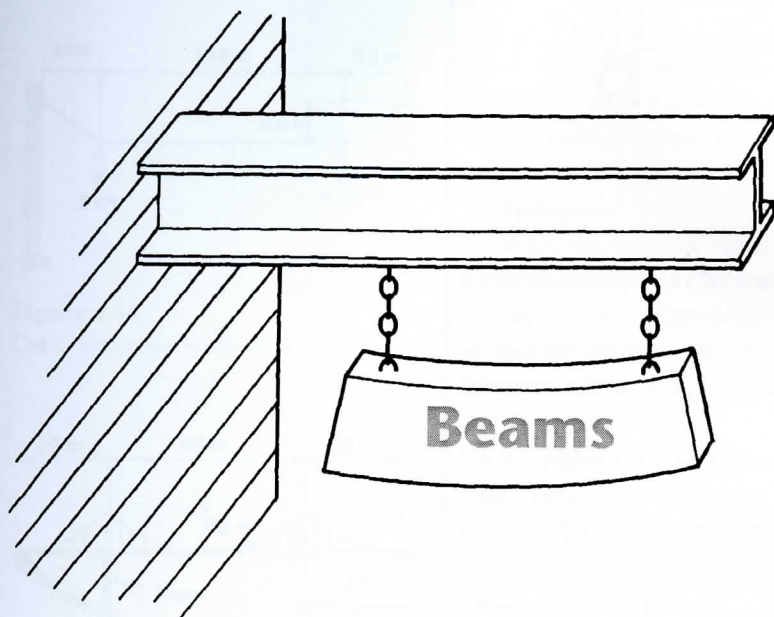


Chapter 7



Learning outcomes

At the end of this chapter:

- You will be able to design the commonest form of building structure – the *beam*
- You will learn how to calculate and draw *shear force* and *bending moment diagrams* for a range of loading patterns and appreciate their importance in the design of real beams
- You will understand the distinction between a *plastic* and an *elastic* approach
- You will be able to apply the calculations outlined in Eurocodes for the design of beams in steel, timber and reinforced concrete
- You will be able to extract the details of standard beams from tables and also be able to calculate the required properties of non-standard beams from first principles

7.1 Introduction

A **beam** is a structural member which is usually the most common type of structural member. Bending occurs when a load is applied perpendicular to the longitudinal axis of the beam at a distance from a support. Commonly, the load is applied downwards. Most beams are supported as shown in figure 7.1a. This is covered in section 2.5. As with planar structures, it is possible to put a beam on supports which are one pin support and one roller support. Another type of support is shown in figure 7.1b. Here the beam must be rigidly built-in to the support. The reaction force has now become a moment. This is covered later. A simply supported beam is shown in figure 7.1c.

With a few exceptions, the design of beams is shown in figure 7.1d and is beyond the scope of this book. **Other beams.** There are many other types of solution by the three basic types of support.

7.2 Examples

Figure 7.2a shows the steel-framed multi-storey building available from local structural engineers. Some of these are tabulated in the book. They consist of two flanges and a web. The cross-section is efficient because the material is concentrated in the flanges. This will be proved later.

Bridges often have a reinforced concrete form of bridge is the most common. The plates together to form a box section.

Alternatively, multi-span concrete frames. For

7.1 Introduction

A **beam** is a structural member subject to **bending** and is probably the most common structural element that designers have to cope with. Bending occurs in a member when a component of load is applied perpendicular to the member axis, and some distance from a support. Bending causes **curvature** of a member. Commonly, beams are horizontal, and loads vertically downwards. Most beams span between two or more fixed points as shown in *figure 7.1a*. The types of support were discussed in *section 2.5*. As with plane trusses there must be at least three reactive forces to put a beam into equilibrium. In *figure 7.1a* the supports are one pin and one roller, and this is called **simply supported**. Another common beam type is the **cantilever**, as shown in *figure 7.1b*. Here one end is unsupported but the other must be rigidly built-in to prevent rotation. The third reaction force has now become a moment or couple, but more of that later. A simply supported beam can have a cantilevered end as shown in *figure 7.1c*.

With a few exceptions, beams with extra supports such as that shown in *figure 7.1d* are statically indeterminate and their analysis is beyond the scope of this book. These are known as **continuous beams**. There are too many unknown support reactions for solution by the three basic equilibrium equations.

7.2 Examples of beams and beam types

Figure 7.2a shows the use of a standard **universal beam** in a steel-framed multi-storey building. Universal beams are widely available from local stock holders in a range of sizes and weights. Some of these are tabulated in the *Appendix*. The beam is made up of two flanges and a web – *figure 7.2a*. The 'I' shape of the beam cross-section is efficient, as the steel works more effectively if it is concentrated in the flanges at the top and bottom of the beam. This will be proved later.

Bridges often have a requirement for a long-span beam. A modern form of bridge is the steel box-girder, made by welding flat steel plates together to form a hollow stiffened box (*figure 7.2b*).

Alternatively, multi-storey buildings can have reinforced concrete frames. *Figure 7.2c* shows a typical cross-section

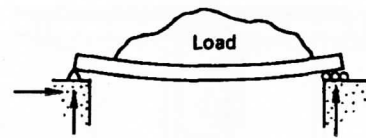


Figure 7.1a
A simply supported beam

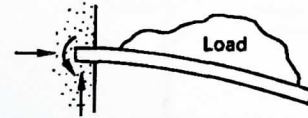


Figure 7.1b
A cantilever beam

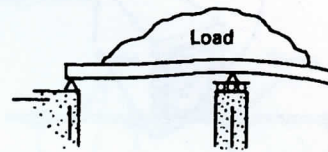


Figure 7.1c
A simply supported beam with a cantilever end

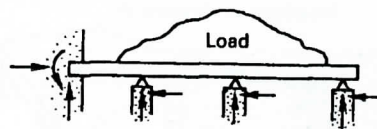


Figure 7.1d
A continuous beam

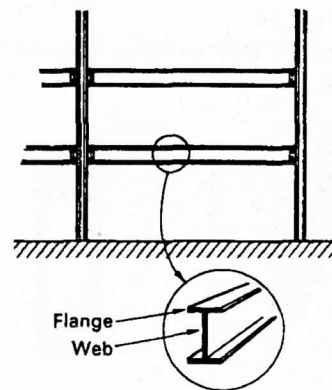


Figure 7.2a
Universal beams in a steel-framed building

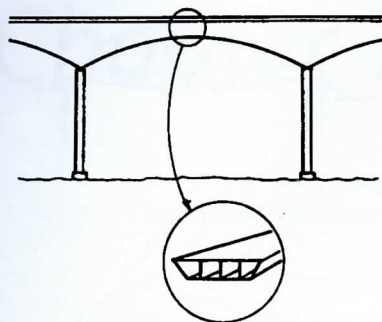


Figure 7.2b
A steel box-girder bridge

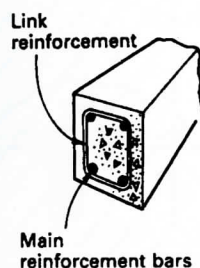


Figure 7.2c
A reinforced concrete beam

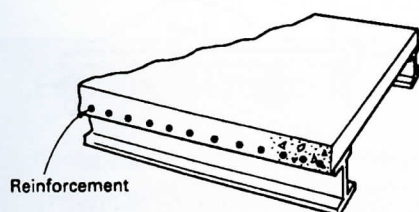


Figure 7.2d
A concrete slab spanning between steel beams

through a reinforced concrete beam. We saw in *chapter 3* that concrete is reasonably strong in compression but weak in tension. In fact we usually ignore the tensile strength of concrete altogether, and provide steel reinforcement to resist the tensile forces. The main reinforcement bars must occur on the tensile side of the beam.

