

Maths (Lecture 12)

Complex Numbers

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What is a complex number?

Any number, k such that $k^2 \geq 0$ is called a Real number and belongs to the set of real numbers, i.e., $k \in \mathbb{R}$.

To deal with a number k such that $k^2 < 0$ requires the introduction of a new symbol (number) i defined as $i^2 = -1$ (alternatively, $i = \sqrt{-1}$).

What is a complex number?

A **complex number** is an ordered pair of real numbers, usually denoted by z or w etc. So if a, b are real numbers then we designate a complex number through:

$$z = a + ib$$

where i is a symbol obeying the rule

$$i^2 = -1$$

If $z = a + ib$ then $\operatorname{Re}(z) = a$ and $\operatorname{Im}(z) = b$

Both the real and imaginary parts of a complex number are **real**.

What is a complex number?

Two complex numbers $z = a + ib$ and $w = c + id$ are said to be **equal** if and only if both their real parts are the same and both their imaginary parts are the same, that is

$$a = c \quad \text{and} \quad b = d$$

The **modulus** of a complex number $z = a + ib$ is denoted by $|z|$ and is defined by

$$|z| = \sqrt{a^2 + b^2}$$

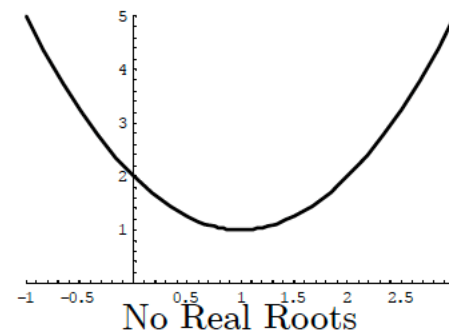
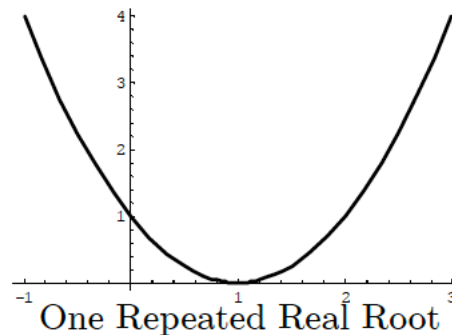
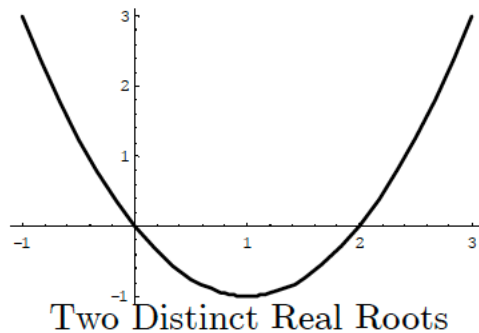
so that the modulus is always a non-negative real number.

The Need For Complex Numbers

the two roots of the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

When $b^2 - 4ac > 0$ then these two roots are real and distinct; graphically they are where the curve $y = ax^2 + bx + c$ cuts the x-axis. When $b^2 - 4ac = 0$ then we have one real root and the curve just touches the x-axis here. But what happens when $b^2 - 4ac < 0$?



The Need For Complex Numbers

Example

Solve the following equation $x^2 + 2x + 5 = 0$

$$\frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

$$-1 \pm 2\sqrt{-1}$$

$$x = -1 + 2i$$

$$x = -1 - 2i$$

Complex conjugate

If $z = a + ib$ is any complex number then the complex conjugate of z is denoted by z^* and is defined by $z^* = a - ib$. For example if $z = 2 - 3i$ then $z^* = 2 + 3i$.

$$\operatorname{Re}(z) = \frac{z + z^*}{2} \quad \text{and} \quad \operatorname{Im}(z) = \frac{i(z^* - z)}{2}$$

Example:

If $z = -2 + i$ find expressions for $\operatorname{Re}(z^*)$ and $\operatorname{Im}(i(z^* - z))$.

