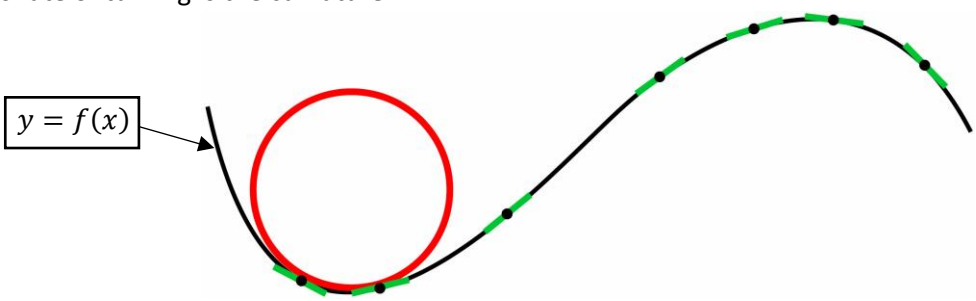
Slope and Deflection of Beams

In practically all engineering applications limitations are placed upon the performance and behaviour of components and normally they are expected to operate within certain limits of, for example, stress or deflection. The stress limits are normally set so that the component does not yield or fail under the most severe load conditions which it is likely to meet in service. In certain structural or machine linkage designs, however, maximum stress levels may not be the most severe condition for the component in question. In such cases it is the limitation in the maximum deflection which places the most severe restriction on the operation or design of the component. It is evident, therefore, that methods are required to accurately predict the deflection of members under lateral loads since it is those form of loading which will generally produce the greatest deflection of beams.

I. Relationship between loading, shear force, bending moment, slope and deflection.

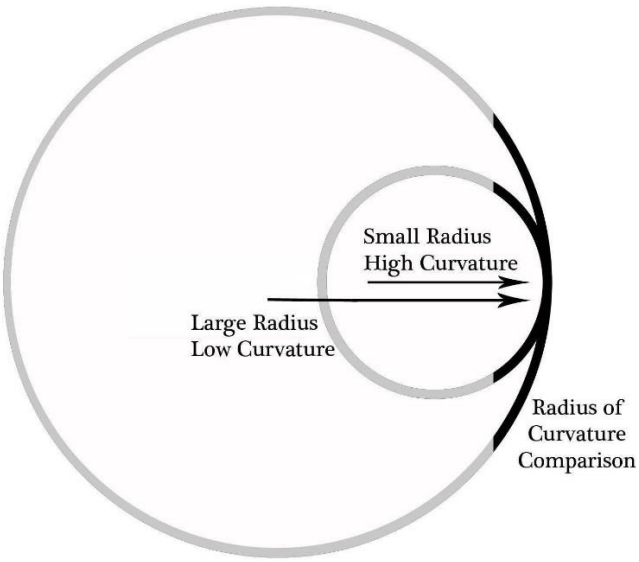
At each point on a curve, with equation $y = f(x)$, the tangent line turns at a certain rate. A measure of this rate of turning is the curvature K.



From analytical geometry, it can be demonstrated that the curvature K can be expressed as follows,

$$\mathrm{K} = \left| \frac{f''(x)}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}} \right|$$

Also, it can be considered that the curvature of a circle of radius R should be large if R is small and small if R is large.



Thus the curvature of a circle is defined to be the reciprocal of the radius, that is,

$$\mathrm{K} = \frac{1}{R}$$

Therefore,

$$\mathrm{K} = \frac{1}{R} = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}$$

For small deflection dy/dx is small so that $(dy/dx)^2$ is negligibly small compared with unity. Therefore the above curvature equation reduces to,

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

According to the Simple Bending Equation,

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \quad \text{then} \quad \frac{1}{R} = \frac{M}{EI}$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

That is,

$$\boxed{EI\frac{d^2y}{dx^2} = M}$$

This is the differential equation for the deflection of beams.