Slabs are widely used for floors and roofs in buildings. They are designed simply as wide but shallow beams. *Figure 7.2d* shows a concrete slab spanning between steel beams. Again the reinforcement must be on the tensile side of the slab.

Extra economy can be achieved by joining a slab to a beam, so that they function as one combined unit. *Figure 7.2e* shows two examples of this. Firstly there is a reinforced concrete 'T' beam, where the slab is cast monolithically with the beam. Secondly there is **composite construction**, where a concrete slab is joined to a steel beam by **shear connectors** welded to the top of the beam at regular intervals.

A variant of the above is **stressed skin construction** where plates are fixed to both the top and bottom of a beam. *Figure 7.2f* shows the aluminium alloy floor of an aircraft, where rivets have been used to join the components together.

Small timber beams are often used in the floor construction of houses, where they are known as **joists** (*figure 7.2g*).

For larger timber beams either **plywood box beams** or **laminated timber beams** can be used (*figure 7.2h*). Laminated beams are formed from strips of timber which are glued together. They can be sanded and varnished to produce a high quality appearance, but they tend to be rather expensive.

7.3 The concept of shear force and bending moment

Stresses within a member due to bending are clearly more complex than those from simple axial tension. An important step in evaluating the stresses within a beam is to determine the magnitude of the shear force and bending moment at the point under consideration. Students often seem to find these concepts difficult,

but it is worth persevering as it can be very rewarding.

7.3.1 Cantilever with

Consider the cantilever point load of magnitude through the beam at so consider the equilibrium the cut (*figure 7.3b*). The **must** ensure the eq Considering the equilibriure in this way is a c section under considera the free body floating v applied, the free body v ence of stresses at the c here with the method where the forces in the structure into equilibrium

Clearly a force, V , is (*figure 7.3c*). In this case distance along the beam to the axis of the beam,

We have still not sa stands, the free body wo To prevent this we need in *figure 7.3d*. In this case

M is termed the **bending moment** the distance from the lo $= PL$ at the wall support support must be capable be avoided.

If values of V and M able scale, we get the **sl bending moment** dia bending moment and sl we use them to determin

but it is worth persevering with them as, with a little practice, they can be very rewarding.

7.3.1 Cantilever with point load

Consider the cantilever beam shown in *figure 7.3a* which has a point load of magnitude P at the end. Make an imaginary cut through the beam at some point distance x from the end and consider the equilibrium of the section of beam to the right of the cut (*figure 7.3b*). The sum of the stresses on the cut surface **must** ensure the equilibrium of this piece of beam. Considering the equilibrium of an isolated section of a structure in this way is a common and powerful technique. The section under consideration is known as a **free body**. Think of the free body floating weightlessly in space. When the load is applied, the free body would move if it were not for the presence of stresses at the cut surface. (There is an obvious parallel here with the method of sections analysis of plane frames, where the forces in the cut members must put a section of the structure into equilibrium.)

Clearly a force, V , is required to satisfy vertical equilibrium (*figure 7.3c*). In this case obviously $V = P$ and is independent of the distance along the beam, x . This force V , which is always normal to the axis of the beam, is termed **shear force**.

We have still not satisfied equilibrium, however, and, as it stands, the free body would spin around in a clockwise direction. To prevent this we need a couple or moment, M , applied as shown in *figure 7.3d*. In this case, taking moments about the cut:

$$M = P \times x$$

M is termed the **bending moment** and as expected increases as the distance from the load increases. It reaches a peak value of $M = PL$ at the wall support. Clearly, in the case of cantilevers, the support must be capable of resisting this moment if collapse is to be avoided.

If values of V and M are plotted for all values of x , to some suitable scale, we get the **shear force diagram** (*figure 7.3e*) and the **bending moment diagram** (*figure 7.3f*). We shall see later that bending moment and shear force diagrams are important because we use them to determine the actual sizes of beams.

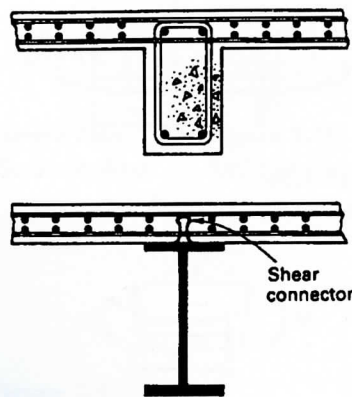


Figure 7.2e
A concrete 'T' beam and composite construction

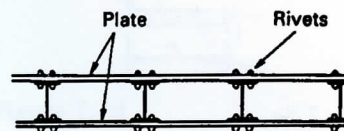


Figure 7.2f
The aluminium floor of an aircraft

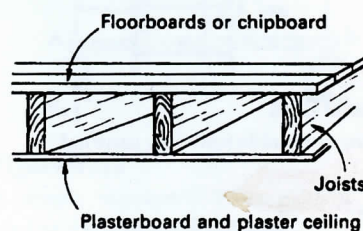


Figure 7.2g
Typical house floor construction using timber joists

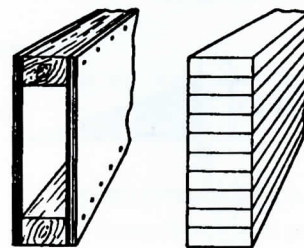
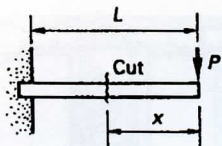
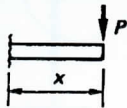


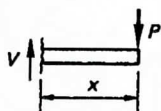
Figure 7.2h
A plywood box beam and a laminated timber frame

**Figure 7.3a**

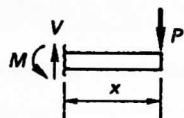
A cantilever beam with a point load

**Figure 7.3b**

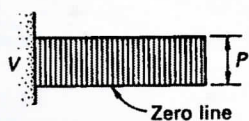
Free body diagram

**Figure 7.3c**

Vertical equilibrium satisfied

**Figure 7.3d**

Moment equilibrium also satisfied

**Figure 7.3e**

Shear force diagram

**Figure 7.3f**

Bending moment diagram

7.3.2 Convention for shear force and bending moment diagrams

It is important to be consistent in the way that shear force and bending moment diagrams are drawn. Different designers use different conventions but in this book the following is used throughout:

Clockwise shears (i.e. when the left-hand side of the beam is being pushed upwards and the right-hand side of the beam is being pushed downwards – figure 7.4a) are drawn **above** the zero line.

Anticlockwise shears (figure 7.4b) are drawn **below** the zero line.

Bending moments are always drawn on the **tensile side** of the beam.

