$7 \times 3 - 1 = A(3 + 2) + B(3 - 3)$$

$$\Rightarrow 21 - 1 = A(5) + B(0)$$

$$\Rightarrow 20 = 5A$$

$$\Rightarrow A = 4$$

Thus:

$$\frac{3x^3 - x^2 - 13x - 13}{x^2 - x - 6} = 3x + 2 + \frac{4}{x - 3} + \frac{3}{x + 2}$$

➤ **Example d:**

$$\frac{6x^3}{1 + x}$$

The first consideration is:

Is the numerator of lower degree than the denominator?

No, it is not, so we have to divide out by long division:

$$\begin{array}{r|l} 6x^3 + 0x^2 + 0x + 0 & x + 1 \\ - (6x^3 + 6x^2) & 6x^2 - 6x + 6 \\ \hline -6x^2 + 0x + 0 & \\ - (-6x^2 - 6x) & \\ \hline 6x + 0 & \\ - (6x + 6) & \\ \hline -6 & \end{array}$$

Thus:

$$\frac{6x^3}{1 + x} = 6x^2 - 6x + 6 - \frac{6}{x + 1}$$

3. Denominators with quadratic factors:

Now let's look at a rational algebraic fraction where the denominator contains a quadratic factor that will not factorise into 2 simple factors.

Example: express in partial fractions,

$$\frac{15x^2 - x + 2}{(x - 5)(3x^2 + 4x - 2)}$$

Here the degree of the numerator is less than the degree of the denominator so no initial division is required. However, the denominator contains a quadratic factor that cannot be factorised further into 2 simple factors.

Indeed the discriminant $b^2 - 4ac = 16 - 4 \times 3 \times (-2) = 40$ confirms this because it is not a perfect square. In this situation there is a rule that applies:

An irreducible quadratic factor in the denominator of the original rational expression of the form $ax^2 + bx + c$ gives rise to a partial fraction of the form:

$$\frac{Ax + B}{ax^2 + bx + c}$$

Therefore:

$$\begin{aligned} \frac{15x^2 - x + 2}{(x-5)(3x^2 + 4x - 2)} &= \frac{A}{x-5} + \frac{Bx + C}{3x^2 + 4x - 2} \\ \Rightarrow \frac{15x^2 - x + 2}{(x-5)(3x^2 + 4x - 2)} &= \frac{A(3x^2 + 4x - 2) + (Bx + C)(x-5)}{(x-5)(3x^2 + 4x - 2)} \\ \Rightarrow 15x^2 - x + 2 &= A(3x^2 + 4x - 2) + (Bx + C)(x-5) \\ \Rightarrow 15x^2 - x + 2 &= 3Ax^2 + 4Ax - 2A + Bx^2 - 5Bx + Cx - 5C \\ \Rightarrow 15x^2 - x + 2 &= (3A + B)x^2 + (4A - 5B + C)x - 2A - 5C \\ \Rightarrow \begin{cases} 15 = 3A + B & Eq(1) \\ -1 = 4A - 5B + C & Eq(2) \\ 2 = -2A - 5C & Eq(3) \end{cases} \\ \Rightarrow \begin{cases} 75 = 15A + 5B & Eq(1)(\times 5) \\ -1 = 4A - 5B + C & Eq(2) \\ 2 = -2A - 5C & Eq(3) \end{cases} \end{aligned}$$

$Eq(1)(\times 5) + Eq(2)$ gives:

$$74 = 19A + C \quad Eq(4)$$

$Eq(4)(\times 5)$ gives:

$$370 = 95A + 5C \quad Eq(4)(\times 5)$$

$Eq(4)(\times 5) + Eq(3)$ gives:

$$372 = 93A$$

$$\Rightarrow A = \frac{372}{93} = 4$$

Substituting for A in $Eq(1)$ gives:

$$15 = 3 \times 4 + B$$

$$\Rightarrow B = 15 - 12 = 3$$

Substituting for A in Eq(3) gives:

$$2 = -2 \times 4 - 5C$$

$$\Rightarrow C = \frac{2+8}{-5} = -2$$

Thus:

$$\frac{15x^2 - x + 2}{(x-5)(3x^2 + 4x - 2)} = \frac{4}{x-5} + \frac{3x-2}{3x^2 + 4x - 2}$$

➤ **Example:**

$$\frac{7x^2 - 18x - 7}{(x-4)(2x^2 - 6x + 3)}$$

For the factor $(2x^2 - 6x + 3)$, the discriminant $b^2 - 4ac = 36 - 4 \times 2 \times 3 = 12$ is not a perfect square, therefore $(2x^2 - 6x + 3)$ cannot be factorised. Thus:

$$\frac{7x^2 - 18x - 7}{(x-4)(2x^2 - 6x + 3)} = \frac{A}{x-4} + \frac{Bx+C}{2x^2 - 6x + 3}$$

$$\Rightarrow \frac{7x^2 - 18x - 7}{(x-4)(2x^2 - 6x + 3)} = \frac{A(2x^2 - 6x + 3) + (Bx+C)(x-4)}{(x-4)(2x^2 - 6x + 3)}$$

$$\Rightarrow 7x^2 - 18x - 7 = A(2x^2 - 6x + 3) + (Bx+C)(x-4)$$

$$\Rightarrow 7x^2 - 18x - 7 = 2Ax^2 - 6Ax + 3A + Bx^2 - 4Bx + Cx - 4C$$

$$\Rightarrow 7x^2 - 18x - 7 = (2A+B)x^2 + (-6A-4B+C)x + 3A-4C$$

$$\Rightarrow \begin{cases} 7 = 2A + B & \text{Eq(1)} \\ -18 = -6A - 4B + C & \text{Eq(2)} \\ -7 = 3A - 4C & \text{Eq(3)} \end{cases}$$

$$\Rightarrow \begin{cases} 28 = 8A + 4B & \text{Eq(1)}(\times 4) \\ -18 = -6A - 4B + C & \text{Eq(2)} \\ -7 = 3A - 4C & \text{Eq(3)} \end{cases}$$

