

$(uL/AE) \times u$ (mm)	Member forces
+ 0.065	+ 25.38
+ 0.131	+ 10.87
+ 0.065	- 9.99
$\Sigma = + 0.261$	

kN (Tie)
and are given in the last

4. Beams

4.1 Statically Determinate Beams

Two parameters which are fundamentally important to the design of beams are *shear force* and *bending moment*. These quantities are the result of internal forces acting on the material of a beam in response to an externally applied load system.

4.1.1 Example 4.1: Beam with Point Loads

Consider a simply supported beam as shown in Figure 4.1 carrying a series of secondary beams each imposing a point load of 4 kN.

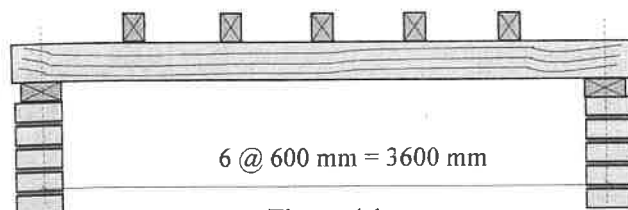


Figure 4.1

This structure can be represented as a line diagram as shown in Figure 4.2:

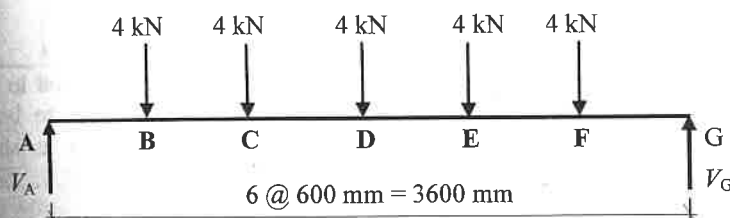


Figure 4.2

Since the externally applied force system is in equilibrium, the three equations of static equilibrium must be satisfied, i.e.

+ve $\uparrow \Sigma F_y = 0$ The sum of the vertical forces must equal zero.

+ve $\curvearrowright \Sigma M = 0$ The sum of the moments of all forces about *any* point on the plane of the forces must equal zero.

+ve $\rightarrow \Sigma F_x = 0$ The sum of the horizontal forces must equal zero.

The assumed positive directions are as indicated. In this particular problem there are no externally applied horizontal forces and consequently the third equation is not required. (Note: It is still necessary to provide horizontal restraint to a structure since it can be subject to a variety of load cases, some of which may have a horizontal component.)

Consider the vertical equilibrium of the beam:

$$+ve \uparrow \Sigma F_y = 0$$

$$+ V_A - (5 \times 4.0) + V_G = 0$$

$$\therefore V_A + V_G = 20 \text{ kN} \quad \text{Equation (1)}$$

Consider the rotational equilibrium of the beam:

$$+ve \curvearrowright \Sigma M_A = 0$$

Note: The sum of the moments is taken about one end of the beam (end A) for convenience. Since one of the forces (V_A) passes through this point it does not produce a moment about A and hence does not appear in the equation. It should be recognised that the sum of the moments could have been considered about *any* known point in the same plane.

$$+ (4.0 \times 0.6) + (4.0 \times 1.2) + (4.0 \times 1.8) + (4.0 \times 2.4) + (4.0 \times 3.0) - (V_G \times 3.6) = 0$$

$$\therefore V_G = 10 \text{ kN} \quad \text{Equation (2)}$$

Substituting into Equation (1) gives

$$\therefore V_A = 10 \text{ kN}$$

This calculation was carried out considering only the externally applied forces, i.e.

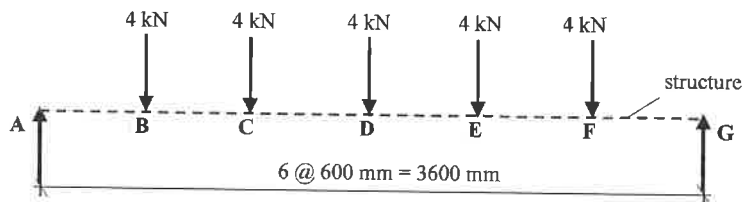


Figure 4.3

The structure itself was ignored, however the applied loads are transferred to the end supports through the material fibres of the beam. Consider the beam to be cut at section X-X producing two sections each of which is in equilibrium as shown in Figure 4.4.

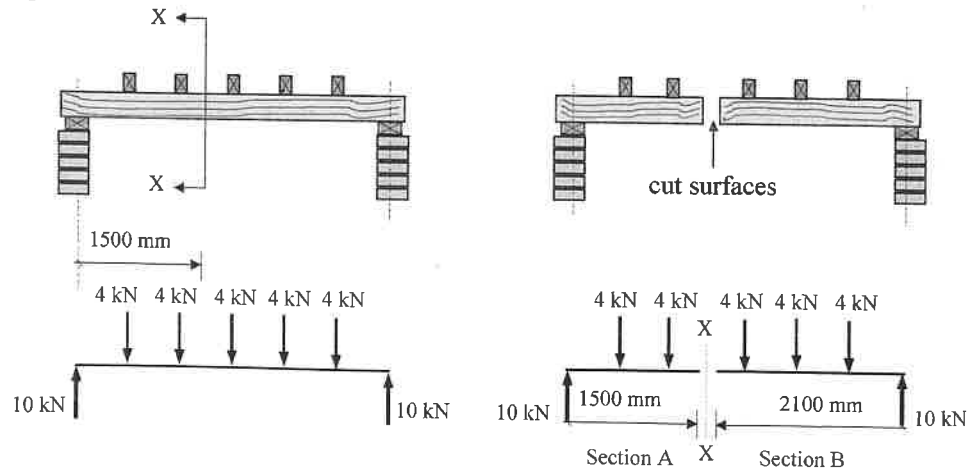


Figure 4.4

Clearly if the two sections are in equilibrium there must be internal forces acting on the cut surfaces to maintain this; these forces are known as the **shear force** and the **bending moment**, and are illustrated in Figure 4.5