Answer

$$\operatorname{Re}(z^*) = -2 \text{ and } \operatorname{Im}(i(z^* - z)) = 0$$

Exercise

Given two complex numbers, $z_1 = 2 + 3i$ and $z_2 = 2 - 3i$

Find the following;

1. $Im(z_1)$

2. $Re(z_2)$

3. $Re(z_1) - Re(z_2)$

4. $Im(z_1) + 2Re(z_2)$

Solution

$$z_1 = 2 + 3i$$

$$z_2 = 2 - 3i$$

$$1. \operatorname{Im}(z_1) = 3$$

$$2. \operatorname{Re}(z_2) = 2$$

$$3. \operatorname{Re}(z_1) - \operatorname{Re}(z_2) = 2 - 2 = 0$$

$$4. \operatorname{Im}(z_1) + 2\operatorname{Re}(z_2) = 3 + 2(2) = 7$$

The algebra of complex numbers

Given $z_1 = a + ib$ and $z_2 = c + id$

1. Addition & subtraction

$$z_1 + z_2 = a + c + (b + d)i$$

$$z_1 - z_2 = a - c + (b - d)i$$

2. Multiplication

$$\begin{aligned} z_1 z_2 &= (a + ib)(c + id) \\ &= ac + (ad + bc)i + bdi^2 \\ &= ac - bd + (ad + bc)i \end{aligned}$$

Example

Given two complex numbers, $z_1 = 2 + 3i$ and $z_2 = 2 - 3i$ find the following;

1. $z_1 + z_2$
2. $z_1 - z_2$
3. $\text{Im}(z_1 + z_2)$
4. $2z_1 + 3z_2$

Solution:

$$1. z_1 + z_2 = 2 + 3i + 2 - 3i = 4$$

$$2. z_1 - z_2 = 2 + 3i - (2 - 3i) = 6i$$

$$3. \text{Im}(z_1 + z_2) = \text{Im}(4) = 0$$

$$4. 2z_1 + 3z_2 = 2(2 + 3i) + 3(2 - 3i) = 10 - 3i$$

Examples

Evaluate $(2 + i)(3 - i)$

$$\begin{aligned}(2 + i)(3 - i) &= 2(3) - 2i + 3i - i^2 \\ &= 6 + i - (-1) \\ &= 7 + i\end{aligned}$$

Evaluate $\text{Im}\{(3 - 2i)^2\}$

$$\begin{aligned}(3 - 2i)^2 &= (3 - 2i)(3 - 2i) \\ &= 3(3) - 3(2i) - 2i(3) + (2i)^2 \\ &= 9 - 12i - 4 \\ &= 5 - 12i\end{aligned}$$

$$\text{Im}(5 - 12i) = -12$$

Examples

Show, for any complex number $z = a + ib$ that $zz^* = |z|^2$.

Answer

By definition $|z| = \sqrt{a^2 + b^2}$, so that $|z|^2 = a^2 + b^2$. Now $zz^* = a^2 + b^2$ so that $zz^* = |z|^2$.

Exercise

Evaluate the following;

1 i^3

2 $2i(3 - 2i)$

3 $(2 - 3i)(2 + 3i)$

Solution

1 $i^3 = i(i^2) = -i$

2 $2i(3 - 2i) = 6i - 4i^2 = 4 + 6i$

3 $(2 - 3i)(2 + 3i) = 4 - 9i^2 = 13$

Exercise

If $z = -2 + i$ and $w = 3 + 2i$ find expressions for

(a) $z + 2w$, (b) $|z - w|$ and (c) zw

Solution

$$(a) \quad z + 2w = 4 + 5i$$

$$(b) \quad z - w = -5 - i \quad \rightarrow \quad |z - w| = \sqrt{(-5)^2 + (-1)^2} = \sqrt{26}$$

$$(c) \quad zw = -6 + 3i - 4i + 2i^2 = -8 - i$$

Dividing one complex number by another

Here we consider the operation of dividing one complex number $z = a + ib$ by another, $w = c + id$:

$$\frac{z}{w} = \frac{a + ib}{c + id}$$

$$\frac{z}{w} = \frac{zw^*}{ww^*}$$

To make denominator real we multiply denominator and numerator by the conjugate of the denominator.