Therefore:

Hogging moments (figure 7.4c) are drawn **above** the line.

Sagging moments (figure 7.4d) are drawn **below** the line.

We shall now look at two more commonly occurring cases.

7.3.3 Simply supported beam with point load at mid-span

The beam has a span of L and a point load of P at mid-span (figure 7.5a).

The first step is to find the reactions. In this case from symmetry:

$$R_A = R_B = \frac{P}{2}$$

We can now draw the shear force diagram. We will start from R_A and work to the right, considering vertical equilibrium as we go. Inspection of the free body (figure 7.5b) indicates a clockwise shear which means that the diagram is plotted above the line. However, as we go beyond the mid-span load the direction of the shear force is reversed, and it becomes anticlockwise (figure 7.5c). It can be seen that the value of the shear force is $V = P/2$ throughout in order to satisfy vertical equilibrium. The final shear force diagram is as shown (figure 7.5d).

At this stage we can see that the shear force diagram

Start at the left-hand end and draw the direction of each shear force on the right-hand side.

So in this case we start at the left-hand end where the reaction is vertically upwards representing a positive shear. There is no change until we reach the mid-span where we descend an amount $P/2$ to the reaction R_B where we return to the baseline.

Turning to bending moment, this is a sagging beam with a point load at mid-span, therefore plot the diagram below the baseline. The bending moment, M , varies linearly from zero at the ends to a maximum at the mid-point where $x = L/2$.

$$(x < L/2)$$

Beyond the mid-point we have:

$$(x > L/2) \quad M = \frac{P}{2}x - \frac{P}{2}(x - L/2)$$

At the mid-span point it is:

$$(x = L/2)$$

If these values are now plotted, the final bending moment diagram is shown in figure 7.5e.

7.3.4 Simply supported beam with uniformly distributed load

We shall now consider the case of a uniformly distributed load. The dashed line in figure 7.6a represents the beam. Particularly with

At this stage we can deduce a simple procedure for ensuring that the shear force diagram always complies with our convention:

Start at the left-hand side of the structure and follow the direction of each force as you work towards the right-hand side.

So in this case we start at the left-hand reaction R_A and draw a line vertically upwards representing $P/2$. As we move to the right there is no change until we reach the central point load, where we descend an amount P . There is no further change until we reach the reaction R_B where we move up $P/2$. This should always take us back to the baseline and hence provides a check.

Turning to bending moment, it should firstly be obvious that this is a sagging beam with the bottom of the beam in tension. We therefore plot the diagram below the line. The magnitude of the bending moment, M , varies with the distance along the beam, x . Up to the mid-point we can see from figure 7.5e that:

$$(x < L/2) \quad M = \frac{P \times x}{2}$$

Beyond the mid-point we can see from figure 7.5f that:

$$(x > L/2) \quad M = (P/2 \times x) - [P \times (x - L/2)]$$

$$M = P \times (L/2 - x/2)$$

At the mid-span point itself:

$$(x = L/2) \quad M_{\max} = \frac{PL}{4}$$

If these values are now plotted for the whole beam, then the bending moment diagram shown in figure 7.5g results.

7.3.4 Simply supported beam with uniformly distributed load

We shall now consider the case of a beam of span L which carries a uniformly distributed load (UDL) of w kN per m (figure 7.6a). The dashed line in figure 7.6a shows the deflected shape of the beam. Particularly with more complex cases, it is often helpful to



Figure 7.4a
Clockwise shear

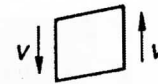


Figure 7.4b
Anticlockwise shear

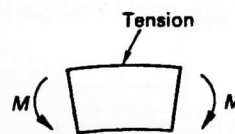


Figure 7.4c
Hogging moment

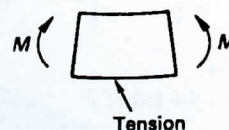


Figure 7.4d
Sagging moment

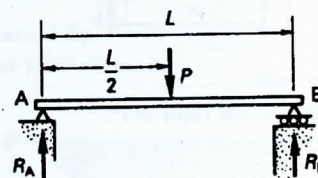


Figure 7.5a
A simply supported beam with a point load at mid-span

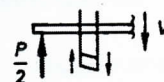


Figure 7.5b
Free body to the right of P (clockwise shear)

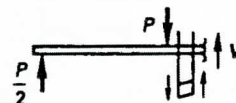


Figure 7.5c
Free body to the left of P (anticlockwise shear)

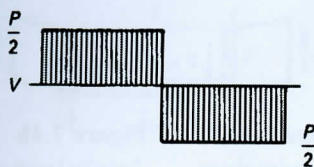


Figure 7.5d
Shear force diagram

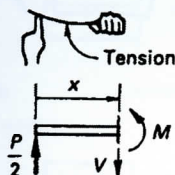


Figure 7.5e
Free body to the left of P

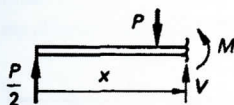


Figure 7.5f
Free body to the right of P

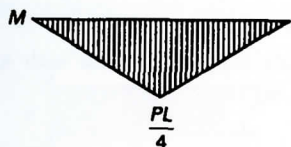


Figure 7.5g
Bending moment diagram

start by drawing the expected deflected shape. The tension side of the beam, on which the bending moment is drawn, is usually obvious.

Shear forces The free body diagram (figure 7.6b) shows how the shear force, V , now varies with x as the gradually applied downward load affects the vertical equilibrium of the free body. The shear force must be equal to the difference between the downward load and the upward reaction. Thus as x increases, V gradually reduces until it is zero at mid-span (where the load and the reaction are equal). Beyond mid-span there is a change of sign. You can obtain the value of V at any point in the beam by simply substituting the appropriate value of x into the following equation:

$$V = \frac{wL}{2} - (w \times x)$$

Plotting this results in the shear force diagram shown in figure 7.6c. As expected, the slope of the shear force line is equal to the magnitude of the UDL.

Bending moments The magnitude of the UDL load is $w x$. When taking moments we consider the UDL to be concentrated at its mid-point. Remember we always take moments about the cut end of the free body. Therefore all distances are measured from this point when we take moments. The distance from the cut to the mid-point of the UDL is thus $x/2$.

$$M = (wL/2 \times x) - (wx \times x/2)$$

$$M = \frac{w \times (Lx - x^2)}{2}$$

The above is the equation of a parabola and when plotted produces the bending moment diagram shown in figure 7.6d.

At mid-span when $x = L/2$ we find:

$$M_{\max} = \frac{wL^2}{8}$$

7.3.5 General rules bending moment

In practice, bending moment diagrams are often plotted by using plotting equations, as shown in figure 7.3. The peak values of the bending moment at the ends of the beam, a point load at mid-span, and the average value over the whole span, should be noted.

We can conclude from

Between point loads

Throughout a UDL

The peak value of the bending moment occurs at the point where the shear force is zero.

This last point can be proved mathematically. The magnitude of the maximum bending moment can be proved it mathematically. the proof can skip the following steps.

Proof that a peak bending moment occurs at a point

Consider a free body of a beam supporting a UDL of magnitude w . The shear force and bending moment at the left end are V and M . At the right end, the shear force is $V + \delta V$ and the bending moment is $M + \delta M$.