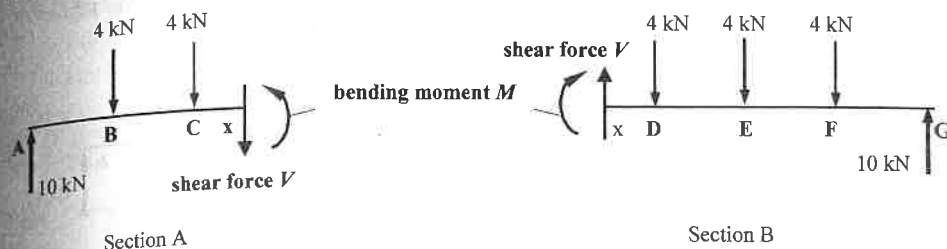


Figure 4.5

The force V and moment M are equal and opposite on each surface. The magnitude and direction of V and M can be determined by considering two equations of static equilibrium for either of the cut sections; both will give the same answer.

Consider the left-hand section with the 'assumed' directions of the internal forces V and M as shown in Figure 4.6.

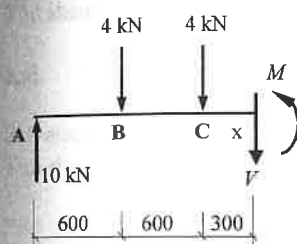


Figure 4.6

$$+\text{ve } \uparrow \Sigma F_y = 0$$

$$+ 10 - 4.0 - 4.0 - V = 0 \quad \therefore V = 2 \text{ kN}$$

$$+\text{ve } \curvearrowright \Sigma M_A = 0$$

$$+ (4.0 \times 0.6) + (4.0 \times 1.2) - (V \times 1.5) - M = 0$$

$$\therefore M = 10.2 \text{ kNm}$$

4.1.2 Shear Force Diagrams

In a statically determinate beam, the numerical value of the shear force can be obtained by evaluating the algebraic sum of the vertical forces to one side of the section being considered. The convention adopted in this text to indicate positive and negative shear forces is shown in Figure 4.7.

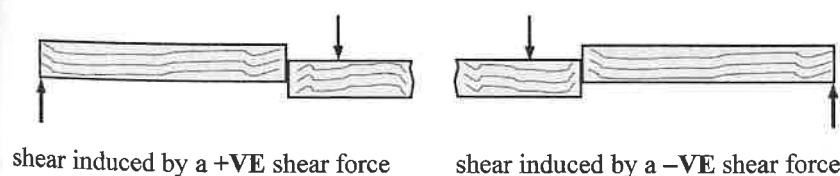
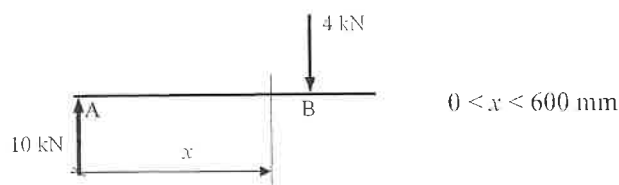


Figure 4.7

The calculation carried out to determine the shear force can be repeated at various locations along a beam and the values obtained plotted as a graph; this graph is known as the *shear force diagram*. The shear force diagram indicates the variation of the shear force along a structural member.

Consider any section of the beam between A and B:

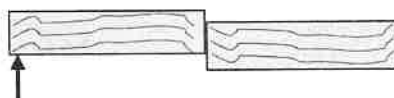


Note: The value immediately under the point load at the cut section is not being considered.

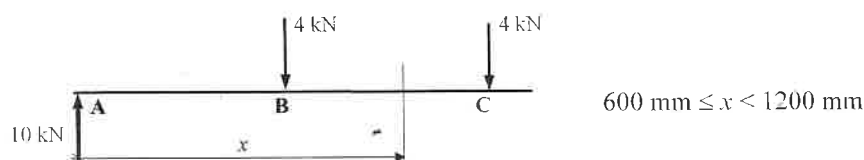
The shear force at any position $x = \Sigma$ vertical forces to one side
 $= +10.0 \text{ kN}$

This value is a constant for all values of x between zero and 600 mm, the graph will therefore be a horizontal line equal to 10.0 kN. This force produces a +ve shear effect, i.e.

+ve shear effect



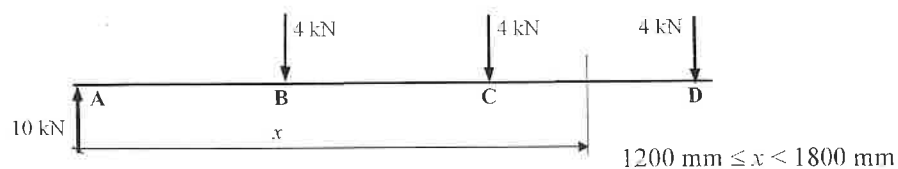
Consider any section of the beam between B and C:



The shear force at any position $x = \Sigma$ vertical force to one side
 $= +10.0 - 4.0 = 6.0 \text{ kN}$

This value is a constant for all values of x between 600 mm and 1200 mm, the graph will therefore be a horizontal line equal to 6.0 kN. This force produces a +ve effect shear effect.

