

WITTGENSTEIN

games. Still, as in the *Tractatus*, Wittgenstein believes that the metaphysician must be shown that he has given no meaning to certain signs in his utterances; but the way to do this, in the *Investigations*, is to show that the signs have no part to play in a language-game. Wittgenstein still denies the possibility of philosophical theses; the aim of philosophy is a therapeutic one, to cure us from talking nonsense and being tormented by problems for which there is no solution. When philosophy achieves clarity, this is not by the solution of philosophical problems but by their disappearance. 'Why is philosophy so complicated?' Wittgenstein once asked himself. His answer sums up his conception of its nature.

Philosophy unties the knots in our thinking, which we have foolishly put there; but to do that it must make movements which are just as complicated as those knots. Although the result of philosophy is simple, its method cannot be if it is to arrive at that result.

The complexity of philosophy is not in its subject matter, but in our knotted understanding.¹

1. p. 52; translated by Norman Malcolm, *The Philosophical Review*, Vol. LXXVI, p. 229.

Anthony Kern 'Wittgenstein' (1916, Penguin)

The Legacy of Frege and Russell

Wittgenstein's philosophy of language cannot be understood without some knowledge of the work of Frege and Russell and earlier logicians on which he built. This chapter will attempt to provide this background information.

The picture theory of meaning, which was sketched in the previous chapter, is a theory of the nature of the proposition. We must ask what Wittgenstein meant by this word 'proposition' (*Satz*) which he took over from a philosophical tradition. As a first approximation, 'proposition' means the same as 'sentence', and indeed Wittgenstein's German word can be translated by either English word, and was so translated by him when he was writing in English. Thus the sentences 'Caesar invaded Gaul', 'Thrushes' eggs are blue', 'Two and three make five', 'Either John will come tomorrow or he won't' would all be naturally said to be propositions. However, the command 'Polly, put the kettle on' and the wish 'O to be in England, now that April's there' and the question 'Who is Sylvia?', though perfectly good sentences, would not normally be called propositions. On the other hand, a sentence such as 'If Eclipse wins the Derby then I shall be rich' would be thought of by a philosopher as a proposition built up out of two other propositions 'Eclipse wins the Derby' and 'I shall be rich' by means of a connecting link 'if... then...'; whereas to the non-philosopher 'If Eclipse wins the Derby' would more naturally be described as a *clause*, a hypothetical clause, than as a sentence, or a sentence plus part of a connective. So it seems then that not all sentences are propositions, and not all propositions are sentences: summing up thus far, a proposition is an indicative sentence capable of standing on its own, whether or not it is actually doing so.

But there are further complications. 'Brutus killed Caesar'

WITTGENSTEIN

and 'Brutus occidit Caesarem' are two different sentences: one is an English sentence and the other is a Latin sentence. But because they mean the same in two different languages they would not be regarded as specimens of two different propositions. So in addition to the qualifications already made we must say that a proposition is a sentence considered with respect to its meaning and not (say) with respect to its sound when spoken or appearance when written.

Some philosophers have preferred to say, not that a proposition is a sentence (of a certain kind, considered in a certain way), but that a sentence expresses a proposition, which is thought of as a more abstract and less tangible entity. On this view, for example, the second law of thermodynamics is a proposition which may be expressed in many different forms and in many different languages; it always was a proposition long before there were any human beings to discover it, think of it, or talk about it.¹ Throughout his life Wittgenstein was opposed to such a conception of the proposition. The philosophy of language, he insisted in the *Philosophical Investigations*, deals with the spatial and temporal phenomenon of language, not with some fiction outside space and time. But the philosopher talks about words and sentences 'as we talk about the chess pieces when we are stating the rules of the game, not describing their physical properties' (p11, 108). What he says here fits his usage also in his earlier writings.