Now consider the free body of the beam, summing moments about the left end:

$$(\Sigma M \text{ about A})$$

$$M + V\delta x$$

Neglecting multiples of δx^2 on both sides:

7.3.5 General rules concerning shear force and bending moment diagrams

In practice, bending moment diagrams are not usually drawn by plotting equations, as suggested above, but by simply evaluating the peak moments at key points and then sketching the intermediate shape. Peak values for common cases, such as $M = PL/4$ for a point load at mid-span and $M = wL^2/8$ for a UDL over the whole span, should be remembered.

We can conclude from the above cases that:

- Between point loads – Shear force is constant.
- Bending moment varies as a straight line.
- Throughout a UDL – Shear force varies as a straight line.
- Bending moment varies as a parabola.

The peak value of the bending moment occurs at the point in the beam where the shear force passes through zero.

This last point can be very useful for locating the position and magnitude of the maximum bending moment. We shall now prove it mathematically. Those readers who are not interested in the proof can skip the following section.

Proof that a peak value of bending moment occurs at a point of zero shear

Consider a free body consisting of a short length, δx , of beam supporting a UDL of magnitude w/m (figure 7.7). The values of shear force and bending moment increase from V and M at one end to $V + \delta V$ and $M + \delta M$ at the other.

Now consider the moment equilibrium of the free body by summing moments about point A:

(ΣM about A)

$$M + V\delta x - (w\delta x \times \delta x/2) + M + \delta M$$

Neglecting multiples of small quantities and cancelling M from both sides:

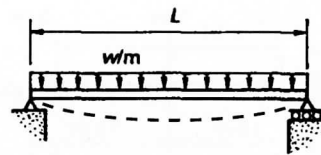


Figure 7.6a

A simply supported beam with a UDL

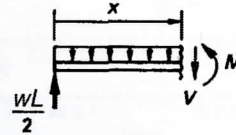


Figure 7.6b

Free body

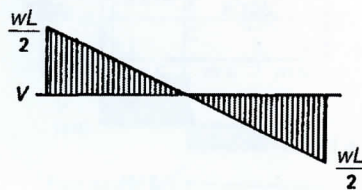


Figure 7.6c

Shear force diagram

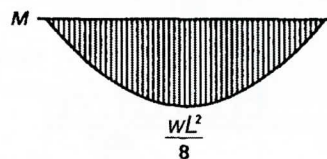


Figure 7.6d

Bending moment diagram

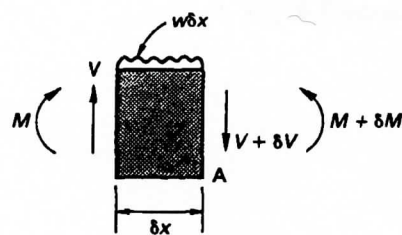


Figure 7.7

A short length of beam

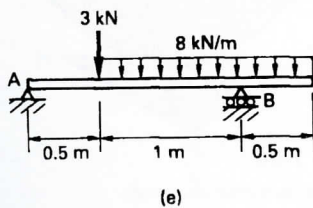
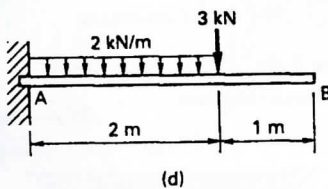
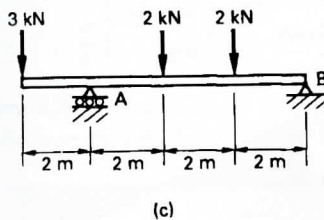
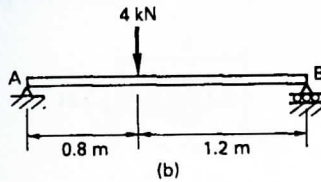
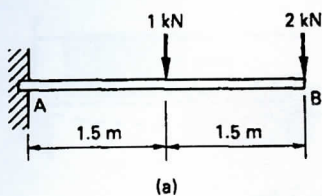


Figure 7.8

$$V\delta x = \delta M$$

As δx approaches zero this can be written as the differential:

$$V = \frac{dM}{dx} \quad [7.1]$$

In words, this means that the shear force is equal to the rate of change of the bending moment diagram. Thus when the shear force is zero the slope of the bending moment is zero, indicating either a maximum or minimum value.

EXAMPLE 7.1

For each of the beams shown in figure 7.8a–e draw the deflected shape, shear force and bending moment diagrams. Indicate the magnitude of all peak values.

Solution

Beam (a)

As usual the first step is to find the reactions. This is clearly a cantilever beam, and there is no reaction at the end B. Therefore the upward reaction at end A must, from vertical equilibrium, equal the total downward force on the beam, i.e. 3 kN. So starting at A we draw a line vertically upwards representing 3 kN (figure 7.9a). Moving rightwards there is no change in vertical load until we reach the first point load where we descend 1 kN. Continuing rightwards there is no further change until the end B where there is a further downward force of 2 kN. Thus the line returns to the baseline indicating that vertical equilibrium is satisfied, i.e. the sum of the downward forces equals the upward reaction.

As far as the bending moment is concerned, because of the point loads, we would expect it to consist of a series of straight lines. We can also see by inspection that these particular loads would cause the beam to deflect downwards at B, producing tension in the top face. We therefore plot the diagram on the top. If we consider the free-body diagram shown, it is clear that the bending moment:

$$M = 2 \times x$$

Hence it has a zero value at the mid-point of the beam. The point loads into account.

$$M_A =$$

Remember, if we want the bending moment to be measured from A.

Comment – This result tells us that the beam must be capable of resisting a...

Beam (b)

As before we must first find the reactions before we can start to draw the diagrams.

$$(\Sigma M \text{ about A})$$

We could find R_A in a similar way, but there is an easier way. Since the beam is in equilibrium, the sum of the vertical forces must be zero.

$$(\Sigma V = 0)$$

Hence

We are now in a position to draw the diagrams. We start at the left-hand end of the beam (figure 7.9b).

The peak bending moment occurs at the point load, C, and from the diagram we can see that the peak value is 4 kNm.

$$M_C =$$

Clearly we would get the same result if we considered the equilibrium of the beam to the right of C.

$$M_C$$

This time the bottom of the beam is in tension.

Hence it has a zero value at B (where $x = 0$) and a value of 3 kNm at the mid-point of the beam. At the support A we must take both point loads into account. Hence:

$$\begin{aligned} M_A &= (2 \times 3) + (1 \times 1.5) \\ &= 7.5 \text{ kNm} \end{aligned}$$

Remember, if we want the moment at A, then all distances must be measured from A.

Comment – This result tells us that both the wall and the beam connection at A must be capable of resisting a moment of 7.5 kNm if collapse is to be avoided.