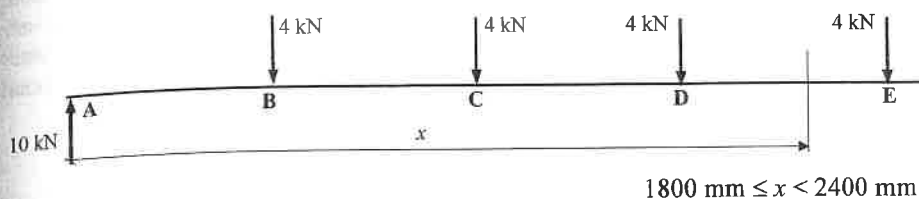
Similarly for any section between C and D:



be repeated at various
; this graph is known as
riation of the shear force

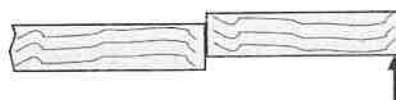
The shear force at any position $x = \Sigma$ vertical forces to one side
 $= +10.0 - 4.0 - 4.0 = 2.0 \text{ kN}$

Consider any section of the beam between D and E:



The shear force at any position $x = \Sigma$ vertical forces to one side
 $= +10.0 - 4.0 - 4.0 - 4.0 = -2.0 \text{ kN}$

In this case the shear force is negative:



Similarly between E and F $2400 \text{ mm} < x < 3000 \text{ mm}$
The shear force at any position $x = \Sigma$ vertical forces to one side
 $= +10.0 - 4.0 - 4.0 - 4.0 - 4.0 = -6.0 \text{ kN}$

and

between F and G $3000 \text{ mm} < x < 3600 \text{ mm}$
The shear force at any position $x = \Sigma$ vertical forces to one side
 $= +10.0 - 4.0 - 4.0 - 4.0 - 4.0 - 4.0 = -10.0 \text{ kN}$

In each of the cases above the value has not been considered at the point of application of the load.

Consider the location of the applied load at B shown in Figure 4.8.

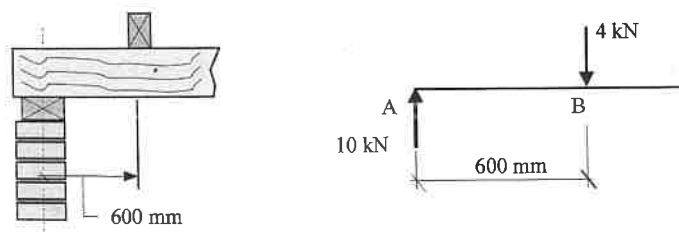


Figure 4.8

The 4.0 kN is not instantly transferred through the beam fibres at B but instead over the width of the actual secondary beam. The change in value of the shear force between $x < 600 \text{ mm}$ and $x > 600 \text{ mm}$ occurs over this width, as shown in Figure 4.9.

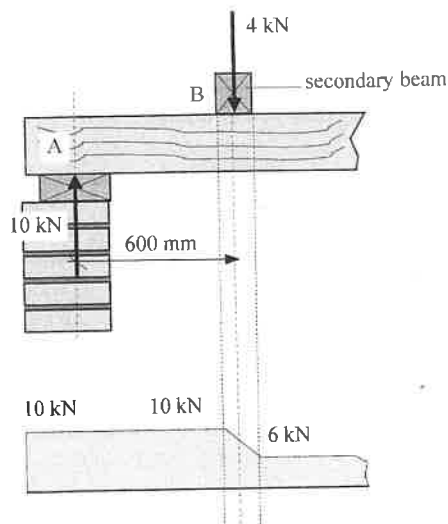


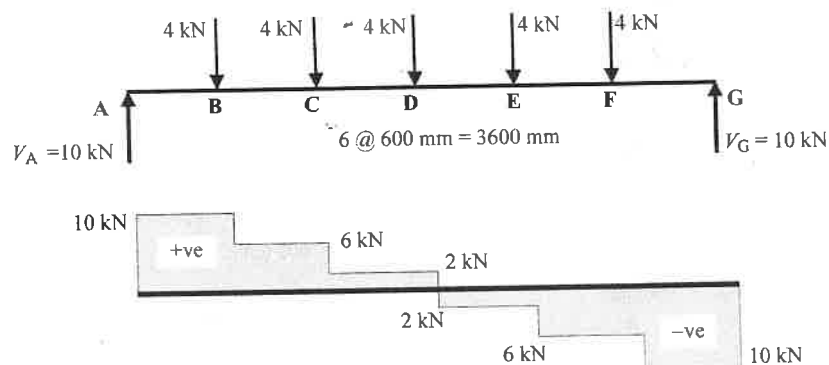
Figure 4.9

The width of the secondary beam is insignificant when compared with the overall span, and the shear force is assumed to change instantly at this point, producing a vertical line on the shear force diagram as shown in Figure 4.10.



Figure 4.10

The full shear force diagram can therefore be drawn as shown in Figure 4.11.



Shear Force Diagram

Figure 4.11

The same result can be obtained by considering sections from the right-hand side of the beam.

4.1.3 Bending

In a static system, the bending moment at any point is caused by the sum of the shear forces acting on either side of the point. The conventional sign convention for bending moment is shown in Figure 4.12.

Bending

Note: The sign of the bending moment is being considered. As with the location of the bending moment, the bending moment is considered.

In this case, the bending moment is considered.

4.1.3 Bending Moment Diagrams

In a statically determinate beam the numerical value of the bending moment (i.e. moments caused by forces which tend to bend the beam) can be obtained by evaluating the algebraic sum of the moments of the forces to one side of a section. In the same manner as with shear forces either the left-hand or the right-hand side of the beam can be considered. The convention adopted in this text to indicate positive and negative bending moments is shown in Figures 4.12(a) and (b).