And since $Q = dM/dx$ we get,

$$Q = EI\frac{d^3y}{dx^3}$$

And since $w = dQ/dx$ we get,

$$w = EI\frac{d^4y}{dx^4}$$

<u>The results can then be summarised as follows</u>,

$$\boldsymbol{deflection = y}$$

$$\boldsymbol{slope = \frac{dy}{dx}}$$

$$\boldsymbol{Bending\ Moment\ M(x) = EI\frac{d^2y}{dx^2}}$$

$$\boldsymbol{Shear\ Force\ Q(x) = EI\frac{d^3y}{dx^3}}$$

$$\boldsymbol{Load\ distribution\ w(x) = EI\frac{d^4y}{dx^4}}$$

In order that the above results agree algebraically, i.e. that positive slope have the normal mathematical interpretation of the positive sign and that BM and SF convention are consistent with those introduced in the bending beams chapter, it is imperative that the sign convention illustrated in the following figure be adopted.

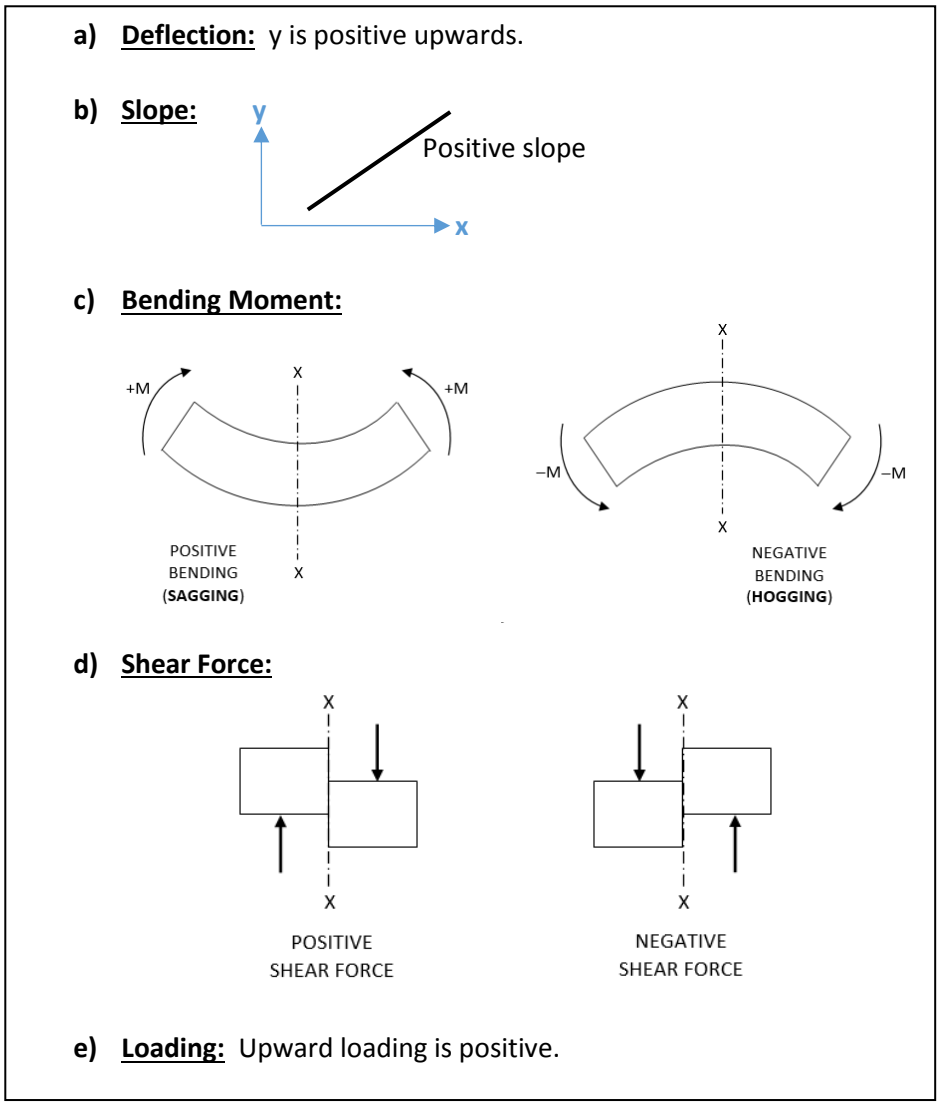


Figure: Sign conventions for load, SF, BM, slope and deflection.