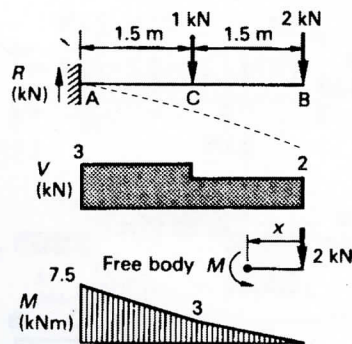


Figure 7.9a

Solution to beam a

Beam (b)

As before we must first find the unknown reactions at A and B before we can start to draw the shear force diagram.

$$\begin{aligned} (\Sigma M \text{ about A}) \quad & 4 \times 0.8 = R_B \times 2 \\ & \text{clockwise} \quad \text{anticlockwise} \\ \text{Hence } R_B &= 1.6 \text{ kN} \uparrow \end{aligned}$$

We could find R_A in a similar way, by taking moments about B, but there is an easier way. Simply use vertical equilibrium.

$$\begin{aligned} (\Sigma V = 0) \quad & 4 = R_A + 1.6 \\ & \text{downwards} \quad \text{upwards} \\ \text{Hence } R_A &= 2.4 \text{ kN} \uparrow \end{aligned}$$

We are now in a position to draw the shear force diagram. Again start at the left-hand end and follow the direction of the force (figure 7.9b).

The peak bending moment will occur at the position of the point load, C, and from the free body, is given by:

$$\begin{aligned} M_C &= 1.2 \times R_B = 1.2 \times 1.6 \\ &= 1.92 \text{ kNm} \end{aligned}$$

Clearly we would get the same answer if, as an alternative, we had considered the equilibrium of the left-hand end:

$$\begin{aligned} M_C &= 0.8 \times R_A = 0.8 \times 2.4 \\ &= 1.92 \text{ kNm} \end{aligned}$$

This time the bottom of the beam is in tension.

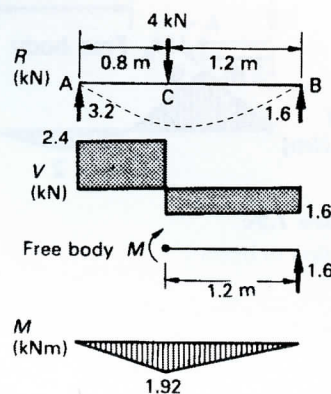


Figure 7.9b

Solution to beam b

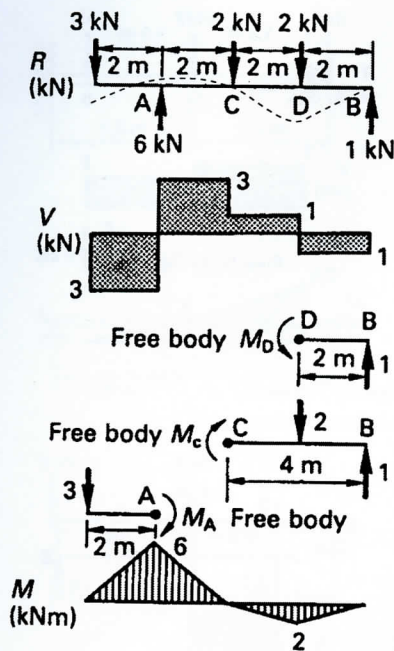


Figure 7.9c
Solution to beam c

Beam (c)

First find the reactions.

$$(\Sigma M \text{ about A}) \quad (2 \times 2) + (2 \times 4) = (3 \times 2) + (R_B \times 6)$$

clockwise anticlockwise

$$R_B = 1 \text{ kN } \uparrow$$

$$(\Sigma V = 0) \quad 3 + 2 + 2 = R_A + 1$$

downwards upwards

$$R_A = 6 \text{ kN } \uparrow$$

Comment – Examination of the above calculations shows that if the 3 kN load on the cantilever was doubled, the reaction at B would reduce to zero as the moments became balanced about A. A further increase in the load would result in R_B becoming negative, which implies that the beam must be held down at B in order to maintain equilibrium. Thus when calculating reactions, if a negative sign results, it simply means that the direction of the reaction was originally guessed wrongly.

Hence the shear force diagram can be drawn (figure 7.9c).

We can now determine the bending moment at the significant points by breaking the beam, and considering the equilibrium of the resulting free body by summing the moments to one side of the break. We choose either the left-hand or right-hand side depending upon which is the simpler calculation.

$$M_D = -1 \times 2 = -2$$

$$M_C = (-1 \times 4) + (2 \times 2) = 0$$

$$M_A = -3 \times 2 = -6$$

If these values are plotted and joined up by straight lines we have the bending moment diagram.

Comment – In the preceding calculations clockwise moments have been considered positive and anticlockwise moments negative. However, inspection of the bending moment diagram reveals no apparent connection between these signs and the side of the beam on which the bending moment is plotted. This is because we changed from considering the right-hand end to considering the left-hand end. Remember that our convention is always to plot the diagram on the tension side.

Beam (d)

This is a cantilever, since the load is a UDL.

Reaction, R_B

The gradually applied load represents a downward vertical drop. The last method shows the shear force diagram.

Because of the UDL,

In the above term for the UDL (kN/m) by the length of the beam from A to the centre of the UDL, giving bending moment

Comment – It is an easy way of calculating the bending moment at any point, and determining the reaction into equilibrium.

Beam (e)

Firstly evaluate the reactions.

$$(\Sigma M \text{ about A}) \quad (3 \times 0.5)$$

$$(\Sigma V = 0)$$

The shear force diagram shows the zero shear point from the reaction at A and make sure that you are aware of the fact that the diagram is the same as we have already shown. The values of bending moment at the zero shear point from

Beam (d)

This is a cantilever, similar to beam (a), but with the added complication of a UDL.

$$\text{Reaction, } R_A = 3 + (2 \times 2) = 7 \text{ kN down}$$

The gradually applied load (at the rate of 2 kN/m), i.e. over 2 m, represents a downward force of 4 kN. We then have a 3 kN vertical drop. The last metre is unloaded and hence remains at zero on the shear force diagram (figure 7.9d).

Because of the UDL, the bending moment will be curved.

$$\begin{aligned} M_A &= (3 \times 2) + (2 \times 2 \times 1) \\ &= 10 \text{ kNm} \end{aligned}$$

In the above term for the UDL we multiply the magnitude (2 kN/m) by the length over which it acts (2 m) by the distance from A to the centre of the UDL load (1 m). Figure 7.9d shows the resulting bending moment diagram.

Comment – It is an easy matter to plot the actual shape of the curve by evaluating the bending moment at several intermediate points. Simply cut the structure at any point, and determine the value of M to put the resulting free body into equilibrium.