Bending inducing **tension on the underside** of a beam is considered **positive**.

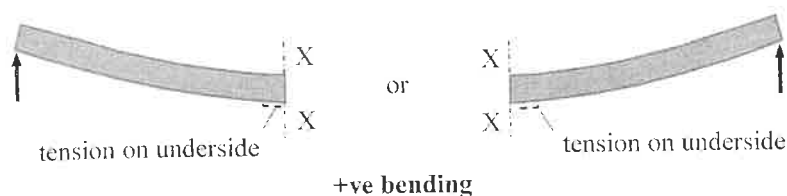


Figure 4.12 (a)

Bending inducing **tension on the top** of a beam is considered **negative**.

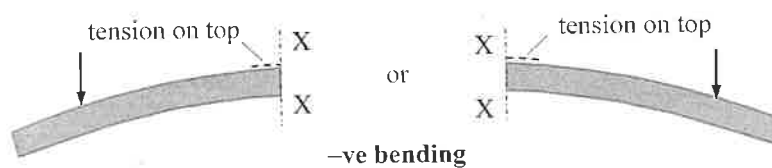
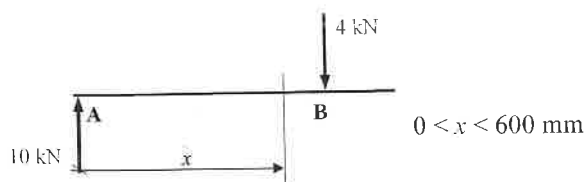


Figure 4.12 (b)

Note: Clockwise/anti-clockwise moments do **not** define +ve or -ve **bending** moments. The sign of the bending moment is governed by the location of the tension surface at the point being considered.

As with shear forces the calculation for bending moments can be carried out at various locations along a beam and the values plotted on a graph; this graph is known as the 'bending moment diagram'. The bending moment diagram indicates the variation in the bending moment along a structural member.

Consider sections between A and B of the beam as before:



In this case when $x = 600$ mm the 4.0 kN load passes through the section being considered and does not produce a bending moment, and can therefore be ignored.

Bending moment = Σ algebraic sum of the moments of the forces to one side of a section,
 $= \Sigma (\text{Force} \times \text{lever arm})$

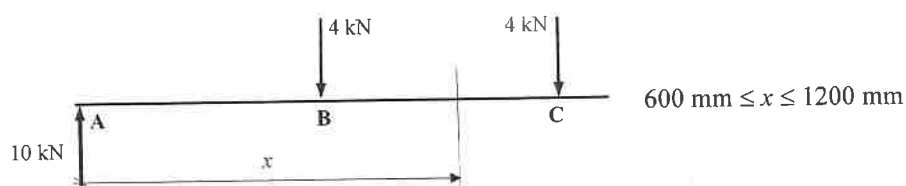
$$M_x = 10.0 \times x = 10.0 x \text{ kNm}$$

Unlike the shear force, this expression is not a constant and depends on the value of 'x' which varies between the limits given. This is a linear expression which should be reflected in the calculated values of the bending moment.

$$\begin{aligned} x = 0 & \quad M_x = 10.0 \times 0 = \text{zero} \\ x = 200 \text{ mm} & \quad M_x = 10.0 \times 0.2 = 2.0 \text{ kNm} \\ x = 400 \text{ mm} & \quad M_x = 10.0 \times 0.4 = 4.0 \text{ kNm} \\ x = 600 \text{ mm} & \quad M_x = 10.0 \times 0.6 = 6.0 \text{ kNm} \end{aligned}$$

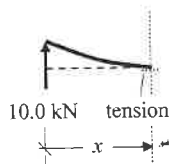
Clearly the bending moment increases linearly from zero at the simply supported end to a value of 6.0 kNm at point B.

Consider sections between B and C of the beam:

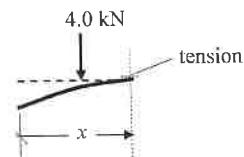


Bending moment = Σ algebraic sum of the moments of the forces to 'one' side of a section
 $M_x = + (10.0 \times x) - (4.0 \times [x - 0.6])$

bending effect of
10.0 kN load is +ve



bending effect of
4.0 kN load is -ve



$$\begin{aligned} x = 800 \text{ mm} & \quad M_x = + (10.0 \times 0.8) - (4.0 \times 0.2) = 7.2 \text{ kNm} \\ x = 1000 \text{ mm} & \quad M_x = + (10.0 \times 1.0) - (4.0 \times 0.4) = 8.4 \text{ kNm} \\ x = 1200 \text{ mm} & \quad M_x = + (10.0 \times 1.2) - (4.0 \times 0.6) = 9.6 \text{ kNm} \end{aligned}$$

As before the bending moment increases linearly, i.e. from 7.2 kNm at $x = 800$ mm to a value of 9.6 kNm at point C.

Since the variation is linear it is only necessary to evaluate the magnitude and sign of the bending moment at locations where the slope of the line changes, i.e. each of the point load locations.