Eq(1)($\times 4$) + Eq(2) gives:

$$10 = 2A + C \quad \text{Eq(4)}$$

Eq(4)($\times 4$) gives:

$$40 = 8A + 4C \quad \text{Eq(4)}(\times 4)$$

Eq(4)($\times 4$) + Eq(3) gives:

$$33 = 11A$$

$$\Rightarrow A = \frac{33}{11} = 3$$

Substituting for A in Eq(1) gives:

$$7 = 2 \times 3 + B$$

$$\Rightarrow B = 7 - 6 = 1$$

Substituting for A in Eq(3) gives:

$$-7 = 3 \times 3 - 4C$$

$$\Rightarrow C = \frac{-7 - 9}{-4} = 4$$

Thus:

$$\frac{7x^2 - 18x - 7}{(x - 4)(2x^2 - 6x + 3)} = \frac{3}{x - 4} + \frac{x + 4}{2x^2 - 6x + 3}$$

4. **Denominators with repeated simple factors:**

Now let's look at a rational algebraic fraction where the denominator contains a repeated simple factor.

Example: express in partial fractions,

$$\frac{35x - 14}{(7x - 2)^2}$$

In this situation the following rule applies:

Repeated factors in the denominator of the algebraic expression of the form $(ax + b)^n$ give partial fractions of the form:

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \dots + \frac{A_n}{(ax + b)^n} = \sum_{i=1}^n \frac{A_i}{(ax + b)^i}$$

Therefore:

$$\frac{35x - 14}{(7x - 2)^2} = \frac{A}{7x - 2} + \frac{B}{(7x - 2)^2}$$

$$\Rightarrow \frac{35x - 14}{(7x - 2)^2} = \frac{A(7x - 2) + B}{(7x - 2)^2}$$

$$\Rightarrow 35x - 14 = A(7x - 2) + B$$

$$\Rightarrow 35x - 14 = 7Ax - 2A + B$$

$$\Rightarrow \begin{cases} 35 = 7A \\ -14 = -2A + B \end{cases}$$

$$\Rightarrow \begin{cases} A = 5 \\ -14 = -2 \times 5 + B \end{cases}$$

$$\Rightarrow \begin{cases} A = 5 \\ B = -4 \end{cases}$$

Thus:

$$\frac{35x - 14}{(7x - 2)^2} = \frac{5}{7x - 2} - \frac{4}{(7x - 2)^2}$$

➤ **Example a:**

$$\frac{42x + 44}{(6x + 5)^2}$$

$$\frac{42x + 44}{(6x + 5)^2} = \frac{A}{6x + 5} + \frac{B}{(6x + 5)^2}$$

$$\Rightarrow \frac{42x + 44}{(6x + 5)^2} = \frac{A(6x + 5) + B}{(6x + 5)^2}$$

$$\Rightarrow 42x + 44 = A(6x + 5) + B$$

$$\Rightarrow 42x + 44 = 6Ax + 5A + B$$

$$\Rightarrow \begin{cases} 42 = 6A \\ 44 = 5A + B \end{cases}$$

$$\Rightarrow \begin{cases} A = 7 \\ 44 = 5 \times 7 + B \end{cases}$$

$$\Rightarrow \begin{cases} A = 7 \\ B = 9 \end{cases}$$

Thus:

$$\frac{42x + 44}{(6x + 5)^2} = \frac{7}{6x + 5} + \frac{9}{(6x + 5)^2}$$

➤ **Example b:**

$$\frac{20x^2 - 54x + 35}{(2x - 3)^3}$$

$$\begin{aligned}
& \frac{20x^2 - 54x + 35}{(2x - 3)^3} = \frac{A}{2x - 3} + \frac{B}{(2x - 3)^2} + \frac{C}{(2x - 3)^3} \\
\Rightarrow & \frac{20x^2 - 54x + 35}{(2x - 3)^3} = \frac{A(2x - 3)^2 + B(2x - 3) + C}{(2x - 3)^3} \\
\Rightarrow & 20x^2 - 54x + 35 = A(2x - 3)^2 + B(2x - 3) + C \\
\Rightarrow & 20x^2 - 54x + 35 = A(4x^2 - 12x + 9) + B(2x - 3) + C \\
\Rightarrow & 20x^2 - 54x + 35 = 4Ax^2 - 12Ax + 9A + 2Bx - 3B + C \\
\Rightarrow & 20x^2 - 54x + 35 = 4Ax^2 + (-12A + 2B)x + 9A - 3B + C \\
\Rightarrow & \begin{cases} 20 = 4A \\ -54 = -12A + 2B \\ 35 = 9A - 3B + C \end{cases} \\
\Rightarrow & \begin{cases} A = 5 \\ -54 = -12 \times 5 + 2B \\ 35 = 9 \times 5 - 3B + C \end{cases} \\
\Rightarrow & \begin{cases} A = 5 \\ B = 3 \\ 35 = 9 \times 5 - 3 \times 3 + C \end{cases} \\
\Rightarrow & \begin{cases} A = 5 \\ B = 3 \\ C = -1 \end{cases}
\end{aligned}$$

Thus:

$$\frac{20x^2 - 54x + 35}{(2x - 3)^3} = \frac{5}{2x - 3} + \frac{3}{(2x - 3)^2} - \frac{1}{(2x - 3)^3}$$