

# COMPLEX NUMBERS

## Imaginary Quantity

The square root of a negative real number is called an imaginary quantity or imaginary number.  
e.g.,  $\sqrt{-3}$ ,  $\sqrt{-7/2}$

The quantity  $\sqrt{-1}$  is an imaginary number, denoted by 'i', called iota.

## Integral Powers of Iota (i)

$$i = \sqrt{-1}, i^2 = -1, i^3 = -i, i^4 = 1$$

$$\text{So, } i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i, i^{4n+4} = i^{4n} = 1$$

In other words,

$$i^n = (-1)^{n/2}, \text{ if } n \text{ is an even integer}$$

$$i^n = (-1)^{(n-1)/2} \cdot i, \text{ if } n \text{ is an odd integer}$$

## Complex Number

A number of the form  $z = x + iy$ , where  $x, y \in \mathbb{R}$ , is called a complex number

The numbers  $x$  and  $y$  are called respectively real and imaginary parts of complex number  $z$ .

$$\text{i.e., } x = \operatorname{Re}(z) \text{ and } y = \operatorname{Im}(z)$$

## Purely Real and Purely Imaginary Complex Number

A complex number  $z$  is a purely real if its imaginary part is 0.

$$\text{i.e., } \operatorname{Im}(z) = 0. \text{ And purely imaginary if its real part is 0 i.e., } \operatorname{Re}(z) = 0.$$

## Equality of Complex Numbers

Two complex numbers  $z_1 = a_1 + ib_1$  and  $z_2 = a_2 + ib_2$  are equal, if  $a_1 = a_2$  and  $b_1 = b_2$  i.e.,  $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$  and  $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$ .

## Algebra of Complex Numbers

### 1. Addition of Complex Numbers

Let  $z_1 = (x_1 + iy_1)$  and  $z_2 = (x_2 + iy_2)$  be any two complex numbers, then their sum defined as

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

### Properties of Addition

(i) Commutative  $z_1 + z_2 = z_2 + z_1$

(ii) Associative  $(z_1 + z_2) + z_3 = + (z_2 + z_3)$

(iii) Additive Identity  $z + 0 = z = 0 + z$

Here, 0 is additive identity.

### 2. Subtraction of Complex Numbers

Let  $z_1 = (x_1 + iy_1)$  and  $z_2 = (x_2 + iy_2)$  be any two complex numbers, then their difference is defined as

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

### 3. Multiplication of Complex Numbers

Let  $z_1 = (x_1 + iy_1)$  and  $z_2 = (x_2 + iy_2)$  be any two complex numbers, then their multiplication is defined as

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

### Properties of Multiplication

(i) **Commutative**  $z_1 z_2 = z_2 z_1$

(ii) **Associative**  $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

(iii) **Multiplicative Identity**  $z \cdot 1 = z = 1 \cdot z$

Here, 1 is multiplicative identity of an element  $z$ .

(iv) **Multiplicative Inverse** Every non-zero complex number  $z$  there exists a complex number  $z_1$  such that  $z \cdot z_1 = 1 = z_1 \cdot z$

(v) **Distributive Law**

(a)  $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$  (left distribution)

(b)  $(z_2 + z_3)z_1 = z_2 z_1 + z_3 z_1$  (right distribution)

### 4. Division of Complex Numbers

Let  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  be any two complex numbers, then their division is defined as

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)}$$

$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

where  $z_2 \neq 0$ .

### Conjugate of a Complex Number

If  $z = x + iy$  is a complex number, then conjugate of  $z$  is denoted by  $\bar{z}$

i.e.,  $\bar{z} = x - iy$

### Properties of Conjugate

- (i)  $\overline{(\bar{z})} = z$
  - (ii)  $z + \bar{z} \Leftrightarrow z$  is purely real
  - (iii)  $z - \bar{z} \Leftrightarrow z$  is purely imaginary
  - (iv)  $\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$
  - (v)  $\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$
  - (vi)  $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
  - (vii)  $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
  - (viii)  $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
  - (ix)  $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$
  - (x)  $z_1 \bar{z}_2 \pm \bar{z}_1 z_2 = 2 \operatorname{Re}(\bar{z}_1 z_2) = 2 \operatorname{Re}(z_1 \bar{z}_2)$
  - (xi)  $\overline{(z)^n} = (\bar{z})^n$
  - (xii) If  $z = f(z_1)$ , then  $\bar{z} = f(\bar{z}_1)$
  - (xiii) If  $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ , then  $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$
- where  $a_i, b_i, c_i; (i = 1, 2, 3)$  are complex numbers.
- (xiv)  $z \bar{z} = \{\operatorname{Re}(z)\}^2 + \{\operatorname{Im}(z)\}^2$