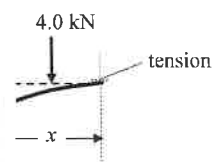
one side of a section.

is on the value of 'x'
ion which should be

ly supported end to a

$1 \leq x \leq 1200$ mm

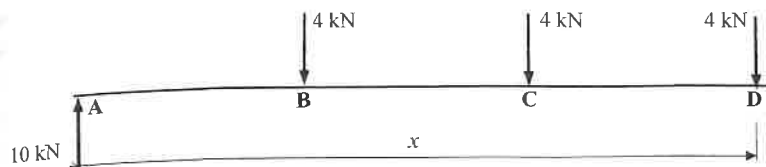
'one' side of a section



n at $x = 800$ mm to a

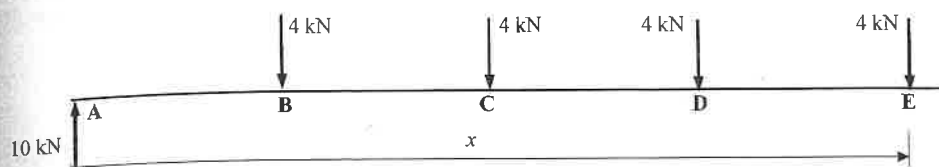
magnitude and sign of the
i.e. each of the point

Consider point D:



$$x = 1800 \text{ mm} \quad M_x = (10.0 \times 1.8) - (4.0 \times 1.2) - (4.0 \times 0.6) = 10.8 \text{ kNm}$$

Consider point E:

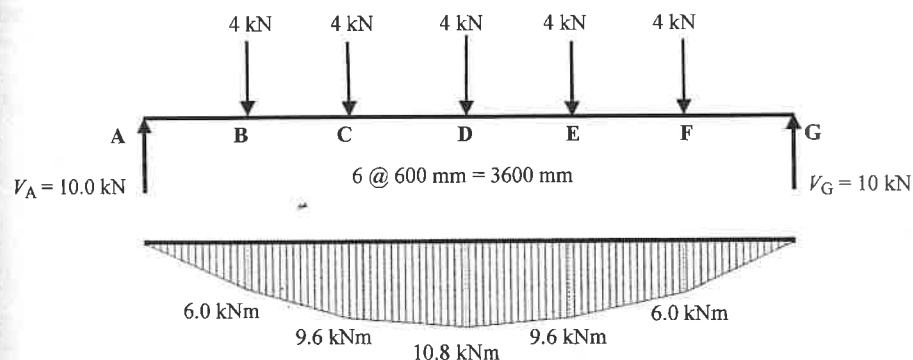


$$x = 2400 \text{ mm} \quad M_x = (10.0 \times 2.4) - (4.0 \times 1.8) - (4.0 \times 1.2) - (4.0 \times 0.6) = 9.6 \text{ kNm}$$

Similarly at point F:

$$x = 3000 \text{ mm} \quad M_x = (10.0 \times 3.0) - (4.0 \times 2.4) - (4.0 \times 1.8) - (4.0 \times 1.2) - (4.0 \times 0.6) = 6.0 \text{ kNm}$$

The full bending moment diagram can therefore be drawn as shown in Figure 4.13.

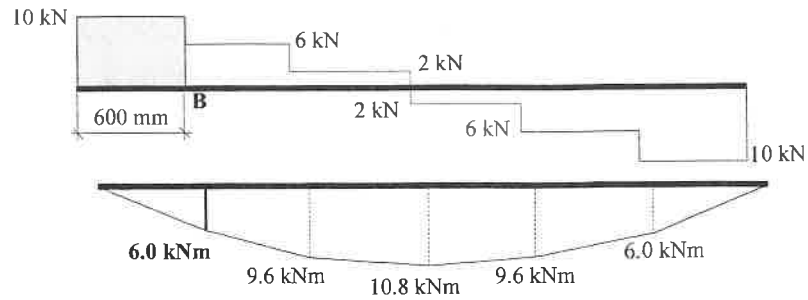


Bending Moment Diagram

Figure 4.13

The same result can be obtained by considering sections from the right-hand side of the beam. The value of the bending moment at any location can also be determined by evaluating the area under the shear force diagram.

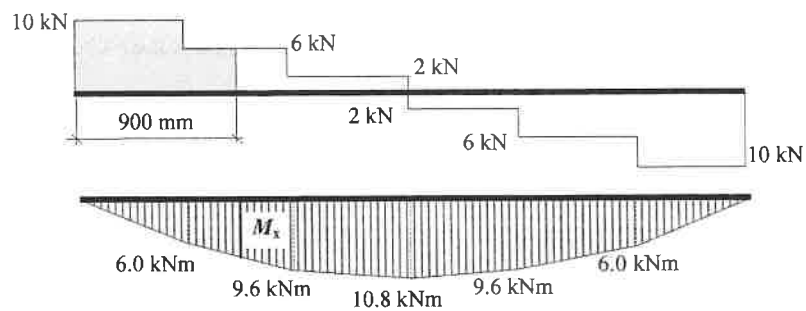
Consider point B:



Bending moment at B = shaded area on the shear force diagram

$$M_B = (10.0 \times 0.6) = 6.0 \text{ kNm as before}$$

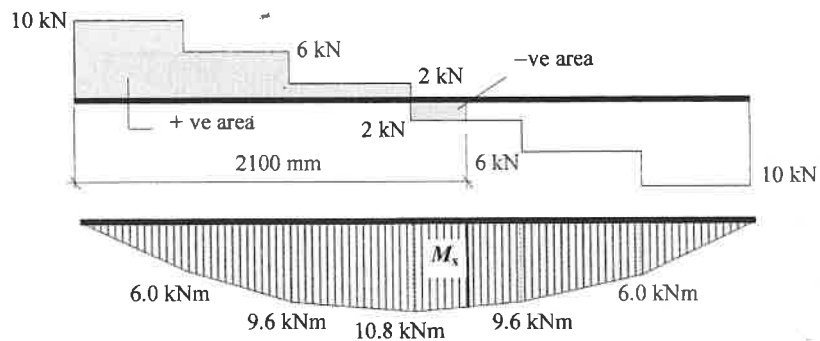
Consider a section at a distance of $x = 900 \text{ mm}$ along the beam between D and E:



Bending moment at x = shaded area on the shear force diagram

$$M_x = (10.0 \times 0.6) + (6.0 \times 0.3) = 7.8 \text{ kNm as before}$$

Consider a section at a distance of $x = 2100 \text{ mm}$ along the beam between D and E:



Bending moment at x = shaded area on the shear force diagram

$$M_x = (10.0 \times 0.6) + (6.0 \times 0.6) + (2.0 \times 0.6) - (2.0 \times 0.3) = 10.2 \text{ kNm}$$

(Note: A maximum bending moment occurs at the same position as a zero shear force.)