II. Direct Integration method.

If the value of the BM at any point on the beam is known in terms of x, the distance along the beam, and provided that the equation ***applies along the complete beam***, the integration of the differential equation for the deflection of beams will yield slopes and deflection at any point. That is,

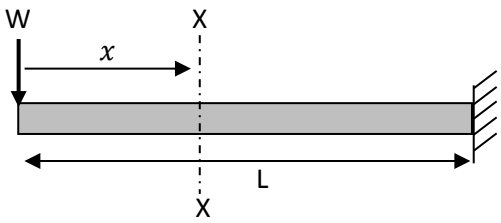
$$EI\frac{d^2y}{dx^2} = M \qquad \therefore \qquad EI\frac{dy}{dx} = \int M\,dx + A$$

$$\therefore \quad EIy = \iint (M\,dx)\,dx + Ax + B$$

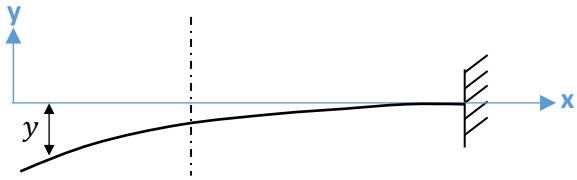
Where A and B are constants of integration evaluated from known conditions of slope and deflection for particular values of x.

1. <u>Cantilever with concentrated load at the end:</u>
   Determine the deflection curve equation and the maximum deflection for the cantilever as shown in the following figure (a).
   The load W causes the cantilever to deflect such that its neutral axis takes up the curved shape shown in figure (b).
   The axis system is chosen so that the origin coincides with the tip where the load is applied and where the deflection y is clearly maximum.

(a)



(b)



The Bending Moment M at section XX is,

$$M_{XX} = -Wx$$

Substituting for M in the differential equation for the deflection of beams we obtain,

$$EI\frac{d^2y}{dx^2} = -Wx$$

Integrating with respect to x gives,

$$EI\frac{dy}{dx} = -W\frac{x^2}{2} + A$$

Where A is a constant of integration which is obtained from a known condition of slope at a particular value of x.
At $x = L$ we know that $dx/dy = 0$, therefore,

$$EI \times 0 = -W\frac{L^2}{2} + A \qquad \therefore \qquad A = \frac{WL^2}{2}$$

Hence the differential equation for the deflection of beams after 1st integration can be written,

$$EI\frac{dy}{dx} = \frac{-W}{2}x^2 + \frac{WL^2}{2}$$

Integrating again gives,

$$EIy = \frac{-W}{2}\frac{x^3}{3} + \frac{WL^2}{2}x + B$$

Where B is a constant of integration which is obtained from a known condition of deflection at a particular value of x.
At $x = L$ we know that $y = 0$, therefore,

$$EI \times 0 = \frac{-W}{2}\frac{L^3}{3} + \frac{WL^2}{2}L + B \qquad \therefore \qquad B = \frac{WL^3}{6} - \frac{WL^3}{2} = \frac{-WL^3}{3}$$

Hence the equation of the deflection curve of the cantilever is,

$$EIy = \frac{-W}{6}x^3 + \frac{WL^2}{2}x - \frac{WL^3}{3}$$

That is,

$$\boxed{y = \frac{1}{EI}\left(\frac{-W}{6}x^3 + \frac{WL^2}{2}x - \frac{WL^3}{3}\right)}$$

Equation of the deflection curve

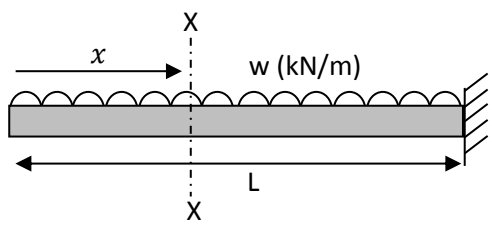
The maximum deflection y_{max} at the free end is obtained by setting $x = 0$ in the equation of the deflection curve. Thus,

$$\boxed{y_{max} = \frac{-WL^3}{3EI}}$$

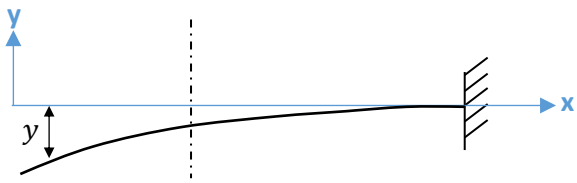
Which is clearly negative, i.e. downwards.

2. <u>Cantilever with Uniformly Distributed Load:</u>
Determine the deflection curve equation and the maximum deflection for the cantilever carrying a UDL as shown in the following figure (a).
The load w (kN/m) causes the cantilever to deflect such that its neutral axis takes up the curved shape shown in figure (b).
The axis system is chosen so that the origin coincides with the free tip and where the deflection y is clearly maximum.

