

Introduction to Matrices

A green pencil with a red eraser is pointing towards a matrix diagram. The matrix is represented by a grid of elements a_{ij} where i is the row index and j is the column index. The elements are arranged in a grid with rows and columns labeled $1, 2, 3, \dots, n$. The matrix is enclosed in a large bracket on the left side, labeled A . The background is dark gray.

Matrices

- In many engineering calculations, it is often necessary to handle large amounts of data. Matrices are used to store this data in a form that makes it relatively easy to carry out appropriate calculations

Matrices

- A matrix consists of 'elements' which are often integers and are arranged in rows and columns. The number of rows and columns that a matrix has is called the 'dimension' of the matrix.

$$\begin{pmatrix} 3 & 2 & -1 \\ 4 & 8 & -2 \end{pmatrix}$$

- In this example, we have 2 rows (rows go across) and 3 columns (columns go down) and is so called a 2x3 matrix.
- Remember, when you specify the dimension of a matrix, the number of rows are always specified first and the number of columns second

Manipulation of Matrices

- You can only add or subtract matrices if they are the same dimension (i.e. same shape). Further, the dimension of the resulting matrix will be the same as the original ones.

$$A = \begin{pmatrix} 2 & -1 \\ 6 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$$

Manipulation of Matrices

Let's find $A+B$.

- All we do here is add the corresponding elements.

$$A = \begin{pmatrix} 2 & -1 \\ 6 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2 & -1 \\ 6 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 2 + -1 & -1 + 2 \\ 6 + 0 & 3 + -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 6 & 0 \end{pmatrix}$$

Manipulation of Matrices

Let's find A-B.

- All we do here is subtract the corresponding elements.

$$A = \begin{pmatrix} 2 & -1 \\ 6 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 2 & -1 \\ 6 & 3 \end{pmatrix} - \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 2 - (-1) & -1 - 2 \\ 6 - 0 & 3 - (-3) \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 6 & 6 \end{pmatrix}$$

Manipulation of Matrices

Let's find B-A.

- All we do here is subtract the corresponding elements.

$$A = \begin{pmatrix} 2 & -1 \\ 6 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$$

$$\begin{aligned} B - A &= \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} -1 - 2 & 2 - (-1) \\ 0 - 6 & -3 - 3 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 3 \\ -6 & -6 \end{pmatrix} \end{aligned}$$

Manipulation of Matrices

- Scalar multiplication is when any shape of matrix is multiplied by a single number.

$$A = \begin{pmatrix} 2 & 3 \\ -4 & 6 \\ 0 & -2 \end{pmatrix}$$

Manipulation of Matrices

Let's find $3A$.

- All we do here is multiply each element by 3.

$$A = \begin{pmatrix} 2 & 3 \\ -4 & 6 \\ 0 & -2 \end{pmatrix}$$

$$3A = \begin{pmatrix} 3 \times 2 & 3 \times 3 \\ 3 \times -4 & 3 \times 6 \\ 3 \times 0 & 3 \times -2 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ -12 & 18 \\ 0 & -6 \end{pmatrix}$$

Manipulation of Matrices

- General matrix multiplication is when one matrix is multiplied by another matrix.
- Two matrices can only be multiplied together if the number of columns in the first one is equal to the number of rows in the second one.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} j & k & l \\ m & n & p \\ q & r & s \end{pmatrix}$$

Manipulation of Matrices

- General matrix multiplication is when one matrix is multiplied by another matrix.
- Two matrices can only be multiplied together if the number of columns in the first one is equal to the number of rows in the second one.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} j & k & l \\ m & n & p \\ q & r & s \end{pmatrix}$$

3 × 3

3 × 3

As long as the two **innermost** numbers are the same, then you can multiply the matrices together.

The numbers on the **outside** gives the dimension of the matrix you will end up with.

Manipulation of Matrices

- General matrix multiplication is when one matrix is multiplied by another matrix.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} j & k & l \\ m & n & p \\ q & r & s \end{pmatrix} = \begin{pmatrix} aj + bm + cq & ak + bn + cr & al + bp + cs \\ dj + em + fq & dk + en + fr & dl + ep + fs \\ gj + hm + iq & gk + hn + ir & gl + hp + is \end{pmatrix}$$

Manipulation of Matrices

Let's find $A \times B$.

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A \times B = \begin{pmatrix} 2 \times 1 + 1 \times 1 + 4 \times 1 & 2 \times 2 + 1 \times 0 + 4 \times 0 & 2 \times 3 + 1 \times 4 + 4 \times 1 \\ 0 \times 1 + 3 \times 1 + 1 \times 1 & 0 \times 2 + 3 \times 0 + 1 \times 0 & 0 \times 3 + 3 \times 4 + 1 \times 1 \\ 2 \times 1 + 1 \times 1 + 5 \times 1 & 2 \times 2 + 1 \times 0 + 5 \times 0 & 2 \times 3 + 1 \times 4 + 5 \times 1 \end{pmatrix}$$

$$A \times B = \begin{pmatrix} 7 & 4 & 14 \\ 4 & 0 & 13 \\ 8 & 4 & 15 \end{pmatrix}$$

Manipulation of Matrices

Let's find $A \times B$.