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd + (bc - ad)i}{c^2 + b^2}$$

Examples

Evaluate $\frac{10+5i}{1+2i}$

Multiply the numerator and denominator by the conjugate of $1 + 2i$,

$$\frac{(10 + 5i)}{(1 + 2i)} \times \frac{(1 - 2i)}{(1 - 2i)} = \frac{10 - 20i + 5i - 10i^2}{5}$$
$$= 4 - 3i$$

Simplify i^{-1}

$$i^{-1} = \frac{1}{i}$$

$$\frac{(1)}{(i)} \frac{(-i)}{(-i)} = \frac{-i}{-i^2} = -i$$

The conjugate of i is $-i$, since $i = 0 + i$

Examples

Find $\frac{z}{w}$ if $z = 2 - 3i$ and $w = 2 + i$.

Solution

$$\begin{aligned}\frac{z}{w} &= \frac{2 - 3i}{2 + i} = \frac{(2 - 3i) \times (2 - i)}{(2 + i) \times (2 - i)} && \text{rationalising} \\ &= \frac{4 - 3 + i(-6 - 2)}{4 + 1} && \text{multiplying out} \\ &= \frac{1}{5} - \frac{8}{5}i && \text{dividing through}\end{aligned}$$

Exercise

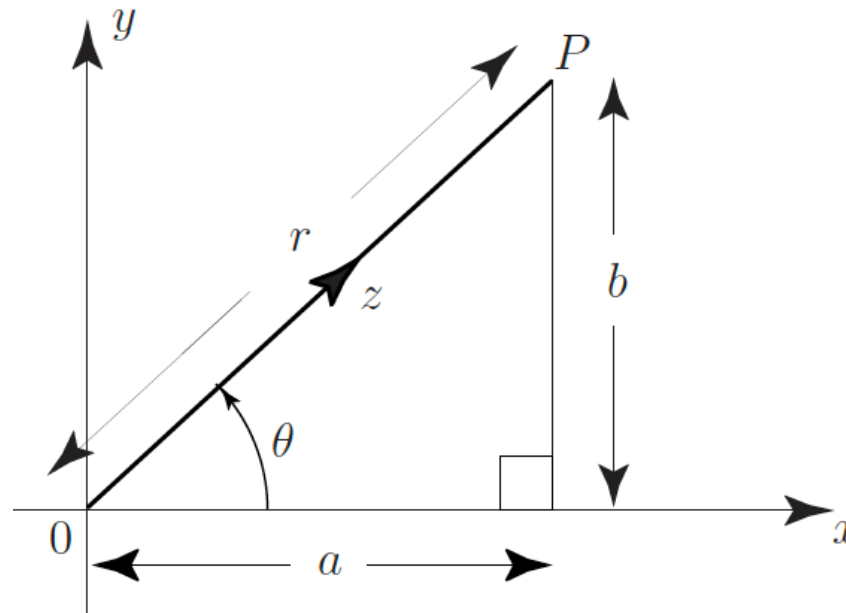
If $z = 3 - i$ and $w = 1 + 3i$ find $\frac{2z + 3w}{2z - 3w}$.

Answer

$$\begin{aligned}\frac{2z + 3w}{2z - 3w} &= \frac{9 + 7i}{3 - 11i} = \frac{(9 + 7i)(3 + 11i)}{(3 - 11i)(3 + 11i)} \\ &= \frac{27 - 77 + (21 + 99)i}{9 + 121} \\ &= -\frac{50}{130} + \frac{120}{130}i = -\frac{5}{13} + \frac{12}{13}i\end{aligned}$$

The polar form of a complex number

the complex number $z = a+ib$ can be represented by a line pointing out from the origin and ending at a point with Cartesian coordinates (a, b) .



The polar form of a complex number

If $z = a + ib$ then

$$z = r(\cos \theta + i \sin \theta)$$

in which

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arg(z) = \tan^{-1} \frac{b}{a}$$

Example

Find the polar coordinate form of (a) $z = 3 + 4i$ (b) $z = -3 - i$

Solution

(a) Here

$$r = |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \quad \theta = \arg(z) = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

so that $z = 5(\cos 53.13^\circ + i \sin 53.13^\circ)$

(b) Here

$$r = |z| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10} \approx 3.16 \quad \theta = \arg(z) = \tan^{-1} \frac{(-1)}{(-3)}$$