4.1.4 Example 4.2: Beam with a Uniformly Distributed Load (UDL)

Consider a simply-supported beam carrying a uniformly distributed load of 5 kN/m, as shown in Figure 4.14.

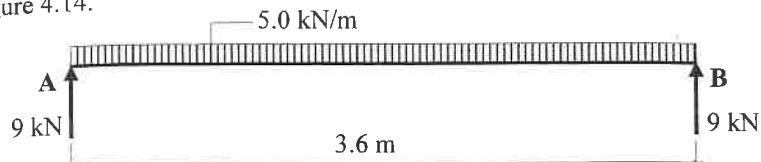
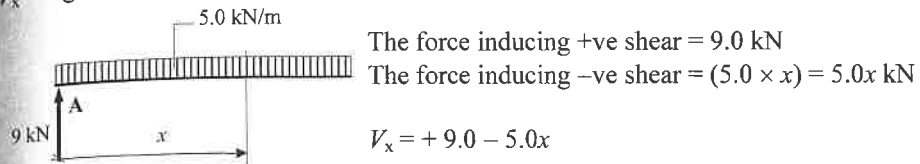


Figure 4.14

The shear force at any section a distance x from the support at A is given by:
 $V_x =$ algebraic sum of the vertical forces



This is a linear equation in which V_x decreases as x increases. The points of interest are at the supports where the maximum shear forces occur, and at the locations where the maximum bending moment occurs, i.e. the point of zero shear.

$$V_x = 0 \text{ when } +9.0 - 5.0x = 0 \quad \therefore x = 1.8 \text{ m}$$

Any intermediate value can be found by substituting the appropriate value of ' x ' in the equation for the shear force; e.g.

$$x = 600 \text{ mm} \quad V_x = +9.0 - (5.0 \times 0.6) = +6.0 \text{ kN}$$

$$x = 2100 \text{ mm} \quad V_x = +9.0 - (5.0 \times 2.1) = -1.5 \text{ kN}$$

The shear force can be drawn as shown in Figure 4.15.

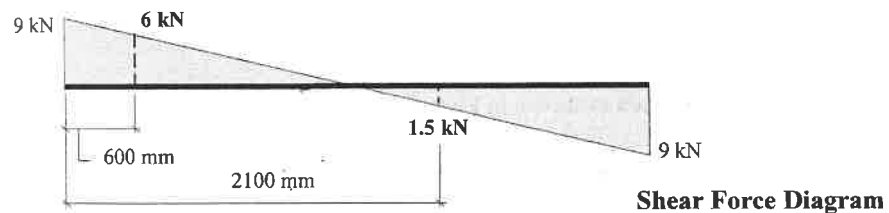
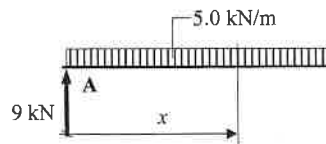


Figure 4.15

The bending moment can be determined as before, **either** using an equation or evaluating the area under the shear force diagram.

Using an equation:



$$\text{Bending moment at } x: \quad M_x = + (9.0 \times x) - [(5.0 \times x) \times (x/2)] = (9.0x - 2.5x^2)$$

tween D and E:

tween D and E:

s a zero shear force.)

In this case the equation is *not* linear, and the bending moment diagram will therefore be *curved*.

Consider several values:

$$x = 0 \quad M_x = \text{zero}$$

$$x = 600 \text{ mm} \quad M_x = + (9.0 \times 0.6) - (2.5 \times 0.6^2) = 4.5 \text{ kNm}$$

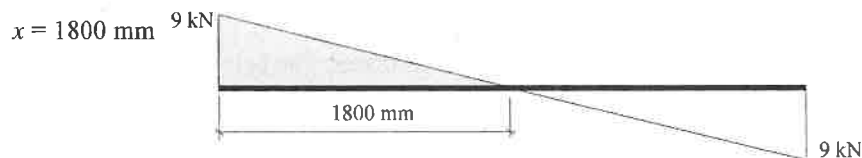
$$x = 1800 \text{ mm} \quad M_x = + (9.0 \times 1.8) - (2.5 \times 1.8^2) = 8.1 \text{ kNm}$$

$$x = 2100 \text{ mm} \quad M_x = + (9.0 \times 2.1) - (2.5 \times 2.1^2) = 7.88 \text{ kNm}$$

Using the shear force diagram:

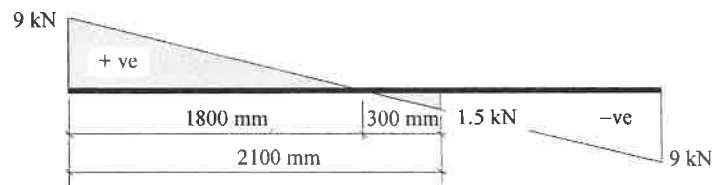


$$M_x = \text{shaded area} = + [0.5 \times (9.0 + 6.0) \times 0.6] = 4.5 \text{ kNm}$$



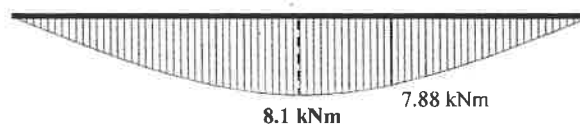
$$M_x = \text{shaded area} = + [0.5 \times 9.0 \times 1.8] = 8.1 \text{ kNm}$$

$$x = 2100 \text{ mm}$$



$$M_x = \text{shaded area} = + [8.1 - (0.5 \times 0.3 \times 1.5)] = 7.88 \text{ kNm}$$

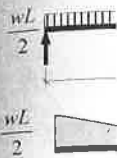
The bending moment diagram is shown in Figure 4.16.