When he wrote the *Tractatus*, though he did not speak of sentences as expressing propositions, he did speak of sentences, and propositions, as expressing thoughts (TLP 3.2). This was indeed one traditional function ascribed to propositions ever since Aristotle, in the *De interpretatione*, wrote that 'spoken sounds are symbols of affections in the soul' (16a3). The content of a man's thoughts or beliefs is capable of expression in sentences: for instance, if James believes that revolution is inevitable, then James's thoughts can be expressed by the

1. Cf. W. and M. Kneale, *The Development of Logic*, Oxford University Press, 1962, p. 361.

THE LEGACY OF FREGE AND RUSSELL

sentence 'revolution is inevitable.' Some people have believed that there are thoughts which are inexpressible in words. Whether this is so or not – and this is a point to which we will have to return – it is clear that the great majority of thoughts are expressible in language; and so a study of the nature of language will also throw light on the nature of thought. The philosopher of language studies both at the same time. Thus Wittgenstein sums up the aim of the *Tractatus*: 'The aim of the book is to set a limit to thought – or rather, not to thought, but to the expression of thoughts' (TLP Preface). The philosophy of the *Tractatus* aims to show what is thinkable by showing what is sayable, to mark the limits of thought by setting the limits of language.

Three features of propositions had interested logicians since the time of Plato and Aristotle: first, that they were composed of subject and predicate; secondly, that (unlike questions and commands) they were capable of being true or false; thirdly, that they stood in relationships to each other, so that from one proposition it was sometimes possible to infer or deduce other propositions. These interests were all present in the theory of syllogism, the study of a certain form of argument which was for centuries regarded as the main business of the logician. Consider, for instance, the following argument:

All platypuses are web-footed creatures
All web-footed creatures can swim
So All platypuses can swim.

Each one of these propositions has, on the face of it, a subject and a predicate (e.g. 'platypuses' or 'all platypuses' appears to be the subject of the first and third proposition, and 'can swim' appears to be the predicate of the second and third propositions). The third proposition, the conclusion of the argument, follows from, or can be inferred or deduced from, the first two propositions which constitute the premisses of the argument. What is meant by this is that if the premisses are true the conclusion must be true also, so that one can safely

pass from asserting the premisses to asserting the conclusion without fear of passing from a true assertion to a false one. I do not myself know whether either of the premisses is true; but I do know that if the premisses are both true then the conclusion is also. What I am ignorant of is a pair of facts in elementary zoology: what I know is a matter of logic. For logic is, at least in part, the study of rules governing the safety (the validity, as it is called) of inferences between propositions.

In the three decades before Wittgenstein went to Cambridge the study of logic had been revolutionized by the work of Frege. Frege formalized the theory of inference in a way which was at once more rigorous and more general in its application than the traditional syllogistic. To do so he invented a symbolic system or 'concept-script' (*Begriffsschrift*), as he called it, to formalize ordinary language on the model of the language of arithmetic.

One feature in which this system differed from traditional logic was that it rejected the normal grammatical distinction between subject and predicate. The subject of 'Caesar conquered Gaul' is not the same as the subject of 'Gaul was conquered by Caesar'; yet whatever inferences can be drawn from the first proposition can be drawn from the second; so the difference between them is irrelevant to the theory of inference.

In Frege's system the grammatical distinction between subject and predicate is replaced by a distinction between function and argument, a distinction drawn from the terminology of mathematics.

An algebraic expression such as ' $x^2 + 1$ ' may be said to represent a function of x : the value of the number represented by the whole expression will vary in accordance with the numeral we substitute for the variable ' x ' in the expression. Thus the value of the function $x^2 + 1$ for the argument 1 is 2, for the argument 2 is 5, since

$$\begin{aligned} 1^2 + 1 &= 1 + 1 = 2 \\ 2^2 + 1 &= 4 + 1 = 5. \end{aligned}$$

22

An expression such as ' $x^2 + 1$ ' represents a function with a single argument. This is shown by the expression's containing only a single variable, ' x '. There are also functions of two or more arguments, such as the functions represented by ' $x^2 + y$ ' and ' $2xy + z^3$ '. In each case the function will have a specific value only if all its argument places are completed, and the value will normally differ for different arguments.

Frege made two innovations: he extended the notion of function to equations as well as to expressions for numbers (e.g. to ' $x^2 = 1$ ' as well as to ' $x^2 + 1$ ') and he extended it to expressions in ordinary language as well as to expressions in mathematical notation. Thus he called 'the capital of x ' the expression of a function, which takes London as its value if Britain is supplied as the argument. And a sentence such as 'Caesar conquered Gaul' he regarded as signifying the result either of completing the function

x conquered Gaul

with the argument Caesar, or of completing the function

Caesar conquered x

with the argument Gaul, or of completing the function

x conquered y

with the arguments Caesar and Gaul in the appropriate places (Frege GB I-3, 21ff.). The incomplete expressions signifying the functions can now be called 'predicates' in a logical, as opposed to a grammatical, sense.