(a)



(b)



The Bending Moment M at section XX is,

$$M_{XX} = -(wx)\frac{x}{2}$$

$$\therefore \quad M_{XX} = \frac{-w}{2}x^2$$

Substituting for M in the differential equation for the deflection of beams we obtain,

$$EI\frac{d^2y}{dx^2} = \frac{-w}{2}x^2$$

Integrating with respect to x gives,

$$EI\frac{dy}{dx} = \frac{-w}{2}\frac{x^3}{3} + A$$

Where A is a constant of integration which is obtained from a known condition of slope at a particular value of x.
At $x = L$ we know that $dx/dy = 0$, therefore,

$$EI \times 0 = \frac{-w}{2}\frac{L^3}{3} + A \qquad \therefore \qquad A = \frac{wL^3}{6}$$

Hence the differential equation for the deflection of beams after 1st integration can be written,

$$EI\frac{dy}{dx} = \frac{-w}{6}x^3 + \frac{wL^3}{6}$$

Integrating again gives,

$$EIy = \frac{-w}{6}\frac{x^4}{4} + \frac{wL^3}{6}x + B$$

Where B is a constant of integration which is obtained from a known condition of deflection at a particular value of x.
At $x = L$ we know that $y = 0$, therefore,

$$EI \times 0 = \frac{-w}{6}\frac{L^4}{4} + \frac{wL^3}{6}L + B \qquad \therefore \qquad B = \frac{wL^4}{24} - \frac{wL^4}{6} = \frac{-wL^4}{8}$$

Hence the equation of the deflection curve of the cantilever is,

$$EIy = \frac{-w}{24}x^4 + \frac{wL^3}{6}x - \frac{wL^4}{8}$$

That is,

$$\boxed{y = \frac{1}{EI}\left(\frac{-w}{24}x^4 + \frac{wL^3}{6}x - \frac{wL^4}{8}\right)}$$

Equation of the deflection curve

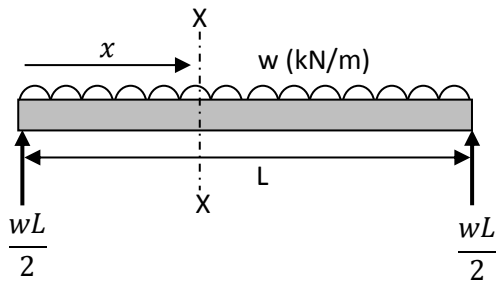
The maximum deflection y_{max} at the free end is obtained by setting $x = 0$ in the equation of the deflection curve. Thus,

$$\boxed{y_{max} = \frac{-wL^4}{8EI}}$$

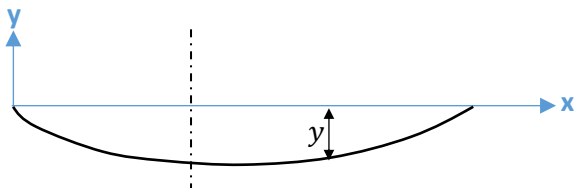
Which is clearly negative, i.e. downwards.

3. <u>Simply supported beam with UDL:</u>
Determine the deflection curve equation and the maximum deflection at mid-span for the simply supported beam carrying a UDL as shown in the following figure (a).
The load w (kN/m) causes the beam to deflect such that its neutral axis takes up the curved shape shown in figure (b).
The axis system is chosen so that the origin coincides with the left tip and where the deflection y is zero.

(a)



(b)



The Bending Moment M at section XX is,

$$M_{XX} = \frac{wL}{2}x - (wx)\frac{x}{2}$$

$$\therefore \quad M_{XX} = -\frac{w}{2}x^2 + \frac{wL}{2}x$$

Substituting for M in the differential equation for the deflection of beams we obtain,

$$EI\frac{d^2y}{dx^2} = -\frac{w}{2}x^2 + \frac{wL}{2}x$$

Integrating with respect to x gives,

$$EI\frac{dy}{dx} = \frac{-w}{2}\frac{x^3}{3} + \frac{wL}{2}\frac{x^2}{2} + A$$

Where A is a constant of integration which is obtained from a known condition of slope at a particular value of x.
At $x = L/2$ we know that $dx/dy = 0$, therefore,

$$EI \times 0 = \frac{-w}{2}\frac{\left(\frac{L}{2}\right)^3}{3} + \frac{wL}{2}\frac{\left(\frac{L}{2}\right)^2}{2} + A \qquad \therefore \quad A = \frac{wL^3}{48} - \frac{wL^3}{16} \qquad \therefore \quad A = \frac{-wL^3}{24}$$