Bending Moment Diagram

Figure 4.16

The UDL loading is a 'standard' load case which occurs in numerous beam designs and can be expressed in general terms using L for the span and w for the applied load/metre or $W_{\text{total}} (= wL)$ for the total applied load, as shown in Figure 4.17.



S



Ben

Clearly both moments. In cantilever 'fixed' support

4.1.5 Example

Consider the uniformly distributed load

Fi

Su

Support Reaction

Consider the

$$M_A + (6.0 \times$$

Consider the

$$+ V_A - (6.0$$

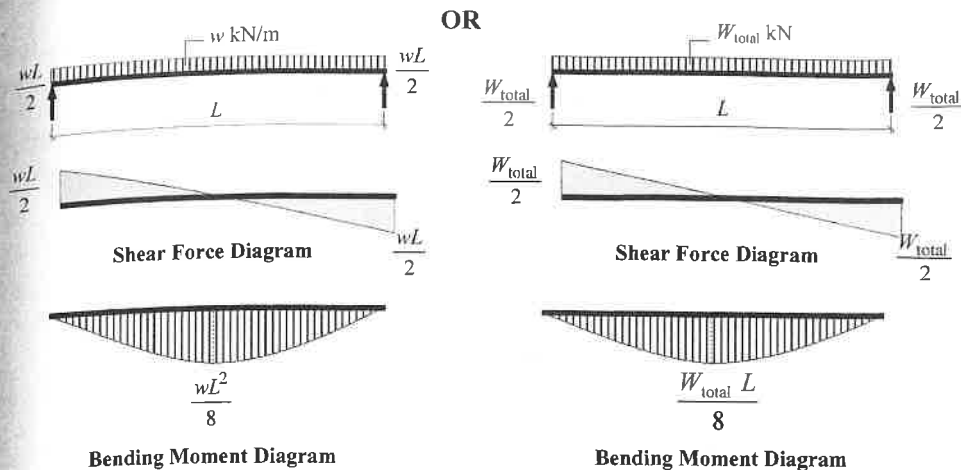


Figure 4.17

Clearly both give the same magnitude of support reactions, shear forces and bending moments.

In cantilever beams, all support restraints are provided at one location, i.e. an 'encastre' or 'fixed' support as shown in Example 4.3.

4.1.5 Example 4.3: Cantilever Beam

Consider the cantilever beam shown in Figure 4.18 which is required to support a uniformly distributed load in addition to a mid-span point load as indicated.

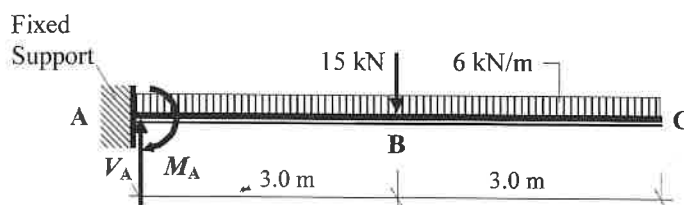


Figure 4.18

Support Reactions

Consider the rotational equilibrium of the beam: $+ve \curvearrowright \Sigma M_A = 0$

$$M_A + (6.0 \times 6.0)(3.0) + (15.0 \times 3.0) = 0 \quad \therefore M_A = -153.0 \text{ kNm}$$

Consider the vertical equilibrium of the beam: $+ve \uparrow \Sigma F_y = 0$

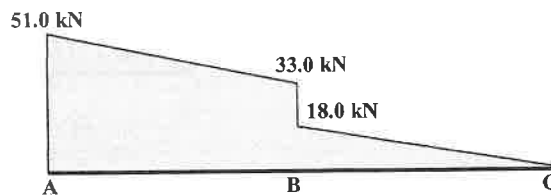
$$+V_A - (6.0 \times 6.0) - 15.0 = 0 \quad \therefore V_A = +51.0 \text{ kN}$$

us beam designs and
applied load/metre or

Shear force at B:

$$V_B = [51.0 - (6.0 \times 3.0)] = 33.0 \text{ kN}$$

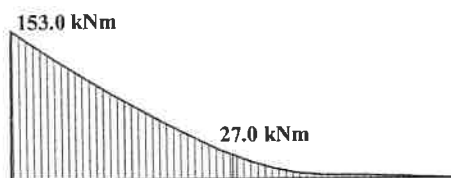
$$\text{and } = (33.0 - 15.0) = 18.0 \text{ kN}$$