### Modulus of a Complex Number

If  $z = x + iy$ , then modulus or magnitude of  $z$  is denoted by  $|z|$  and is given by

$$|z| = \sqrt{x^2 + y^2}.$$

It represents a distance of  $z$  from origin.

In the set of complex number  $C$ , the order relation is not defined i.e.,  $z_1 > z_2$  or  $z_1 < z_2$  has no meaning but  $|z_1| > |z_2|$  or  $|z_1| < |z_2|$  has got its meaning, since  $|z|$  and  $|z_2|$  are real numbers.

### Properties of Modulus

$$(i) |z| \geq 0$$

$$(ii) \text{ If } |z| = 0, \text{ then } z = 0 \text{ i.e., } \operatorname{Re}(z) = 0 = \operatorname{Im}(z)$$

$$(iii) -|z| \leq \operatorname{Re}(z) \leq |z| \text{ and } -|z| \leq \operatorname{Im}(z) \leq |z|$$

$$(iv) |z| = |\bar{z}| = |-z| = |-\bar{z}|$$

$$(v) z\bar{z} = |z|^2$$

$$(vi) |z_1 z_2| = |z_1| |z_2|$$

In general

$$|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$$

$$(vii) \frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|}, \text{ provided } z_2 \neq 0$$

$$(viii) |z_1 \pm z_2| \leq |z_1| + |z_2|$$

In general

$$|z_1 \pm z_2 \pm z_3 \pm \dots \pm z_n| \leq |z_1| + |z_2| + |z_3| + \dots + |z_n|$$

$$(ix) |z_1 \pm z_2| \geq |z_1| - |z_2|$$

$$(x) |z^n| = |z|^n$$

$$(xi) ||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2| \text{ greatest possible value of } |z_1 + z_2| \text{ is } |z_1| + |z_2| \text{ and least possible value of } |z_1 + z_2| \text{ is } ||z_1| - |z_2||$$

$$||z_1| - |z_2||$$

$$(xii) |z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + z_2 \bar{z}_1$$

$$= |z_1|^2 + |z_2|^2 + 2 \operatorname{Re}(z_1 \bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1| |z_2| \cos(\theta_1 - \theta_2)$$

$$(xiii) |z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 - (z_1 \bar{z}_2 + \bar{z}_1 z_2)$$

$$= |z_1|^2 + |z_2|^2 - 2 \operatorname{Re}(z_1 \bar{z}_2)$$

$$= |z_1|^2 + |z_2|^2 - 2|z_1| |z_2| \cos(\theta_1 - \theta_2)$$

$$(xiv) z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2|z_1| |z_2| \cos(\theta_1 - \theta_2)$$

$$(xv) |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

$$(xvi) |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2} \text{ is purely imaginary.}$$

$$(xvii) |az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

where  $a, b \in R$ .

$$(xviii) z \text{ is unimodulus, if } |z| = 1$$

## Reciprocal/Multiplicative Inverse of a Complex Number

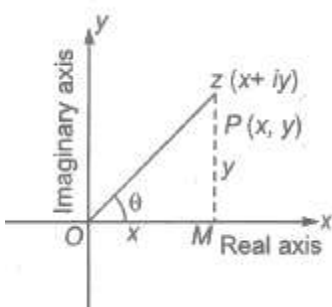
Let  $z = x + iy$  be a non-zero complex number, then

$$\begin{aligned} z^{-1} &= \frac{1}{z} = \frac{1}{x + iy} = \frac{1}{x + iy} \times \frac{x - iy}{x - iy} \\ &= \frac{x - iy}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} + \frac{i(-y)}{x^2 + y^2} \end{aligned}$$

Here,  $z^{-1}$  is called multiplicative inverse of  $z$ .