Hence the differential equation for the deflection of beams after 1st integration can be written,

$$EI\frac{dy}{dx} = \frac{-w}{6}x^3 + \frac{wL}{4}x^2 - \frac{wL^3}{24}$$

Integrating again gives,

$$EIy = \frac{-w}{6}\frac{x^4}{4} + \frac{wL}{4}\frac{x^3}{3} - \frac{wL^3}{24}x + B$$

Where B is a constant of integration which is obtained from a known condition of deflection at a particular value of x.
At $x = 0$ we know that $y = 0$, therefore,

$$EI \times 0 = \frac{-w}{24} \times 0^4 + \frac{wL}{12} \times 0^3 - \frac{wL^3}{24} \times 0 + B \qquad \therefore \quad B = 0$$

Hence the equation of the deflection curve is,

$$EIy = \frac{-w}{24}x^4 + \frac{wL}{12}x^3 - \frac{wL^3}{24}x$$

That is,

$$\boxed{y = \frac{1}{EI}\left(\frac{-w}{24}x^4 + \frac{wL}{12}x^3 - \frac{wL^3}{24}x\right)}$$

Equation of the deflection curve

The maximum deflection y_{max} at mid-span is obtained by setting $x = L/2$ in the equation of the deflection curve. Thus,

$$y_{max} = \frac{1}{EI}\left(\frac{-w}{24}\left(\frac{L}{2}\right)^4 + \frac{wL}{12}\left(\frac{L}{2}\right)^3 - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right)$$

$$\therefore \quad y_{max} = \frac{1}{EI}\left(\frac{-wL^4}{384} + \frac{wL^4}{96} - \frac{wL^4}{48}\right)$$

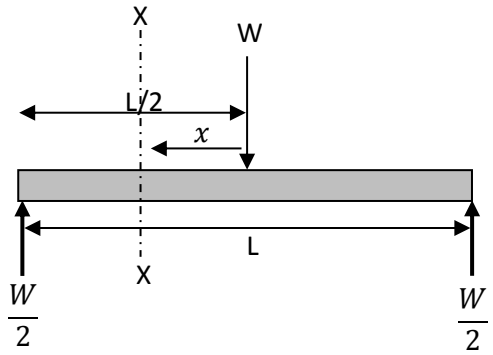
$$\therefore \quad y_{max} = \frac{1}{EI}\left(\frac{-wL^4}{384} + \frac{4wL^4}{384} - \frac{8wL^4}{384}\right)$$

$$\boxed{y_{max} = \frac{-5wL^4}{384EI}}$$

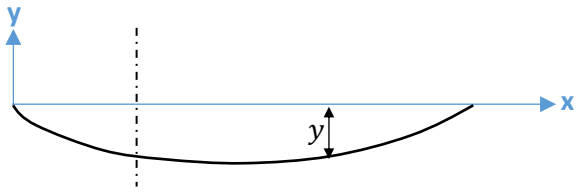
Which is clearly negative, i.e. downwards.

4. <u>Simply supported beam with central concentrated load:</u>
Determine the deflection curve equation and the maximum deflection at mid-span for the simply supported beam carrying a central concentrated load as shown in the following figure (a).
The load W causes the beam to deflect such that its neutral axis takes up the curved shape shown in figure (b).

(a)



(b)



In order to obtain a single expression for the Bending Moment which will apply across the complete beam, in this case it is convenient to take the origin for x at the centre.