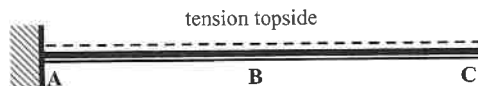
Shear Force Diagram

Bending moment at B:

$$M_B = -(6.0 \times 3.0^2/2) = -27.0 \text{ kNm}$$



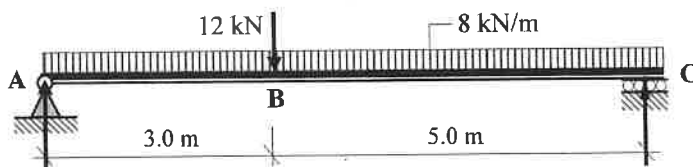
Bending Moment Diagram



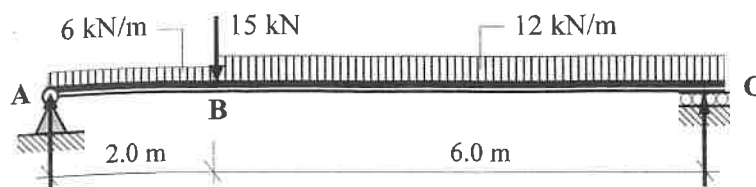
4.1.6 Problems: Statically Determinate Beams – Shear Force and Bending Moment

A series of simply supported beams are indicated in Problems 4.1 to 4.10. Using the applied loading given in each case:

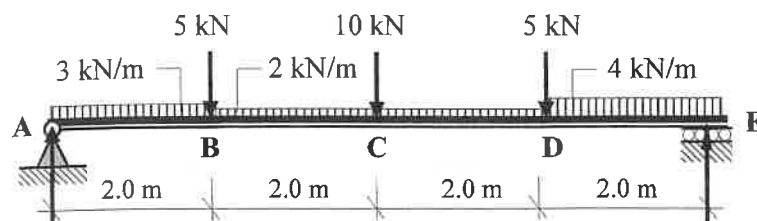
- i) determine the support reactions,
- ii) sketch the shear force diagram and
- iii) sketch the bending moment diagram indicating the maximum value(s).



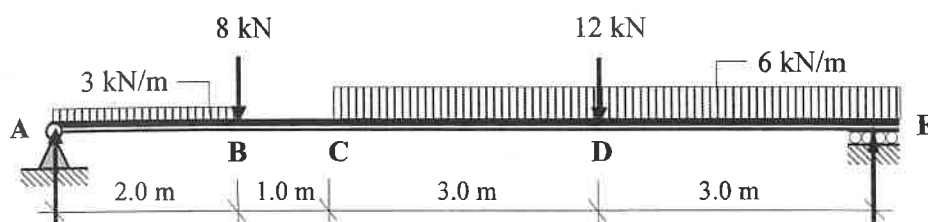
Problem 4.1



Problem 4.2



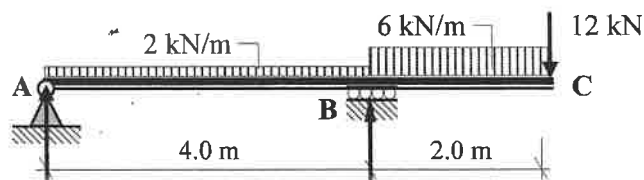
Problem 4.3



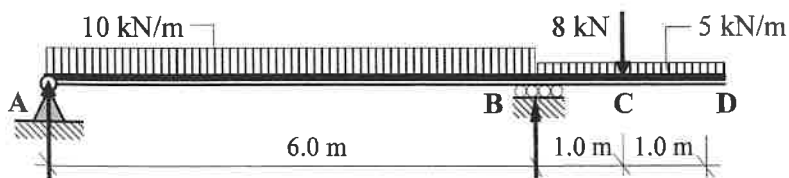
Problem 4.4

d Bending Moment
4.1 to 4.10. Using the

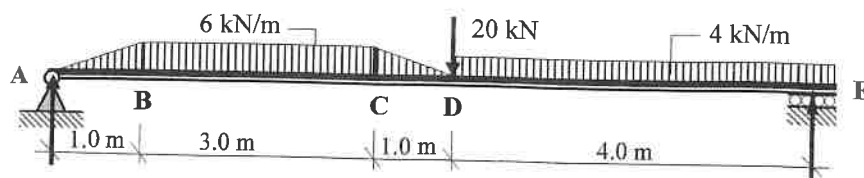
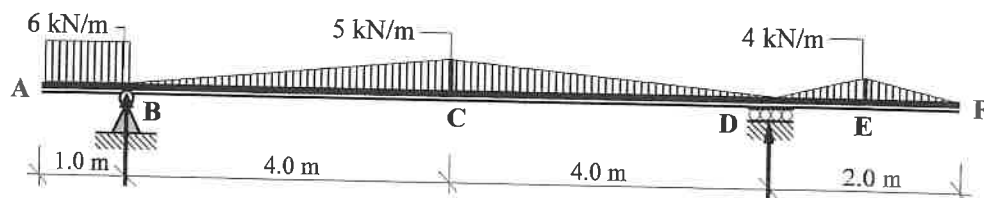
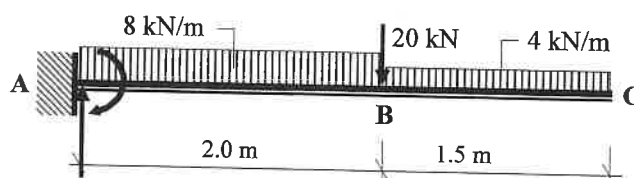
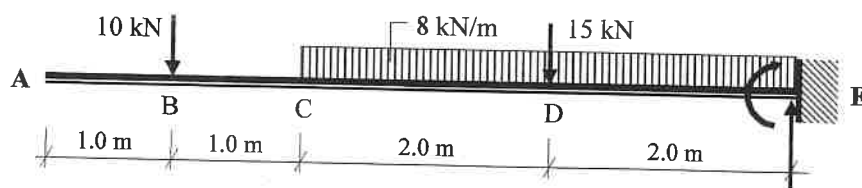
imum value(s).



Problem 4.5



Problem 4.6


Problem 4.7

Problem 4.8

Problem 4.9

Problem 4.10