## Argument of a Complex Number

Any complex number  $z = x + iy$  can be represented geometrically by a point  $(x, y)$  in a plane, called Argand plane or Gaussian plane. The angle made by the line joining point  $z$  to the origin, with the  $x$ -axis is called argument of that complex number. It is denoted by the symbol  $\arg(z)$  or  $\text{amp}(z)$ .



$$\text{Argument}(z) = \theta = \tan^{-1}(y/x)$$

Argument of  $z$  is not unique, general value of the argument of  $z$  is  $2n\pi + \theta$ . But  $\arg(0)$  is not defined.

A purely real number is represented by a point on  $x$ -axis.

A purely imaginary number is represented by a point on  $y$ -axis.

There exists a one-one correspondence between the points of the plane and the members of the set  $C$  of all complex numbers.

The length of the line segment  $OP$  is called the modulus of  $z$  and is denoted by  $|z|$ .

$$\text{i.e., length of } OP = \sqrt{x^2 + y^2}.$$

Principal Value of Argument

The value of the argument which lies in the interval  $(-\pi, \pi]$  is called principal value of argument.

- (i) If  $x > 0$  and  $y > 0$ , then  $\arg(z) = 0$
- (ii) If  $x < 0$  and  $y > 0$ , then  $\arg(z) = \pi - \theta$
- (iii) If  $x < 0$  and  $y < 0$ , then  $\arg(z) = -(\pi - \theta)$
- (iv) If  $x > 0$  and  $y < 0$ , then  $\arg(z) = -\theta$

### **Properties of Argument**

- (i)  $\arg(\bar{z}) = -\arg(z)$
- (ii)  $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$  ( $k = 0, 1$  or  $-1$ )
- In general,  

$$\arg(z_1 z_2 z_3 \dots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \dots + \arg(z_n) + 2k\pi$$
 ( $k = 0, 1$  or  $-1$ )
- (iii)  $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$  ( $k = 0, 1$  or  $-1$ )
- (iv)  $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$
- (v)  $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi$  ( $k = 0, 1$  or  $-1$ )
- (vi)  $\arg(z^n) = n\arg(z) + 2k\pi$  ( $k = 0, 1$  or  $-1$ )
- (vii) If  $\arg\left(\frac{z_2}{z_1}\right) = \theta$ , then  $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$ ,  $k \in I$
- (viii) If  $\arg(z) = 0 \Rightarrow z$  is real
- (ix)  $\arg(z) - \arg(-z) = \begin{cases} \pi, & \text{if } \arg(z) > 0 \\ -\pi, & \text{if } \arg(z) < 0 \end{cases}$
- (x) If  $|z_1 + z_2| = |z_1 - z_2|$ , then  

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$$
- (xi) If  $|z_1 + z_2| = |z_1| + |z_2|$ , then  $\arg(z_1) = \arg(z_2)$
- (xii) If  $|z - 1| = |z + 1|$ , then  $\arg(z) = \pm \frac{\pi}{2}$
- (xiii) If  $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$ , then  $(z) = 1$
- (xiv) If  $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{2}$ , then  $z$  lies on circle of radius unity and centre at origin.
- (xv) (a) If  $z = 1 + \cos \theta + i \sin \theta$ , then  

$$\arg(z) = \frac{\theta}{2} \text{ and } |z| = 2 \cos \frac{\theta}{2}$$
 (b) If  $z = 1 + \cos \theta - i \sin \theta$ , then  

$$\arg(z) = -\frac{\theta}{2} \text{ and } |z| = 2 \cos \frac{\theta}{2}$$
 (c) If  $z = 1 - \cos \theta + i \sin \theta$ , then  

$$\arg(z) = \frac{\pi}{2} - \frac{\theta}{2} \text{ and } |z| = 2 \sin \frac{\theta}{2}$$
 (d) If  $z = 1 - \cos \theta - i \sin \theta$ , then  

$$\arg(z) = \frac{\pi}{4} - \frac{\theta}{2} \text{ and } |z| = \sqrt{2} \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)$$
- (xvi) If  $|z_1| \leq 1, |z_2| \leq 1$ , then  
 (a)  $|z_1 - z_2|^2 \leq (|z_1| - |z_2|)^2 + [\arg(z_1) - \arg(z_2)]^2$   
 (b)  $|z_1 + z_2|^2 \geq (|z_1| + |z_2|)^2 - [\arg(z_1) - \arg(z_2)]^2$