The Bending Moment M at section XX is,

$$M_{XX} = \frac{W}{2}\left(\frac{L}{2} - x\right)$$

$$\therefore \quad M_{XX} = \frac{WL}{4} - \frac{W}{2}x$$

Substituting for M in the differential equation for the deflection of beams we obtain,

$$EI\frac{d^2y}{dx^2} = -\frac{W}{2}x + \frac{WL}{4}$$

Integrating with respect to x gives,

$$EI\frac{dy}{dx} = \frac{-W}{2}\frac{x^2}{2} + \frac{WL}{4}x + A$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{-W}{4}x^2 + \frac{WL}{4}x + A$$

Where A is a constant of integration which is obtained from a known condition of slope at a particular value of x.
At $x = 0$ we know that $dx/dy = 0$, therefore,

$$EI \times 0 = \frac{-W}{4} \times 0^2 + \frac{WL}{4} \times 0 + A \qquad \therefore \qquad 0 = 0 + 0 + A \qquad \therefore \qquad A = 0$$

Hence the differential equation for the deflection of beams after 1st integration can be written,

$$EI\frac{dy}{dx} = \frac{-W}{4}x^2 + \frac{WL}{4}x$$

Integrating again gives,

$$EIy = \frac{-W}{4}\frac{x^3}{3} + \frac{WL}{4}\frac{x^2}{2} + B$$

$$\therefore \quad EIy = \frac{-W}{12}x^3 + \frac{WL}{8}x^2 + B$$

Where B is a constant of integration which is obtained from a known condition of deflection at a particular value of x.
At $x = L/2$ we know that $y = 0$, therefore,

$$EI \times 0 = \frac{-W}{12} \times \left(\frac{L}{2}\right)^3 + \frac{WL}{8} \times \left(\frac{L}{2}\right)^2 + B \qquad \therefore \qquad B = \frac{WL^3}{12 \times 2^3} - \frac{WL^3}{8 \times 2^2} = \frac{-WL^3}{48}$$

Hence the equation of the deflection curve is,

$$EIy = \frac{-W}{12}x^3 + \frac{WL}{8}x^2 - \frac{WL^3}{48}$$

That is,

$$\boxed{y = \frac{1}{EI}\left(\frac{-W}{12}x^3 + \frac{WL}{8}x^2 - \frac{WL^3}{48}\right)}$$

Equation of the deflection curve

The maximum deflection y_{max} at mid-span is obtained by setting $x = 0$ in the equation of the deflection curve. Thus,

$$y_{max} = \frac{1}{EI}\left(\frac{-W}{12} \times 0^3 + \frac{WL}{8} \times 0^2 - \frac{WL^3}{48}\right)$$

$$\therefore \qquad y_{max} = \frac{1}{EI}\left(-\frac{WL^3}{48}\right)$$

$$\boxed{y_{max} = \frac{-WL^3}{48EI}}$$

Which is clearly negative, i.e. downwards.

III. Macauley's Method.

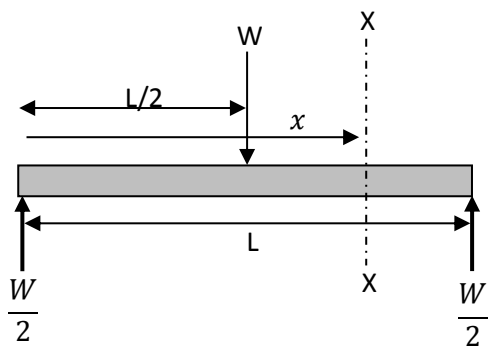
The direct integration method used in the previous examples can only be used when a single expression for the Bending Moment applies along the **complete** length of the beam. In general this is not the case, and the method has to be adapted to cover all loading conditions.

An alternative approach is to introduce so-called **singularity** or **half-range** functions. Such functions were first applied to beam deflection problems by Macauley in 1919 and hence the method is frequently known as Macauley's Method.

We now introduce a quantity $[x-a]$ and define it to be zero if $(x-a) \leq 0$, that is $x \leq a$, and to be simply $(x-a)$ if $x > a$. The quantity $[x-a]$ is known as a singularity or half-range function and is defined to have a value only when the argument is positive in which case the square brackets behave in an identical manner as ordinary parentheses.

Also the **singularity functions**, or Macauley terms, **must be integrated with respect to themselves**.

Let's consider the latter example (simply supported beam with a central concentrated load) with taking the origin at the left tip of the beam.



The Bending Moment M at section XX is,

$$M_{XX} = EI\frac{d^2y}{dx^2} = \frac{W}{2}x - W\left[x - \frac{L}{2}\right]$$

Integrating with respect to x gives,

$$EI\frac{dy}{dx} = \frac{W}{2}\frac{x^2}{2} - W\frac{\left[x-\frac{L}{2}\right]^2}{2} + A$$

$$\therefore \quad EI\frac{dy}{dx} = \frac{W}{4}x^2 - \frac{W}{2}\left[x - \frac{L}{2}\right]^2 + A$$

Where A is a constant of integration which is obtained from a known condition of slope at a particular value of x.