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \\ 1 & 0 & 1 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} 1 \times 2 + 2 \times 0 + 3 \times 2 & 1 \times 1 + 2 \times 3 + 3 \times 1 & 1 \times 4 + 2 \times 1 + 3 \times 5 \\ 1 \times 2 + 0 \times 0 + 4 \times 2 & 1 \times 1 + 0 \times 3 + 4 \times 1 & 1 \times 4 + 0 \times 1 + 4 \times 5 \\ 1 \times 2 + 0 \times 0 + 1 \times 2 & 1 \times 1 + 0 \times 3 + 1 \times 1 & 1 \times 4 + 0 \times 1 + 1 \times 5 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} 8 & 10 & 21 \\ 10 & 5 & 24 \\ 4 & 2 & 9 \end{pmatrix}$$

Manipulation of Matrices

- Notice AB is not the same as BA , and so we say that matrix multiplication is 'non-commutative'
- Therefore, it is very important not to mess about with the order that matrices are written down.

Manipulation of Matrices

- If A is a square matrix, then the determinant is denoted by $|A|$.
- It is a number which gives the magnitude of the matrix.
- It is very useful when solving simultaneous linear equations.
- Determinants are only defined for square matrices (matrices with the same number of rows and columns).

Manipulation of Matrices

- If you have a 2x2 matrix such as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- The determinant is defined as:

$$|A| = ad - bc$$

Manipulation of Matrices

Let's find $|A|$

$$A = \begin{pmatrix} 2 & -1 \\ 6 & -8 \end{pmatrix}$$

$$|A| = (2 \times -8) - (-1 \times 6) = -16 + 6 = -10$$

Manipulation of Matrices

- Cramer's Rule is a mathematical method to solve systems of linear equations using determinants.

Manipulation of Matrices

Problem Set up

Suppose you have a system of linear equations like this:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Manipulation of Matrices

Problem Set up

Suppose you have a system of linear equations like this:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Manipulation of Matrices

Step 1: Write the System in Matrix Form

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The matrix of co-efficients:

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The variable vector: $\begin{bmatrix} x \\ y \end{bmatrix}$

The constants vector: $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Manipulation of Matrices

Step 2: Find the Determinant of A

$$|A| = a_1b_2 - a_2b_1$$

Manipulation of Matrices

Step 3: Replace columns to solve for x and y

Solve for x.

Replace the first column of A with the constants vector, while keeping the second column the same.

$$A_x = \begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}$$

Compute the determinant of A_x

$$|A_x| = c_1 b_2 - c_2 b_1$$

Then calculate x as:

$$x = \frac{|A_x|}{|A|}$$

Manipulation of Matrices

Step 3: Replace columns to solve for x and y

Solve for y.

Replace the second column of A with the constants vector, while keeping the first column the same.

$$A_y = \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}$$

Compute the determinant of A_y

$$|A_y| = a_1 c_2 - a_2 c_1$$

Then calculate y as:

$$y = \frac{|A_y|}{|A|}$$

Manipulation of Matrices

Example:

Solve:

$$\begin{array}{l} 2x + 3y = 8 \\ 4x - y = 2 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$

$$|A| = a_1b_2 - a_2b_1 = (2 \times -1) - (4 \times 3) = -2 - 12 = -14$$

Manipulation of Matrices

Example:

$$A_x = \begin{bmatrix} 8 & 3 \\ 2 & -1 \end{bmatrix}$$

$$|A_x| = c_1 b_2 - c_2 b_1 = (8 \times -1) - (2 \times 3) = -8 - 6 = -14$$

$$x = \frac{|A_x|}{|A|} = \frac{-14}{-14} = 1$$

Manipulation of Matrices

Example:

$$A_y = \begin{bmatrix} 2 & 8 \\ 4 & 2 \end{bmatrix}$$

$$|A_y| = a_1c_2 - a_2c_1 = (2 \times 2) - (4 \times 8) = 4 - 32 = -28$$

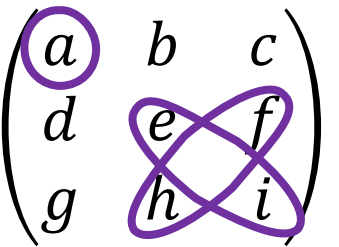
$$y = \frac{|A_y|}{|A|} = \frac{-28}{-14} = 2$$

Solution:

x=1 and y=2

Manipulation of Matrices

- If you have a 3x3 matrix such as:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$


- The determinant is defined as:

$$|A| = a(ei - hf) - b(di - fg) + c(dh - eg)$$


Manipulation of Matrices

- If you have a 3x3 matrix such as:

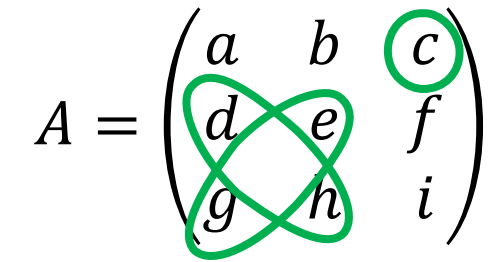
$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

- The determinant is defined as:

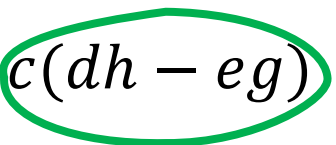
$$|A| = a(ei - hf) - b(di - fg) + c(dh - eg)$$

Manipulation of Matrices

- If you have a 3x3 matrix such as:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$


- The determinant is defined as:

$$|A| = a(ei - hf) - b(di - fg) + c(dh - eg)$$


Manipulation of Matrices

Let's find $|A|$

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 4 & -5 & 6 \\ -2 & 0 & 4 \end{pmatrix}$$

$$|A| = -1(-5 \times 4 - 6 \times 0) - 2(4 \times 4 - 6 \times -2) + 3(4 \times 0 - -5 \times -2)$$

$$|A| = -1(-20) - 2(28) + 3(-10) = 20 - 56 - 30 = -66$$

Manipulation of Matrices

Problem Set up

Suppose you have a system of linear equations like this:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Manipulation of Matrices

Step 1: Write the System in Matrix Form

The matrix of co-efficients:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

The variable vector: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The constants vector: $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Manipulation of Matrices

Step 2: Find the Determinant of A

$$|A| = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

Manipulation of Matrices

Step 3: Replace columns to solve for x, y and z

Solve for x.

Replace the first column of A with the constants vector, while keeping the remaining columns the same.

Then calculate x as:

$$x = \frac{|A_x|}{|A|}$$

$$A_x = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}$$

Compute the determinant of A_x

$$|A_x| = d_1(b_2c_3 - b_3c_2) - b_1(d_2c_3 - c_2d_3) + c_1(d_2b_3 - b_2d_3)$$

Manipulation of Matrices

Step 3: Replace columns to solve for x, y and z

Solve for y.

Replace the middle column of A with the constants vector, while keeping the remaining columns the same.

Then calculate y as:

$$y = \frac{|A_y|}{|A|}$$

$$A_y = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}$$

Compute the determinant of A_x

$$|A_y| = a_1(d_2c_3 - d_3c_2) - d_1(a_2c_3 - c_2a_3) + c_1(a_2d_3 - d_2a_3)$$

Manipulation of Matrices

Step 3: Replace columns to solve for x, y and z

Solve for z.

Replace the final column of A with the constants vector, while keeping the remaining columns the same.

Then calculate z as:

$$z = \frac{|A_z|}{|A|}$$

$$A_z = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$$

Compute the determinant of A_z

$$|A_z| = a_1(b_2d_3 - b_3d_2) - b_1(a_2d_3 - d_2a_3) + d_1(a_2b_3 - b_2a_3)$$

Manipulation of Matrices

Example:

Solve:

$$\begin{array}{l} x + 3y + z = 8 \\ 2x + y + 3z = 7 \\ x + y - z = 2 \end{array} \Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|A| = 1((1 \times -1) - (1 \times 3)) - 3((2 \times -1) - (3 \times 1)) + 1((2 \times 1) - (1 \times 1))$$

$$|A| = -4 + 15 + 1 = 12$$

Manipulation of Matrices

Example:

$$A_x = \begin{bmatrix} 8 & 3 & 1 \\ 7 & 1 & 3 \\ 2 & 1 & -1 \end{bmatrix}$$

$$|A_x| = 8((1 \times -1) - (1 \times 3)) - 3((7 \times -1) - (3 \times 2)) + 1((7 \times 1) - (1 \times 2))$$

$$|A_x| = -32 + 39 + 5 = 12$$

$$x = \frac{|A_x|}{|A|} = \frac{12}{12} = 1$$

Manipulation of Matrices

Example:

$$A_Y = \begin{bmatrix} 1 & 8 & 1 \\ 2 & 7 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$|A_Y| = 1((7 \times -1) - (2 \times 3)) - 8((2 \times -1) - (3 \times 1)) + 1((2 \times 2) - (7 \times 1))$$

$$|A_Y| = -13 + 40 - 3 = 24$$

$$Y = \frac{|A_Y|}{|A|} = \frac{24}{12} = 2$$

Manipulation of Matrices

Example:

$$A_Z = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 1 & 7 \\ 1 & 1 & 2 \end{bmatrix}$$

$$|A_Z| = 1((1 \times 2) - (1 \times 7)) - 3((2 \times 2) - (7 \times 1)) + 8((2 \times 1) - (1 \times 1))$$

$$|A_Z| = -5 + 9 + 8 = 12$$

$$Z = \frac{|A_Z|}{|A|} = \frac{12}{12} = 1$$

Manipulation of Matrices

Example:

Solution:

$x=1$, $y=2$ and $z=1$