## Square Root of a Complex Number

If  $z = x + iy$ , then

$$\begin{aligned}\sqrt{z} = \sqrt{x + iy} &= \pm \left[ \frac{\sqrt{|z| + x}}{2} + i \frac{\sqrt{|z| - x}}{2} \right], \text{ for } y > 0 \\ &= \pm \left[ \frac{\sqrt{|z| + x}}{2} - i \frac{\sqrt{|z| - x}}{2} \right], \text{ for } y < 0\end{aligned}$$

## Polar Form

If  $z = x + iy$  is a complex number, then  $z$  can be written as

$$z = |z| (\cos \theta + i \sin \theta) \text{ where, } \theta = \arg(z)$$

this is called polar form.

If the general value of the argument is  $0$ , then the polar form of  $z$  is

$$z = |z| [\cos(2n\pi + \theta) + i \sin(2n\pi + \theta)], \text{ where } n \text{ is an integer.}$$

## Eulerian Form of a Complex Number

If  $z = x + iy$  is a complex number, then it can be written as

$$z = re^{i\theta}, \text{ where}$$

$$r = |z| \text{ and } \theta = \arg(z)$$

This is called Eulerian form and  $e^{i\theta} = \cos\theta + i \sin\theta$  and  $e^{-i\theta} = \cos\theta - i \sin\theta$ .

## De-Moivre's Theorem

A simplest formula for calculating powers of complex number known as De-Moivre's theorem.

If  $n \in \mathbb{I}$  (set of integers), then  $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$  and if  $n \in \mathbb{Q}$  (set of rational numbers), then  $\cos n\theta + i \sin n\theta$  is one of the values of  $(\cos \theta + i \sin \theta)^n$ .

(i) If  $\frac{p}{q}$  is a rational number, then

$$(\cos \theta + i \sin \theta)^{p/q} = \left( \cos \frac{p}{q} \theta + i \sin \frac{p}{q} \theta \right)$$

(ii)  $\frac{1}{\cos \theta + i \sin \theta} = (\cos \theta - i \sin \theta)$

(iii) More generally, for a complex number  $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$

$$\begin{aligned} z^n &= r^n (\cos \theta + i \sin \theta)^n \\ &= r^n (\cos n\theta + i \sin n\theta) = r^n e^{in\theta} \end{aligned}$$

(iv)  $(\sin \theta + i \cos \theta)^n = \left[ \cos \left( \frac{n\pi}{2} - n\theta \right) + i \sin \left( \frac{n\pi}{2} - n\theta \right) \right]$

(v)  $(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \dots (\cos \theta_n + i \sin \theta_n)$   
 $= \cos (\theta_1 + \theta_2 + \dots + \theta_n) + i \sin (\theta_1 + \theta_2 + \dots + \theta_n)$

(vi)  $(\sin \theta \pm i \cos \theta)^n \neq \sin n\theta \pm i \cos n\theta$

(vii)  $(\cos \theta + i \sin \phi)^n \neq \cos n\theta + i \sin n\phi$

## The nth Roots of Unity

The nth roots of unity, it means any complex number  $z$ , which satisfies the equation  $z^n = 1$  or  $z = (1)^{1/n}$

or  $z = \cos(2k\pi/n) + i\sin(2k\pi/n)$ , where  $k = 0, 1, 2, \dots, (n - 1)$

## Properties of nth Roots of Unity

1. nth roots of unity form a GP with common ratio  $e^{(i2\pi/n)}$ .
2. Sum of nth roots of unity is always 0.
3. Sum of nth powers of nth roots of unity is zero, if  $p$  is a multiple of  $n$ .
4. Sum of  $p$ th powers of nth roots of unity is zero, if  $p$  is not a multiple of  $n$ .
5. Sum of  $p$ th powers of nth roots of unity is  $n$ , if  $p$  is a multiple of  $n$ .
6. Product of nth roots of unity is  $(-1)^{(n-1)}$ .
7. The nth roots of unity lie on the unit circle  $|z| = 1$  and divide its circumference into  $n$  equal parts.

