

Differential Equations

Examples: $\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} - 3y = 0$

We need a characteristic equation

$$m^2 + 2m - 3 = 0$$

factorise: $(m+3)(m-1) = 0$

Solve $m_1 = -3 \quad m_2 = 1$

So we have 2 'real roots' to the equation

So we use the general solution

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

So we get

$$y = Ae^{-3x} + Be^x$$

(overdamped)

Example 2

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = 0$$

Characteristic equation :

$$M^2 + 2M + 1 = 0$$

Factorise :

$$(M+1)^2$$

$$M_1 = M_2 = -1$$

So we need to use the general solution :

$$y = (A + Bx) e^{mx}$$

So we get :

$$y = (A + Bx) e^{-x}$$

(Critically Damped)

Example 3

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0$$

Characteristic equation:

$$M^2 + M + 1 = 0$$

factorise ???

So we need the formula

$$M = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Put in values:

$$M = \frac{-1 \pm \sqrt{1^2 - 4}}{2}$$

$$M = \frac{-1 \pm \sqrt{-3}}{2}$$

$$M = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$$

$$M = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

So we have a complex root.

Equate this with $M = \alpha + i\beta$

$$\text{So } \alpha = -\frac{1}{2} \quad \beta = \frac{\sqrt{3}}{2}$$

We use these with:

$$y = e^{\alpha x} [A \cos(\beta x) + B \sin(\beta x)]$$

So we get:

$$y = e^{-\frac{x}{2}} \left[A \cos\left(\frac{\sqrt{3}}{2} x\right) + B \sin\left(\frac{\sqrt{3}}{2} x\right) \right]$$

$\Rightarrow$ Underdamped

These 'general solutions' give the 'shape' of the solution, but not the exact response --- we need to know A and B

To find A and B, we need the initial conditions of the system, this is displacement and velocity,

So consider the example:



$$\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 10x = 0$$

$$ma + cv + kx = 0$$

This time we need to know a few more things:

Initial conditions: At $t=0$, $x=0$, $\frac{dx}{dt} = 3$

So the mass starts at rest, moving at 3 m/s

Solution: The characteristic equation is

$$m^2 + 7m + 10 = 0$$

$$(m+5)(m+2) = 0$$

$$\text{so } m_1 = -5 \quad m_2 = -2$$

We have 2 real and different roots:

So the general solution is:

$$x = Ae^{m_1 t} + Be^{m_2 t}$$

$$x = Ae^{-5t} + Be^{-2t}$$

So we now need to deal with A and B

So firstly we use the initial conditions
at $t=0, x=0$

$$0 = Ae^0 + Be^0$$

$$A + B = 0 \quad \text{because } e^0 = 1$$

Now we use the other initial condition $t=0, \frac{dx}{dt} = 3$

So we need to differentiate --

$$x = Ae^{-5t} + Be^{-2t}$$

$$\frac{dx}{dt} = -5Ae^{-5t} - 2Be^{-2t}$$

Now put in $t=0, \frac{dx}{dt} = 3$

$$3 = -5Ae^0 - 2Be^0$$

So we have $A + B = 0$ (1)

and $-5A - 2B = 3$ (2)

We know from (1) that $A = -B$
So substitute this in ...

$$-5(-B) - 2B = 3$$

$$5B - 2B = 3$$

$$3B = 3$$

$$B = 1$$

So we have $B = 1$ and $A = -1$
Substitute this back in to $x = Ae^{-5t} + Be^{-2t}$

$$x = -e^{-5t} + e^{-2t}$$

or

$$x = e^{-2t} - e^{-5t}$$

More real examples



$$2 \frac{\partial^2 x}{\partial t^2} + 40 \frac{\partial x}{\partial t} + 128x = 0$$

$$\frac{\partial^2 x}{\partial t^2} + 20 \frac{\partial x}{\partial t} + 64x = 0$$

So we get

$$m^2 + 20m + 64 = 0$$

$$(m+4)(m+16) = 0$$

So we have roots $m_1 = -4$ $m_2 = -16$

$$\text{So } x = Ae^{-4t} + Be^{-16t}$$

If we know some initial conditions:

Assume starting at equilibrium with velocity 0.6 ms^{-1}

$$t=0, x=0, \frac{\partial x}{\partial t} = 0.6$$

$$0 = A + B$$

So now we use at $t=0$, $\frac{dx}{dt} = 0.6$

$$\frac{dx}{dt} = -4Ae^{-4t} - 16Be^{-16t}$$

$$0.6 = -4A - 16B$$

$$t=0 \\ e^0 = 1$$

Since $A = -B$ so $0.6 = 12A$

$$\text{so } A = 0.05$$

So we get

$$x = 0.05e^{-4t} - 0.05e^{-16t}$$

$$\text{or } x = 0.05(e^{-4t} - e^{-16t})$$

One more example:

Motorcycle front end



Mass = 100 kg

Spring stiffness = 18,000 N/m

Damping = 1200 Ns/m

$$100 \frac{d^2 x}{dt^2} + 1200 \frac{dx}{dt} + 18000 x = 0$$

$$\rightarrow 100 m^2 + 1200 m + 18000 = 0$$

$$\rightarrow m^2 + 12m + 180 = 0$$

Factorise?

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-12 \pm \sqrt{12^2 - (4 \times 180)}}{2}$$

$$m = \frac{-6 \pm \sqrt{144 - 720}}{2}$$

$$\text{So } M = -6 \pm i^{24/2}$$

$$M = -6 \pm i 12^{\beta}$$

$$\text{So } x = e^{-6t} [A \cos(12t) + B \sin(12t)]$$

general solution - - -

If at $t=0$, $x=0$ and $\frac{dx}{dt} = 2$

when $t=0$, $x=0$ so

$$x = e^{-6t} [A \cos(12t) + B \sin(12t)]$$

$$0 = A + 0$$

$$\text{So } A = 0$$

Now use at $t=0$ $\frac{dx}{dt} = 2$

from $x = e^{-6t} [\cancel{A \cos(12t)} + B \sin(12t)]$
 Because $A = 0$

$$x = e^{-6t} [B \sin(12t)]$$

$$\dot{x} = \cancel{-6e^{-6t}} [\cancel{B \sin(12t)}] + [\cancel{\sin(12t)} + 12B \cancel{\cos(12t)}] \cancel{e^{-6t}}$$

$$\dot{x} = 12B \cos(12t)$$

$$2 = 12B$$

$$B = \frac{1}{6}$$

So $x = e^{-6t} [\frac{1}{6} \sin(12t)]$