At $x = L/2$ we know that $dx/dy = 0$. Singularity function $\left[x - \frac{L}{2}\right]^2$ is not strictly greater than zero at $x = L/2$ and should therefore by set to zero. Hence,

$$EI \times 0 = \frac{W}{4} \times \left(\frac{L}{2}\right)^2 + A \qquad \therefore \quad 0 = \frac{WL^2}{16} + A \qquad \therefore \quad A = \frac{-WL^2}{16}$$

Hence the differential equation for the deflection of beams after 1st integration can be written,

$$EI\frac{dy}{dx} = \frac{W}{4}x^2 - \frac{W}{2}\left[x - \frac{L}{2}\right]^2 - \frac{WL^2}{16}$$

Integrating again gives,

$$EIy = \frac{W}{4}\frac{x^3}{3} - \frac{W}{2}\frac{\left[x - \frac{L}{2}\right]^3}{3} - \frac{WL^2}{16}x + B$$

$$\therefore \quad EIy = \frac{W}{12}x^3 - \frac{W}{6}\left[x - \frac{L}{2}\right]^3 - \frac{WL^2}{16}x + B$$

Where B is a constant of integration which is obtained from a known condition of deflection at a particular value of x.

At $x = 0$ we know that $y = 0$. Singularity function $\left[x - \frac{L}{2}\right]^2$ is not strictly greater than zero at $x = 0$ and should therefore by set to zero. Hence,

$$EI \times 0 = \frac{W}{12} \times (0)^3 - \frac{WL^2}{16} \times 0 + B \qquad \therefore \quad 0 = 0 - 0 + B \qquad \therefore \quad B = 0$$

Hence the equation of the deflection curve is,

$$EIy = \frac{W}{12}x^3 - \frac{W}{6}\left[x - \frac{L}{2}\right]^3 - \frac{WL^2}{16}x$$

That is,

$$\boxed{y = \frac{1}{EI}\left(\frac{W}{12}x^3 - \frac{W}{6}\left[x - \frac{L}{2}\right]^3 - \frac{WL^2}{16}x\right)}$$

Equation of the deflection curve

The maximum deflection y_{max} at mid-span is obtained by setting $x = L/2$ in the equation of the deflection curve. Singularity function $\left[x - \frac{L}{2}\right]^2$ is not strictly greater than zero at $x = L/2$ and should therefore by set to zero. Thus,

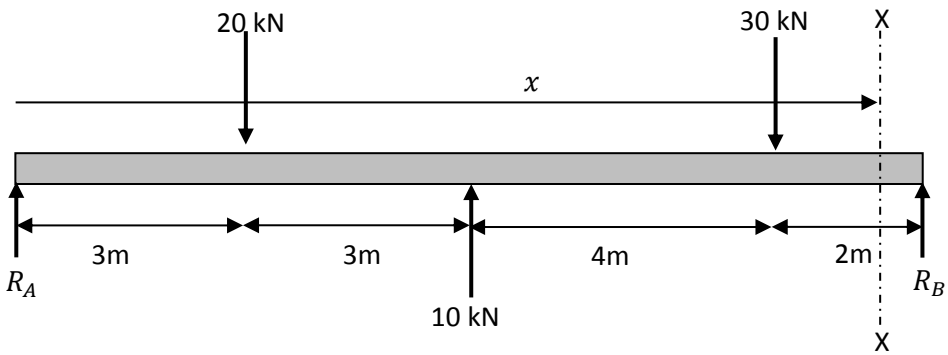
$$y_{max} = \frac{1}{EI}\left(\frac{W}{12} \times \left(\frac{L}{2}\right)^3 - \frac{WL^2}{16} \times \frac{L}{2}\right)$$

$$\therefore \quad y_{max} = \frac{1}{EI}\left(\frac{WL^3}{96} - \frac{WL^3}{32}\right) = \frac{1}{EI}\left(\frac{-WL^3}{48}\right)$$

$$\boxed{y_{max} = \frac{-WL^3}{48EI}}$$

Which is clearly negative, i.e. downwards.