## The Cube Roots of Unity

Cube roots of unity are  $1, \omega, \omega^2$ ,

where  $\omega = -1/2 + i\sqrt{3}/2 = e^{(i2\pi/3)}$  and  $\omega^2 = (-1 - i\sqrt{3})/2$

$$\omega^{3r+1} = \omega, \omega^{3r+2} = \omega^2$$

## Properties of Cube Roots of Unity

(i)  $1 + \omega + \omega^{2r} =$

0, if  $r$  is not a multiple of 3.

3, if  $r$  is a multiple of 3.

(ii)  $\omega^3 = \omega^{3r} = 1$

(iii)  $\omega^{3r+1} = \omega, \omega^{3r+2} = \omega^2$

(iv) Cube roots of unity lie on the unit circle  $|z| = 1$  and divide its circumference into 3 equal parts.

(v) It always forms an equilateral triangle.

(vi) Cube roots of  $-1$  are  $-1, -\omega, -\omega^2$ .

## Important Identities

(i)  $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

(ii)  $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

(iii)  $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(iv)  $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$

(v)  $x^2 + y^2 = (x + iy)(x - iy)$

(vi)  $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(vii)  $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

(viii)  $x^2 + y^2 + z^2 - xy - yz - zx = (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

or  $(x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$

or  $(x\omega + y + z\omega^2)(x\omega^2 + y + z\omega)$

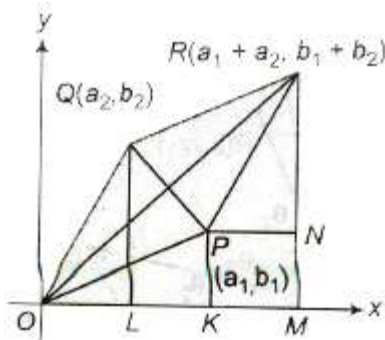
(ix)  $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

## Geometrical Representations of Complex Numbers

### 1. Geometrical Representation of Addition

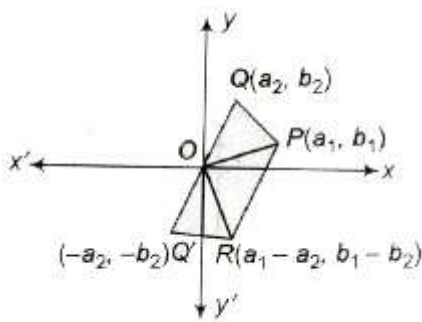
If two points P and Q represent complex numbers  $z_1$  and  $z_2$  respectively, in the Argand plane, then the sum  $z_1 + z_2$  is represented

by the extremity R of the diagonal OR of parallelogram OPRQ having OP and OQ as two adjacent sides.



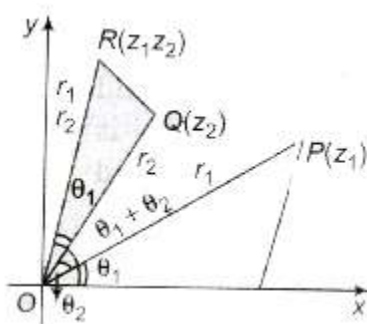
## 2. Geometrical Representation of Subtraction

Let  $z_1 = a_1 + ib_1$  and  $z_2 = a_2 + ia_2$  be two complex numbers represented by points  $P(a_1, b_1)$  and  $Q(a_2, b_2)$  in the Argand plane.  $Q'$  represents the complex number  $(-z_2)$ . Complete the parallelogram  $OPRQ'$  by taking  $OP$  and  $OQ'$  as two adjacent sides.



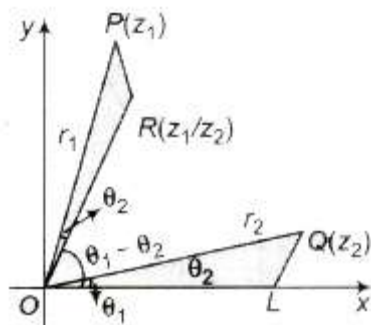
The sum of  $z_1$  and  $-z_2$  is represented by the extremity  $R$  of the diagonal  $OR$  of parallelogram  $OPRQ'$ .  $R$  represents the complex number  $z_1 - z_2$ .