Exercise

Find the polar coordinate form of the complex numbers

(a) $z = -i$ (b) $z = 3 - 4i$

solution

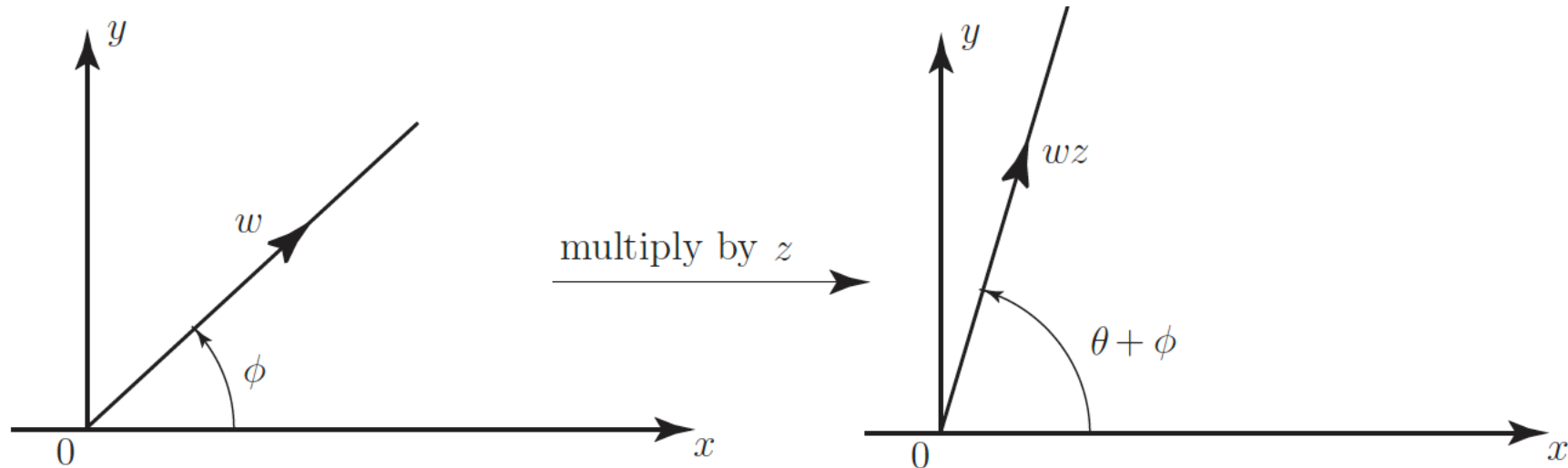
$$z = 1(\cos 270^\circ + i \sin 270^\circ)$$

$$z = 5(\cos 306.87^\circ + i \sin 306.87^\circ)$$

Complex numbers and rotations

If $z = r(\cos \theta + i \sin \theta)$ and $w = t(\cos \phi + i \sin \phi)$ then

$$zw = rt(\cos(\theta + \phi) + i \sin(\theta + \phi)) \quad \frac{z}{w} = \frac{r}{t}(\cos(\theta - \phi) + i \sin(\theta - \phi))$$



The Exponential Form of a Complex Number

We have, so far, considered two ways of representing a complex number:

$$z = a + ib \quad \text{Cartesian form}$$

or

$$z = r(\cos \theta + i \sin \theta) \quad \text{polar form}$$

There is another form of denoting a complex number which the exponential form

$$e^{i\theta} \equiv \cos \theta + i \sin \theta$$

Example

Find complex number expressions, in Cartesian form, for

(a) $e^{i\pi/4}$ (b) e^{-i} (c) $e^{i\pi}$

Solution

(a) $e^{i\pi/4} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$

(b) $e^{-i} = \cos(-1) + i \sin(-1) = 0.540 - i(0.841)$ don't forget: use **radians**

(c) $e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$

The exponential form

Since $z = r(\cos \theta + i \sin \theta)$ and since $e^{i\theta} = \cos \theta + i \sin \theta$ we therefore obtain another way in which to denote a complex number: $z = re^{i\theta}$, called the **exponential form**.

$$z = re^{i\theta} \quad \text{in which} \quad r = |z| \quad \text{and} \quad \theta = \arg(z)$$

$$z = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

Example

Express $z = 3e^{i\pi/6}$ in Cartesian form

Answer

$$\begin{aligned} z = 3e^{i\pi/6} &= 3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right) \\ &= 3(0.8660 + i0.5000) \\ &= 2.60 + 1.50i \quad \text{to 2 d.p.} \end{aligned}$$

Exercises

Express the following complex numbers in exponential form:

(a) $z = 1 - i$ (b) $z = 2 + 3i$ (c) $z = -6$.