1. <u>Problem:</u>
   The beam shown in the following figure is 12 m long with an E value of 208 GN/m^2 and an I value of 82×10^{-6} m^4. Determine slope and defection values in the middle of the beam.



Implementing the conditions for static equilibrium to determine support reactions R_A and R_B:

- *<u>Condition 1:</u>* the algebraic sum of forces is zero that is,

$$\begin{matrix}\text{sum of the } \boldsymbol{upward} \\ \text{component of forces}\end{matrix} = \begin{matrix}\text{sum of the } \boldsymbol{downward} \\ \text{component of forces}\end{matrix}$$

  Thus,

$$R_A + 10 + R_B = 20 + 30$$

- *<u>Condition 2:</u>* the sum of the anticlockwise moments about a point or axis equals the sum of the clockwise moments about the same point or axis. If we choose left tip point A,

$$10 \times 6 + R_B \times 12 = 20 \times 3 + 30 \times 10$$

$$\therefore \quad R_B = \frac{20 \times 3 + 30 \times 10 - 10 \times 6}{12} = 25\ kN$$

And substituting R_B into condition 1 gives,

$$R_A + 10 + 25 = 20 + 30$$

$$\therefore \quad R_A = 20 + 30 - 10 - 25 = 15\ kN$$

Taking the origin at the left hand end of the beam and writing down the bending moment at a section located at the extreme right hand part of the beam in terms of all loads to the left of that section gives,

$$M_{XX} = EI\frac{d^2y}{dx^2} = 15x - 20[x-3] + 10[x-6] - 30[x-10]$$

It must be emphasised that all loads in the right-hand side of the above bending moment equation are in units of kN (i.e. newton × 10^3). In subsequent workings, therefore, it is convenient to carry through this factor as a denominator on the left-hand side of the equation in order that the expressions are dimensionally correct.

Integrating gives,

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{15}{2}x^2 - \frac{20}{2}[x-3]^2 + \frac{10}{2}[x-6]^2 - \frac{30}{2}[x-10]^2 + A$$

Integrating a second time gives,

$$\frac{EI}{10^3}y = \frac{15}{6}x^3 - \frac{20}{6}[x-3]^3 + \frac{10}{6}[x-6]^3 - \frac{30}{6}[x-10]^3 + Ax + B$$

$$\therefore \quad \frac{EI}{10^3}y = \frac{5}{2}x^3 - \frac{10}{3}[x-3]^3 + \frac{5}{3}[x-6]^3 - 5[x-10]^3 + Ax + B$$

Determination of the constant of integration A and B:

- At $x = 0$, $y = 0$. Therefore,

$$\frac{EI}{10^3} \times 0 = \frac{5}{2} \times 0^3 + A \times 0 + B \qquad \therefore \quad 0 = 0 + 0 + B \qquad \therefore \quad B = 0$$

- At $x = 12$, $y = 0$. Therefore,

$$\frac{EI}{10^3} \times 0 = \frac{5}{2} \times 12^3 - \frac{10}{3}[12-3]^3 + \frac{5}{3}[12-6]^3 - 5[12-10]^3 + A \times 12$$

$$\therefore \quad 0 = 4320 - 2430 + 360 - 40 + A \times 12 \qquad \therefore \quad A = \frac{-2210}{12} = -184.1\dot{6}$$

Therefore the slope equation is,

$$\frac{dy}{dx} = \frac{10^3}{EI}\left(\frac{15}{2}x^2 - 10[x-3]^2 + 5[x-6]^2 - 15[x-10]^2 - 184.1\dot{6}\right)$$

And the deflection equation is,

$$y = \frac{10^3}{EI}\left(\frac{5}{2}x^3 - \frac{10}{3}[x-3]^3 + \frac{5}{3}[x-6]^3 - 5[x-10]^3 - 184.1\dot{6}\,x\right)$$

Hence the slope and defection at mid-span, that is $x = 6$, are:

$$\frac{dy}{dx} = \frac{10^3}{208 \times 10^9 \times 82 \times 10^{-6}}\left(\frac{15}{2} \times 6^2 - 10[6-3]^2 - 184.1\dot{6}\right) = -0.00024$$

and,

$$y = \frac{10^3}{208 \times 10^9 \times 82 \times 10^{-6}}\left(\frac{5}{2} \times 6^3 - \frac{10}{3}[6-3]^3 - 184.1\dot{6} \times 6\right) = -38.4\ mm$$