## 3. Geometrical Representation of Multiplication of Complex Numbers



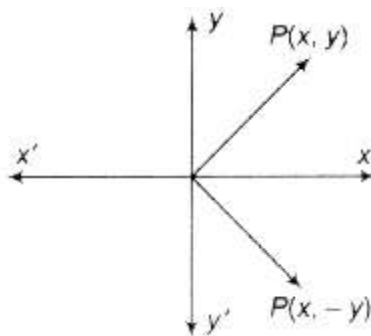
$R$  has the polar coordinates  $(r_1r_2, \theta_1 + \theta_2)$  and it represents the complex numbers  $z_1z_2$ .

## 4. Geometrical Representation of the Division of Complex Numbers



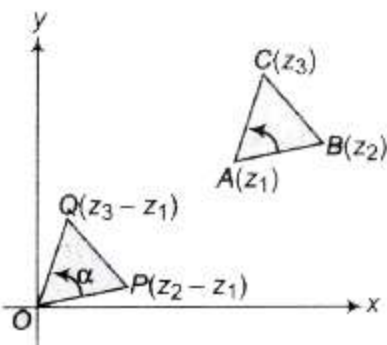
R has the polar coordinates  $(r_1/r_2, \theta_1 - \theta_2)$  and it represents the complex number  $z_1/z_2$ .  $|z|=|z|$  and  $\arg(z) = -\arg(z)$ . The general value of  $\arg(z)$  is  $2n\pi - \arg(z)$ .

If a point P represents a complex number  $z$ , then its conjugate  $\bar{z}$  is represented by the image of P in the real axis.

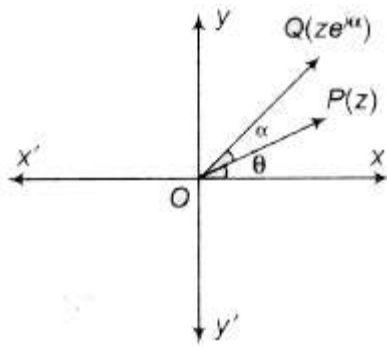


### Concept of Rotation

Let  $z_1, z_2$  and  $z_3$  be the vertices of a  $\Delta ABC$  described in anti-clockwise sense. Draw OP and OQ parallel and equal to AB and AC, respectively. Then, point P is  $z_2 - z_1$  and Q is  $z_3 - z_1$ . If OP is rotated through angle  $\alpha$  in anti-clockwise, sense it coincides with OQ.



### Important Points to be Remembered



(a)  $ze^{i\alpha}$  is the complex number whose modulus is  $r$  and argument  $\theta + \alpha$ .

(b) Multiplication by  $e^{-i\alpha}$  to  $z$  rotates the vector  $OP$  in clockwise sense through an angle  $\alpha$ .

(ii) If  $z_1, z_2, z_3$  and  $z_4$  are the affixes of the points A, B, C and D, respectively in the Argand plane.

(a) AB is inclined to CD at the angle  $\arg [(z_2 - z_1)/(z_4 - z_3)]$ .

(b) If CD is inclined at  $90^\circ$  to AB, then  $\arg [(z_2 - z_1)/(z_4 - z_3)] = \pm(\pi/2)$ .

(c) If  $z_1$  and  $z_2$  are fixed complex numbers, then the locus of a point  $z$  satisfying  $\arg [(z - z_1)/(z - z_2)] = \pm(\pi/2)$ .

### Logarithm of a Complex Number

Let  $z = x + iy$  be a complex number and in polar form of  $z$  is  $re^{i\theta}$ , then

$$\log(x + iy) = \log(re^{i\theta}) = \log(r) + i\theta$$

$$\log(\sqrt{x^2 + y^2}) + i \tan^{-1}(y/x)$$

$$\text{or } \log(z) = \log(|z|) + i \arg(z),$$

In general,

$$z = re^{i(\theta + 2n\pi)}$$

$$\log z = \log|z| + i \arg z + 2n\pi i$$

### Applications of Complex Numbers in Coordinate Geometry

Distance between complex Points

(i) Distance between  $A(z_1)$  and  $B(z_2)$  is given by

$$AB = |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$

(ii) The point P (z) which divides the join of segment AB in the ratio m : n is given by

$$z = (mz_2 + nz_1)/(m + n)$$

If P divides the line externally in the ratio m : n, then

$$z = (mz_2 - nz_1)/(m - n)$$

### **Triangle in Complex Plane**

(i) Let ABC be a triangle with vertices A ( $z_1$ ), B( $z_2$ ) and C( $z_3$ ) then

