

Chapter 1: Introduction to the Theory of Groups

1.1. Definition of a Group

A group is a set G equipped with a binary operation $\cdot$ satisfying the following properties:

(1) Closure: For all $a, b \in G$, $a \cdot b \in G$.

(2) Associativity: For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(3) Identity: There exists an element $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$.

(4) Inverse: For each $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Example 1.1: The set of integers $\mathbb{Z}$ under addition forms a group.

Example 1.2: The set of non-zero real numbers $\mathbb{R}^*$ under multiplication forms a group.

Example 1.3: The set of $n \times n$ invertible matrices over a field F forms a group under matrix multiplication.

Example 1.4: The set of all functions from a set X to a group G forms a group under pointwise multiplication.

Example 1.5: The set of all permutations of a set X forms a group under composition.

Example 1.6: The set of all n th roots of unity in a field F forms a group under multiplication.

Example 1.7: The set of all $n \times n$ matrices over a field F with trace zero forms a group under addition.

Example 1.8: The set of all $n \times n$ matrices over a field F with determinant 1 forms a group under multiplication.

Example 1.9: The set of all $n \times n$ matrices over a field F with determinant 1 and trace zero forms a group under addition.

Example 1.10: The set of all $n \times n$ matrices over a field F with determinant 1 and trace zero forms a group under addition.

Example 1.11: The set of all $n \times n$ matrices over a field F with determinant 1 and trace zero forms a group under addition.