Answer

$$z = \sqrt{2}e^{i7\pi/4} \text{ (or, equivalently, } \sqrt{2}e^{-i\pi/4}\text{)}$$

$$z = \sqrt{13}e^{i(0.9828)}$$

$$z = 6e^{i\pi}$$

Express the following complex numbers in exponential form

(a) $z = 2 - i$ (b) $z = 4 - 3i$ (c) z^{-1} where $z = 2 - 3i$.

Answer

3. (a) $1.5 + i(2.598)$ (b) 1 (c) $-0.707 + i(0.707)$

Express the following complex numbers in exponential form

(a) $z = 2 - i$ (b) $z = 4 - 3i$ (c) z^{-1} where $z = 2 - 3i$.

Answer

(a) $\sqrt{5}e^{i(5.820)}$ (b) $5e^{i(5.6397)}$

(c) $2 - 3i = \sqrt{13}e^{i(5.300)}$ therefore $\frac{1}{2 - 3i} = \frac{1}{\sqrt{13}}e^{-i(5.300)}$

Trigonometric functions

$$e^{i\theta} = \cos \theta + i \sin \theta$$

It follows from this that

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta$$

Now if we add these two relations together we obtain

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

whereas if we subtract the second from the first we have

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

MATLAB

MATLAB Functions

<u>abs</u>	Absolute value and complex magnitude
<u>angle</u>	Phase angle
<u>complex</u>	Create complex array
<u>conj</u>	Complex conjugate
<u>cplxpair</u>	Sort complex numbers into complex conjugate pairs
<u>i</u>	Imaginary unit
<u>imag</u>	Imaginary part of complex number
<u>isreal</u>	Determine whether array uses complex storage
<u>j</u>	Imaginary unit
<u>real</u>	Real part of complex number
<u>sign</u>	Sign function (signum function)
<u>unwrap</u>	Shift phase angles

Example

Suppose we have two complex numbers; $z1 = 3 + 4i$ (b) $z2 = -3 - i$

z = complex(a,b) creates a complex output, z, from two real inputs, such that $z = a + bi$.

```
1
2 - a=3;
3 - b=4;
4
5 - z1 = complex(a,b);
6
7 - a=-3;
8 - b=-1;
9
10 - z2 = complex(a,b);
```

Arithmetic operations

$$W1 = z1 + z2$$

$$W2 = z1 - z2$$

$$W3 = z1 * z2$$

$$W4 = z1 / z2$$

Command Window

W1 =

0.0000 + 3.0000i

W2 =

6.0000 + 5.0000i

W3 =

-5.0000 -15.0000i

W4 =

-1.3000 - 0.9000i

Extract imaginary and real part in a complex number

If $z = a + ib$ then $\text{Re}(z) = a$ and $\text{Im}(z) = b$

Both the real and imaginary parts of a complex number are **real**.

Command Window

```
>> Imaginary_z1=imag(z1)
```

```
Real_z1=real(z1)
```

```
Imaginary_z1 =
```

```
4
```

```
|  
Real_z1 =
```

```
3
```

Phase angle

If $z = a + ib$ then

$$z = r(\cos \theta + i \sin \theta)$$

in which

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arg(z) = \tan^{-1} \frac{b}{a}$$

Command Window

```
>> angle(z1)
```

```
ans =
```

```
0.9273
```

```
>> rad2deg(angle(z1))
```

```
ans =
```

```
53.1301
```

Complex conjugate

```
>> conj(3+5i)

ans =

    3.0000 - 5.0000i
```

Absolute value and complex magnitude

Command Window

```
>> abs(3+5j)

ans =

    5.8310
```

MATLAB Exercise

If $z = -2 + i$ and $w = 3 + 2i$ find expressions for

(a) $z + 2w$, (b) $|z - w|$ and (c) zw

(d) $(z + 5w)^*$, (e) $\text{Im}(3z - w)$, (f) $\text{Re}(3z - w)$,

(g) *angle and modulus of $(2z + 3w)$*