(a) Centroid of the  $\Delta ABC$  is given by

$$z = 1/3(z_1 + z_2 + z_3)$$

(b) Incentre of the  $\Delta ABC$  is given by

$$z = (az_1 + bz_2 + cz_3)/(a + b + c)$$

(ii) Area of the triangle with vertices A( $z_1$ ), B( $z_2$ ) and C( $z_3$ ) is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

For an equilateral triangle,

$$z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$$

(iii) The triangle whose vertices are the points represented by complex numbers  $z_1$ ,  $z_2$  and  $z_3$  is equilateral, if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

i.e.,  $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

### **Straight Line in Complex Plane**

(i) The general equation of a straight line is  $az + \bar{a}z + b = 0$ , where a is a complex number and b is a real number.

- (ii) The complex and real slopes of the line  $az + \bar{a}z = a\bar{a}$  are  $-a/a$  and  $-i[(a + \bar{a})/(a - \bar{a})]$ .
- (iii) The equation of straight line through  $z_1$  and  $z_2$  is  $z = tz_1 + (1 - t)z_2$ , where  $t$  is real.
- (iv) If  $z_1$  and  $z_2$  are two fixed points, then  $|z - z_1| = |z - z_2|$  represents perpendicular bisector of the line segment joining  $z_1$  and  $z_2$ .
- (v) Three points  $z_1, z_2$  and  $z_3$  are collinear, if

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

This is also, the equation of the line passing through  $z_1, z_2$  and  $z_3$  and slope is defined to be  $(z_1 - z_2)/(z_1 - z_2)$

**(vi) Length of Perpendicular** The length of perpendicular from a point  $z_1$  to  $az + \bar{a}z + b = 0$  is given by  $|az_1 + \bar{a}z_1 + b|/2|a|$

(vii)  $\arg(z - z_1)/(z - z_2) = \beta$

Locus is the arc of a circle which the segment joining  $z_1$  and  $z_2$  as a chord.

(viii) The equation of a line parallel to the line  $az + \bar{a}z + b = 0$  is  $az + \bar{a}z + \lambda = 0$ , where  $\lambda \in \mathbb{R}$ .

(ix) The equation of a line parallel to the line  $az + \bar{a}z + b = 0$  is  $az + \bar{a}z + i\lambda = 0$ , where  $\lambda \in \mathbb{R}$ .

(x) If  $z_1$  and  $z_2$  are two fixed points, then  $|z - z_1| = |z - z_2|$  represents perpendicular bisector of the segment joining  $A(z_1)$  and  $B(z_2)$ .

(xi) The equation of a line perpendicular to the line  $z(z_1 - z_2) + \bar{z}(\bar{z}_1 - \bar{z}_2) = |z_1|^2 - |z_2|^2$ .

(xii) If  $z_1, z_2$  and  $z_3$  are the affixes of the points A, B and C in the Argand plane, then

(a)  $\angle BAC = \arg[(z_3 - z_1)/(z_2 - z_1)]$

(b)  $[(z_3 - z_1)/(z_2 - z_1)] = |z_3 - z_1|/|z_2 - z_1| (\cos \alpha + i \sin \alpha)$ , where  $\alpha = \angle BAC$ .

(xiii) If  $z$  is a variable point in the argand plane such that  $\arg(z) = \theta$ , then locus of  $z$  is a straight line through the origin inclined at an angle  $\theta$  with X-axis.

(xiv) If  $z$  is a variable point and  $z_1$  is fixed point in the argand plane such that  $\arg(z - z_1) = \theta$ , then locus of  $z$  is a straight line passing through the point  $z_1$  and inclined at an angle  $\theta$  with the X-axis.