Beam (e)

Firstly evaluate the reactions:

(ΣM about A)

$$\begin{aligned} (3 \times 0.5) + (8 \times 1.5 \times 1.25) &= R_B \times 1.5 \\ R_B &= 11 \text{ kN } \uparrow \end{aligned}$$

($\Sigma V = 0$)

$$\begin{aligned} 3 + (8 \times 1.5) &= R_A + 11 \\ R_A &= 4 \text{ kN } \uparrow \end{aligned}$$

The shear force diagram is shown in figure 7.9e. Look at it carefully and make sure that you understand the shape. A significant point is the fact that the diagram crosses the zero axis at two points, and we have already shown how this indicates maximum or minimum values of bending moment. We can calculate the distance of the zero shear point from B as follows:

$$\text{Distance, } x = 7/8 = 0.875 \text{ m}$$

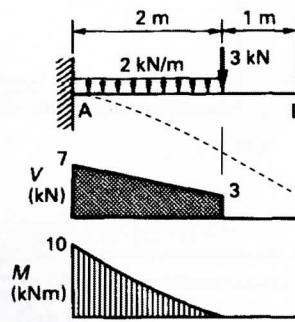


Figure 7.9d

Solution to beam d

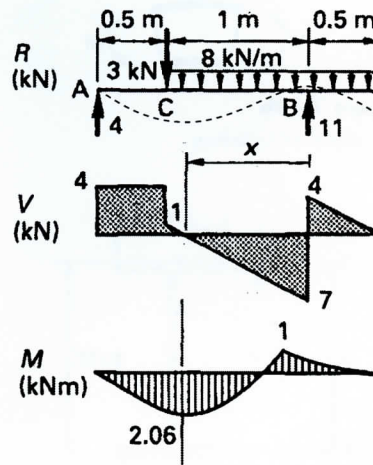


Figure 7.9e

Solution to beam e

The above is simply the value of the shear force at B divided by the slope of the line (which is equal to the UDL).

For the bending moment, firstly calculate M at B by considering the cantilever end to the right:

$$\begin{aligned} M_B &= 8 \times 0.5 \times 0.25 \\ &= 1 \text{ kNm} \end{aligned}$$

Now evaluate the moment at C by considering the left-hand end:

$$M_C = 4 \times 0.5 = 2 \text{ kNm}$$

Lastly evaluate the maximum value of moment which occurs 0.875 m from B:

$$\begin{aligned} M_{\max} &= (8 \times 1.375 \times 1.375/2) - (11 \times 0.875) \\ &= -2.06 \text{ kNm} \end{aligned}$$

The final bending moment diagram is shown in figure 7.9e.

Bending moment diagrams can become quite complicated, however, you should not lose sight of what they mean in physical terms. Quite simply, the shear force and bending moment at any point in a structure are those values necessary to satisfy equilibrium if a free body is considered to one side of the point.

Beams therefore have to resist shear forces which try to deform them as shown in figure 7.10a, and bending moments which try to deform them as shown in figure 7.10b. We shall consider bending moments first, and the next stage is to understand the relationship between the bending moment, M , on a beam and the bending stresses, σ , that this produces in the beam.

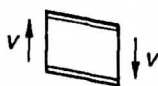


Figure 7.10a
The effect of shear



Figure 7.10b
The effect of bending

7.4 Bending stresses

7.4.1 Approximate bending strength (ignoring the web)

An 'I' beam resists bending by one flange 'pushing' and the other flange 'pulling'. The web also contributes to the bending strength, but it is less effective than the flanges. A crude, but safe, approximation to 'I' beam behaviour is to ignore the contribution of the web to bending strength and hence assume that it is the flanges that do all the work. In practice this would not be an economic

method of design but it is a good starting point for work. In effect, as shown in figure 7.10b, the web functions in the way that it keeps the flanges apart.

The moment, M , at any point is the sum of the forces F_T in tension and F_C in compression times the distance between the centres of these forces. As in the method of sections,

(ΣM about A)

but

therefore
in the above

so for a beam with eq

where

and if the beam is made of a material with a yield stress of σ_y then the ultimate bending moment is

It can clearly be seen that the beam depth and flange width are used where the area moment of inertia is as possible.

EXAMPLE 7.2

A $254 \times 146 \times 31$ kg/m I beam is fixed as a cantilever. What is the maximum bending moment at the end of the cantilever if the allowable stress is 150 N/mm^2 ? Use an approximation that the web contributes nothing to the bending strength.

Solution

The first step is to determine the cross-section (figure 7.11a).

method of design but it gives a useful insight into the way beams work. In effect, as shown in *figure 7.11a* we are assuming that the web functions in the way that diagonal bracing does in a truss. It keeps the flanges apart and carries the shear.

The moment, M , at section $y-y$, produces the two flange forces, F_T in tension and F_C in compression as shown in *figure 7.11b*. The distance between the centres of the top and bottom flanges is D . As in the method of sections for pinned trusses:

$$(\Sigma M \text{ about } A) \quad P \times x = F_T \times D$$

$$\text{but} \quad Px = \text{bending moment} = M$$

$$\text{therefore} \quad M = F_T \times D$$

in the above force, $F_T = \text{stress} \times \text{area}$

so for a beam with equal flanges of area, A ,

$$M = \sigma_t \times A \times D$$

where σ_t = the tensile stress in the flange

and if the beam is made from material with an ultimate yield stress of σ_y then the ultimate moment that the beam can support is:

$$M_{\text{ult}} = \sigma_y \times D \times A$$

It can clearly be seen that the moment capacity increases with beam depth and flange area. This explains why 'I' shaped beams are used where the aim is to concentrate the material as far apart as possible.

EXAMPLE 7.2

A $254 \times 146 \times 31$ kg/m universal beam is used as a 3 m long cantilever. What is the maximum ultimate load that can be applied at the end of the cantilever if the steel yields at a stress of 275 N/mm^2 ? Use an approximate method ignoring the contribution of the web, and ignore the self-weight of the beam.

Solution

The first step is to extract the relevant dimensions of the beam cross-section (*figure 7.12a*) from the table given in the *Appendix*.

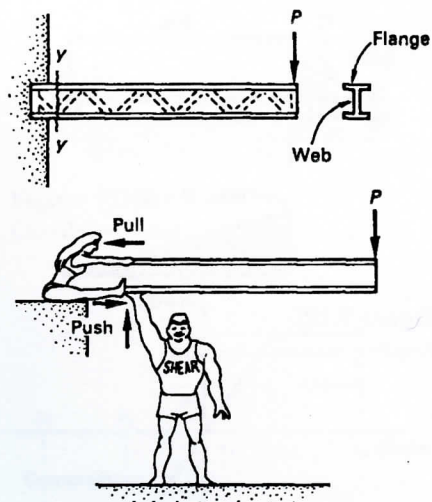


Figure 7.11a

One flange pushes and the other pulls

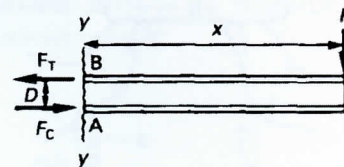


Figure 7.11b

Take moments about the point A

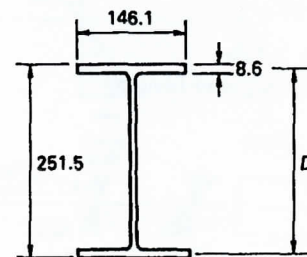


Figure 7.12a

Beam dimensions

Let us now switch to an 'I' beam cantilever as shown in figure 7.14a, which is assumed to be made from a steel with an elastic perfectly plastic stress/strain relationship as shown in figure 7.14b. The aim is to understand what happens to the stresses on the cross-section of the beam at the point of maximum bending moment, $y-y$, as the load, P , is gradually increased until ultimate failure occurs.

