

Formulae Sheet

Vector Operations

Vector Length (Magnitude)

The length of a vector **a** is written using the notation $|\mathbf{a}|$. The formula for the length of a two dimensional vector is $|\mathbf{a}| = \sqrt{x^2 + y^2}$ and in three dimensions it is $|\mathbf{a}| = \sqrt{x^2 + y^2 + z^2}$.

A vector can be normalised (its length made equal to one) by dividing it by its length. The direction remains the same, of course. The notation is $\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$. For example, if vector **a** is (0,6,8) it's length is 10 and the normalised vector, $\hat{a} = (0/10, 6/10, 8/10)$.

Dot product

Another common vector operation is the scalar product, usually called the dot product and written $\mathbf{a} \bullet \mathbf{b}$. It is defined thus:

If $\mathbf{a} = (x_1, y_1, z_1)$ and $\mathbf{b} = (x_2, y_2, z_2)$ then $\mathbf{a} \bullet \mathbf{b} = (x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 \cdot z_2)$.

It is also equivalent to

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\phi$ where ϕ is the angle between the vectors.

Cross product

The cross product of two vectors is written

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

A handy way of remembering the cross product is to write it as a matrix and to see it as a determinant, that is:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where **i**, **j** and **k** are the three orthogonal unit basis vectors in the x, y and z directions.

The result of a cross product is a vector.

The length of the vector cross product can be determined by:

$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin\phi$ where ϕ is the angle between **a** and **b**.

Matrix Operations

The determinant of a matrix A is given by $|A| = x(a_2b_3 - a_3b_2) - y(a_1b_3 - a_3b_1) + z(a_1b_2 - a_2b_1)$,

Where the matrix A =
$$\begin{pmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Multiplication of a (4x4) matrix by another (4x4) is:-

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

$$\begin{pmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42}) & (a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} + a_{14}b_{43}) & (a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42}) & (a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} + a_{24}b_{43}) & (a_{21}b_{14} + a_{22}b_{24} + a_{23}b_{34} + a_{24}b_{44}) \\ (a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41}) & (a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42}) & (a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} + a_{34}b_{43}) & (a_{31}b_{14} + a_{32}b_{24} + a_{33}b_{34} + a_{34}b_{44}) \\ (a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41}) & (a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} + a_{44}b_{42}) & (a_{41}b_{13} + a_{42}b_{23} + a_{43}b_{33} + a_{44}b_{43}) & (a_{41}b_{14} + a_{42}b_{24} + a_{43}b_{34} + a_{44}b_{44}) \end{pmatrix}$$

Multiplication of a (4x4) matrix by a (4x1) matrix is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41}) \\ (a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41}) \\ (a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41}) \end{bmatrix}$$

Transformations:

(Assuming a right handed system and points represented by a column vector)

Rotation about the x-axis:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation:

$$\begin{bmatrix} 1 & 0 & 0 & Tx \\ 0 & 1 & 0 & Ty \\ 0 & 0 & 1 & Tz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about the y-axis:

$$\begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Scaling:

$$\begin{bmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about the z-axis:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lighting Calculations:

Ambient:

$$I_{amb} = K_a I_a$$

.

Diffuse:

$$I_{diff} = K_d I_p \cos(\theta)$$

.

Specular:

$$I_{spec} = K_s I_p \cos^n(\phi)$$

Plane equation:

The equation of a plane is:

$$a_x * n_x + a_y * n_y + a_z * n_z + d = 0$$

or equivalently:

$$\mathbf{a} \cdot \mathbf{n} + d = 0$$

where $\mathbf{a} \cdot \mathbf{n}$ is the dot product of $\mathbf{a}$ with $\mathbf{n}$ and $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{n} = (n_x, n_y, n_z)$.

Here $\mathbf{a}$ is any point on the plane and $\mathbf{n}$ is the normal to the plane, and $d = -\mathbf{x} \cdot \mathbf{n}$ is the dot product of $\mathbf{x}$ and $\mathbf{n}$ where $\mathbf{x}$ is any other point on the plane and $\mathbf{n}$ is the normal to the plane.

If $\mathbf{n}$ is normalized and $\mathbf{a}$ is not on the plane then $\mathbf{a} \cdot \mathbf{n} + d$ gives the shortest distance of the point $\mathbf{a}$ from the plane.