(xv) If  $z$  is a variable point and  $z_1, z_2$  are two fixed points in the Argand plane, then

$$(a) |z - z_1| + |z - z_2| = |z_1 - z_2|$$

Locus of  $z$  is the line segment joining  $z_1$  and  $z_2$ .

$$(b) |z - z_1| - |z - z_2| = |z_1 - z_2|$$

Locus of  $z$  is a straight line joining  $z_1$  and  $z_2$  but  $z$  does not lie between  $z_1$  and  $z_2$ .

$$(c) \arg[(z - z_1)/(z - z_2)] = 0 \text{ or } \pi;$$

Locus  $z$  is a straight line passing through  $z_1$  and  $z_2$ .

$$(d) |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

Locus of  $z$  is a circle with  $z_1$  and  $z_2$  as the extremities of diameter.

### Circle in Complete Plane

(i) An equation of the circle with centre at  $z_0$  and radius  $r$  is

$$|z - z_0| = r$$

$$\text{or } zz - z_0z - z_0z + z_0z_0 = r^2$$

- $|z - z_0| < r$ , represents interior of the circle.
- $|z - z_0| > r$ , represents exterior of the circle.
- $|z - z_0| \leq r$  is the set of points lying inside and on the circle  $|z - z_0| = r$ . Similarly,  $|z - z_0| \geq r$  is the set of points lying outside and on the circle  $|z - z_0| = r$ .
- **General equation of a circle is**

$$zz - az - \bar{a}z + b = 0$$

where  $a$  is a complex number and  $b$  is a real number. Centre of the circle =  $-a$

$$\text{Radius of the circle} = \sqrt{aa - b} \text{ or } \sqrt{|a|^2 - b}$$

(a) Four points  $z_1, z_2, z_3$  and  $z_4$  are concyclic, if

$$[(z_4 - z_1)(z_2 - z_3)]/[(z_4 - z_3)(z_2 - z_1)] \text{ is purely real.}$$

(ii)  $|z - z_1|/|z - z_2| = k \Rightarrow$  Circle, if  $k \neq 1$  or Perpendicular bisector, if  $k = 1$

(iii) The equation of a circle described on the line segment joining  $z_1$  and  $z_2$  as diameter is  $(z - z_1)(z - z_2) + (\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2) = 0$

(iv) If  $z_1$ , and  $z_2$  are the fixed complex numbers, then the locus of a point  $z$  satisfying  $\arg [(z - z_1)/(z - z_2)] = \pm \pi / 2$  is a circle having  $z_1$  and  $z_2$  at the end points of a diameter.

### Conic in Complex plane

(i) Let  $z_1$  and  $z_2$  be two fixed points, and  $k$  be a positive real number.

If  $k > |z_1 - z_2|$ , then  $|z - z_1| + |z - z_2| = k$  represents an ellipse with foci at  $A(z_1)$  and  $B(z_2)$  and length of the major axis is  $k$ .

(ii) Let  $z_1$  and  $z_2$  be two fixed points and  $k$  be a positive real number.

If  $k \neq |z_1 - z_2|$ , then  $|z - z_1| - |z - z_2| = k$  represents hyperbola with foci at  $A(z_1)$  and  $B(z_2)$ .

### Important Points to be Remembered

- $\sqrt{-a} \times \sqrt{-b} \neq \sqrt{ab}$

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$  is possible only, if both  $a$  and  $b$  are non-negative.

So,  $i^2 = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

- is neither positive, zero nor negative.
- Argument of 0 is not defined.
- Argument of purely imaginary number is  $\pi/2$
- Argument of purely real number is 0 or  $\pi$ .
- If  $|z + 1/z| = a$  then the greatest value of  $|z| = a + \sqrt{a^2 + 4}/2$  and the least value of  $|z| = -a + \sqrt{a^2 + 4}/2$
- The value of  $i^i = e^{-\pi/2}$
- The complex number do not possess the property of order, i.e.,  $x + iy < (\text{or}) > c + id$  is not defined.
- The area of the triangle on the Argand plane formed by the complex numbers  $z$ ,  $iz$  and  $z + iz$  is  $1/2|z|^2$ .
- (x) If  $\omega_1$  and  $\omega_2$  are the complex slope of two lines on the Argand plane, then the lines are

(a) perpendicular, if  $\omega_1 + \omega_2 = 0$ .

(b) parallel, if  $\omega_1 = \omega_2$ .