Initially the load, P_1 , is modest so that it only produces strains ϵ_1 well below the yield strain ϵ_y . Consequently the accompanying stresses are also well below yield, as shown in figure 7.14c. It is interesting to compare the effectiveness of the steel at different levels in contributing to the bending resistance of the beam. Level a is close to the neutral axis and level b is within the flange of the beam. Remember, the aim is to provide an anticlockwise moment throughout the cross-section in order to cancel the clockwise moment induced by the load P acting at distance L . Compared to level a the steel at level b is more effective because:

1. It is subject to higher strains and hence stresses.
2. The extra width of the flange means there is a bigger area of steel available for 'pushing' and 'pulling' (force = stress $\times$ area).
3. The moment resisted is obviously proportional to the distance between the forces. Hence the further the steel is from the neutral axis, the more effectively it works.

We shall now increment the load to P_2 , which is the load which just causes the strains in the extreme top and bottom fibres of the beam to reach the yield stress – σ_y (figure 7.14d). This is the highest load at which the deflection of the beam is fully recoverable. If the load is removed the beam will spring back to its original position. The beam is now at the **limit of elastic bending**.

A further small increment of load to P_3 takes the steel near the top and bottom surfaces beyond its yield point. Figure 7.14e shows how, beyond the yield point, as the strain continues to increase the stress remains constant at its yield value. However, closer to the centre of the beam, where the strain is less than σ_y , the steel remains elastic. The beam is not yet at its ultimate failure state, although some permanent deflection has occurred which is not recoverable when the load is removed. This is **elastic-partially plastic bending**.

If the load is incremented further to P_4 the first thing that becomes obvious is the way that relatively small additions to the

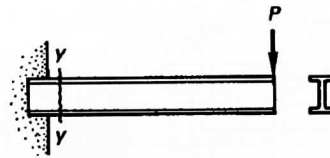


Figure 7.14a
Cantilever beam

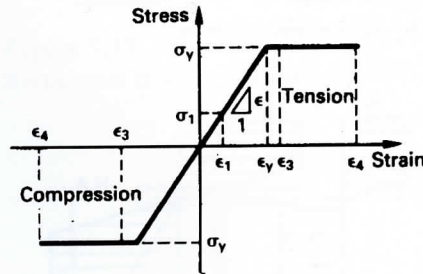


Figure 7.14b
Assumed stress/strain relationship for cantilever material

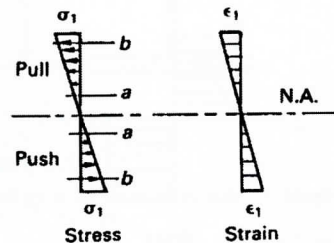


Figure 7.14c
Elastic stresses and strains

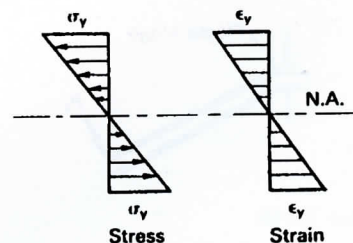
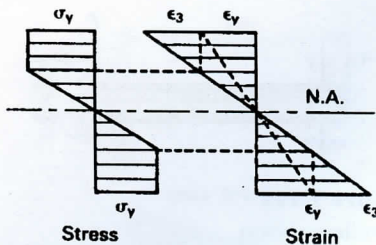
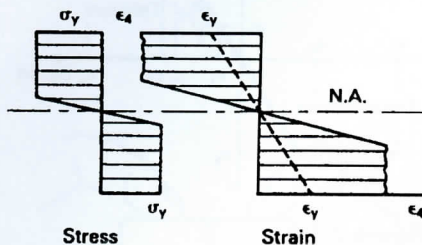


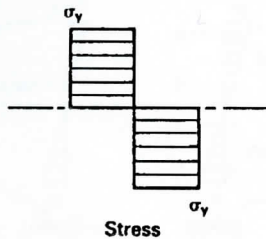
Figure 7.14d
Stresses reach the yield point

**Figure 7.14e**

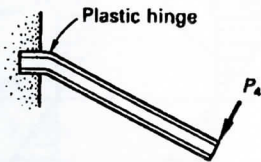
The yield strain exceeded

**Figure 7.14f**

Most of the steel is at the yield stress

**Figure 7.14g**

Full yield

**Figure 7.14h**

A plastic hinge has formed

load produce large deflections. Because of the ductile nature of the mild steel it can be strained to say fifteen times its yield strain, as shown in *figure 7.14f*, ensuring that virtually all the steel has yielded. As already explained, the material close to the neutral axis makes a very small contribution to bending resistance, so for all practical purposes the small remaining elastic zone on the neutral axis can be ignored. A stress distribution as shown in *figure 7.14g* can thus be assumed.

Once **full plastic bending** has been reached the beam would rotate at constant load (in theory) about the point of maximum bending moment which has become a **plastic hinge** (*figure 7.14h*). Obviously ultimate collapse has been reached. For standard 'I' beams the load to cause ultimate failure, P_4 , is about 15% more than the load to cause first yield, P_2 .

7.5 Plastic theory of bending

In limit state design it is of prime importance to ensure that a structure has an adequate safety factor against reaching the ultimate limit state or collapse. As plastic bending is taking place at collapse we must derive a numerical relationship between the collapse moment, the yield stress of the material and the cross-sectional dimensions of the beam.

Initially we shall assume that all beams are provided with lateral restraint. This means that they will not buckle sideways before the full yield stress of the material is reached. Beams without lateral restraint are considered later in *section 7.8*.

7.5.1 Rectangular cross-section in plastic bending

Consider a portion of rectangular beam subject to its ultimate bending moment, M_{ult} , so that it is on the point of collapse. By considering the horizontal equilibrium of the beam free body in *figure 7.15* we can see that tensile force = compressive force and hence that the neutral axis must lie at the mid-height of the beam.

$$\begin{aligned} \text{force} &= \text{stress} \times \text{area} \\ \text{'pull' force} &= \text{'push' force} = \sigma_y \times B \times D/2 \end{aligned}$$

The distance between the centroids of the two forces or **lever arm** = $D/2$

therefore

or

where W_{pl} is the **plastic section modulus** for a rectangular section. This means that it depends on the shape of the beam, and is independent of the material. However, because it is a measure of the bending strength of the beam, it is often expressed in terms of plastic section modulus.

7.5.2 'I' beams in plastic bending

Again, for horizontal bending, the horizontal force and the neutral axis must be in the same line. The plastic section modulus is defined as the sum of the first moments of area about the neutral axis and take moments about the neutral axis to get:

$$M_{ult} = \sigma_y \sum A_i y_i$$

where A_i etc. is the area of the i th element and y_i is the distance from the neutral axis.

However, if the flange

$$M_{ult} = \sigma_y \sum A_i y_i$$

=

In this case the plastic section modulus is the **moment of area** of the cross-section about the neutral axis. Values of W_{pl} are given in the *Appendix*. Therefore, to find the ultimate moment we

therefore

$$M_{\text{ult}} = \sigma_y \times BD^2/4$$

or

$$M_{\text{ult}} = \sigma_y \times W_{\text{pl}} \quad [7.2]$$

where W_{pl} is the **plastic section modulus**, and equals $BD^2/4$ for a rectangular section. The plastic section modulus can be seen to be a **geometric** property of the cross-section of the beam. This means that it depends only upon the size and shape of the beam, and is independent of the beam material. It is important, however, because it is a direct measure of the ultimate plastic bending strength of a beam. We shall now look at the way the plastic section modulus, W_{pl} , varies for differently shaped cross-sections.

7.5.2 'I' beams in plastic bending

Again, for horizontal equilibrium, the total tensile force = total horizontal force and so for a standard 'I' beam with equal flanges the neutral axis must lie at the mid-height of the beam. If we divide the cross-section into four rectangles as shown in figure 7.16 and take moments about the neutral axis for each rectangle we get:

$$M_{\text{ult}} = \sigma_y \times ((A_1 \times Y_1) + (A_2 \times Y_2) + \dots \text{etc.})$$

where A_1 etc. is the area of each rectangle, and Y_1 etc. is the distance from the neutral axis to the centroid of each rectangle.

However, if the flanges are equal, this simplifies to:

$$\begin{aligned} M_{\text{ult}} &= \sigma_y \times 2 \times ((A_1 \times Y_1) + (A_2 \times Y_2)) \\ &= \sigma_y \times \Sigma AY = \sigma_y \times W_{\text{pl}} \end{aligned}$$

In this case the plastic section modulus, W_{pl} , is defined as the **first moment of area** of the section **about the neutral axis**. Values of W_{pl} are given in section tables for standard sections (see the *Appendix*). Therefore to obtain a suitable beam size for a given ultimate moment we simply use:

$$\text{required } W_{\text{pl}} = \frac{M_{\text{ult}}}{\sigma_y} \quad [7.3]$$

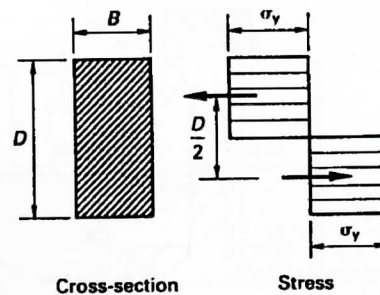


Figure 7.15
Rectangular beam at full yield

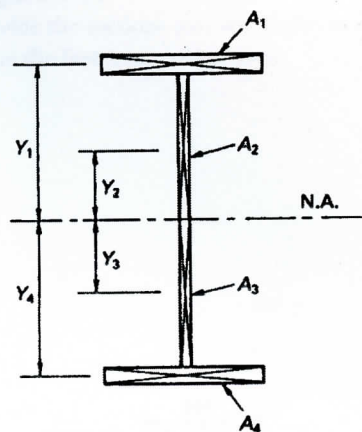


Figure 7.16
Dividing a cross-section into rectangles

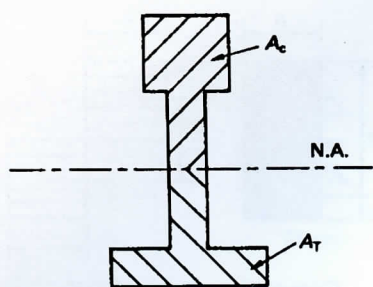


Figure 7.17
A non-standard shape

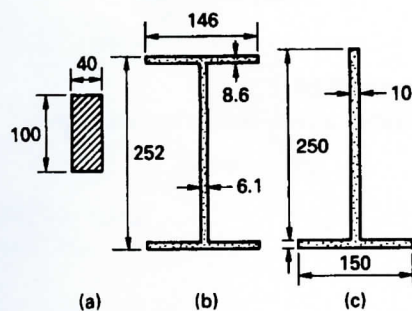
and select a beam from the section tables with an adequate plastic section modulus, W_{pl} . Always aim for a section whose value is equal to, or slightly greater than, the required value. Normally steel is paid for by weight and so the lightest section that will do the job is usually the most economic choice.

7.5.3 General cross-sections in plastic bending

It is now an easy matter to extend these concepts to sections of any cross-sectional shape (figure 7.17). To find the ultimate moment that a beam will support it is simply a case of evaluating the plastic section modulus, W_{pl} , and using equation [7.2]. For horizontal equilibrium the neutral axis must divide the section into two equal areas:

$$A_C = A_T$$

We then proceed to sum the first moments of area about the neutral axis, as before, in order to obtain the plastic section modulus, W_{pl} .



All dimensions in mm

Figure 7.18
Beam cross-sections

EXAMPLE 7.3

The three beam cross-sections shown in figure 7.18a-c all contain the same amount of steel. Find the plastic section modulus, W_{pl} , for each one.

Solution

Comment – The standard section tables (see the Appendix) give the values of W_{pl} in cm^3 . Although these are non-standard SI units, they produce more manageable numbers, and it is proposed that we also work in centimetres.

Beam (a)

From section 7.5.1 $W_{pl} = BD^2/4 = 4 \times 10^2/4$

Answer $W_{pl} = 100 \text{ cm}^3$

Beam (b)

From section 7.5.2 – refer to figure 7.19a.

All dimensions in cm.

Item	Area,
1	14.6×0.86
2	11.74×0.6

Answer

*Comment – The above b...
sal beam, and consequen...
table given in the Appen...
ence is due to rounding...
section.*

Beam (c) – refer

The first step is to ca...
must divide the area...
can be completed.

Total ar

Depth,

All dimensions in c

Item

- 1
- 2
- 3

Answer

*Comment – The abo...
biggest plastic section...
bending.*

All dimensions in cm.

Item	Area, A	y	Ay
1	$14.6 \times 0.86 = 12.56$	12.17	152.9
2	$11.74 \times 0.61 = 7.16$	5.87	42.0
			194.9

Answer

$$W_{pl} = 2 \times 194.9$$

$$W_{pl} = 389.8 \text{ cm}^3$$

Comment – The above beam is a standard $254 \times 146 \times 31 \text{ kg/m}$ universal beam, and consequently we can check this result from the standard section table given in the Appendix. The value given is 396 cm^3 . The small difference is due to rounding error and the presence of fillet radii on the actual section.

Beam (c) – refer to figure 7.19b

The first step is to calculate the depth of the neutral axis, Y , which must divide the area into two equal halves. After this step the table can be completed.

$$\text{Total area} = (1 \times 25) + (1 \times 15) = 40 \text{ cm}^2$$

$$\text{Depth, } Y = \frac{40}{2 \times 1} = 20 \text{ cm}$$

All dimensions in cm.

Item	Area, A	y	Ay
1	$1 \times 20 = 20$	10	200
2	$1 \times 5 = 5$	2.5	12.5
3	$1 \times 15 = 15$	5.5	82.5
			295.0

Answer

$$W_{pl} = 295 \text{ cm}^3$$

Comment – The above results show that, as expected, the 'I' beam has the biggest plastic section modulus, and hence is the most efficient for resisting bending.

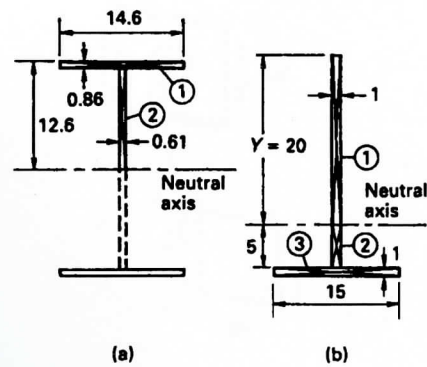


Figure 7.19

Divide the sections into rectangles to evaluate the first moments of area