The value of the function $x^2 + 1$ for any given argument is a number; the value of the function signified by 'the capital of x ' for any given argument of the appropriate kind is a city. What is the value of functions signified by e.g. ' $x = 1$ ' and ' x conquered Gaul'? Frege's answer is ingenious.

Take the expression ' $x^2 = 1$ '. If we replace x successively by $-1, 0, 1, 2$, we get:

23

WITTGENSTEIN

$$\begin{aligned} (-1)^2 &= 1 \\ 0^2 &= 1 \\ 1^2 &= 1 \\ 2^2 &= 1 \end{aligned}$$

Of these equations the first and third are true, the others false. Frege called the truth or falsehood of an equation its 'truth-value' - and this truth-value, he suggested, should be regarded as the value of the function for the different arguments.

I now say: 'the value of our function is a truth-value' and distinguish between the truth-values of what is true and what is false. I call the first, for short, the True; and the second, the False. Consequently, e.g. '2' stands for the True, as say, '2' stands for 4. And '2 = 1' stands for the False (as 'Function and Concept', 29).

Similarly, for Frege, 'Caesar conquered Gaul' stood for the True, and 'Pompey conquered Gaul' stood for the False; the values of the function *x conquered Gaul* for various arguments was always a truth-value.

The analogy which Frege drew between sentences and mathematical expressions has proved a fruitful one in logic; but few logicians or philosophers have been prepared to follow him in this step of regarding sentences as names for objects which can be designated as 'The True' and 'The False'. Wittgenstein, in particular, rejected this; but he followed Frege in replacing the traditional notion of subject and predicate with the concepts of function and argument (TLP 3.333, etc.), and in calling the truth or falsehood of each proposition its 'truth-value' (TLP 4.063, etc.).

In the traditional syllogistic the validity of inferences depended on making the appropriate substitutions in the subject and predicate places in certain permitted patterns of argument, for instance:

No M is P
And
All S is M
So
No S is P.

24

THE LEGACY OF FREGE AND RUSSELL

Frege placed in the centre of logical theory types of inference that did not depend upon breaking up sentences into subject and predicate, or even into function and argument, but which treated them as whole units. The inference 'If birds can fly, then birds have wings; but birds can fly; therefore birds have wings' for instance is of the form

If *p*, then *q*; but *p*; therefore *q*.

Any inference of this form is clearly valid provided we substitute the same proposition for each '*p*' and for each '*q*' respectively; it does not matter what the propositions are about, or whether '*p*' and '*q*' have their subject or predicate in common.

The area of logic in which the internal complexity of propositions is thus ignored is known nowadays as the *propositional calculus*, in contrast to the *predicate calculus* or *functional calculus* which takes into account the manner in which propositions are constructed out of their parts. In the propositional calculus when set out formally two kinds of symbols appear: *variables* (e.g. '*p*', '*q*' and '*r*') to mark places for propositions and *constants* to correspond to the conjunctions such as 'and', 'or', 'if . . . then' which build up complex propositions out of simpler ones, and to the sign of negation ('not' or 'it is not the case that . . .') which will form the contradictory of any proposition.

Though the propositional calculus is only a small part of logic, there are many arguments which can be formalized within it. For instance:

Either the butler killed him or the parlourmaid killed him
The parlourmaid did not kill him
Therefore
The butler killed him

is a valid argument, and its validity has nothing to do with butlers or parlourmaids or murders, but simply with the

25

WITTGENSTEIN

meaning of 'either', 'or' and 'not'. The argument is valid because it is a logical truth that if either p or q , and not q , then p . And indeed to every valid argument of the form 'A, B, therefore C' there corresponds a logical truth of the form 'If A and B, then C' in the manner just illustrated. Frege systematized propositional logic not by systematizing directly the arguments which occur in it, but by systematizing the logical truths which correspond to the valid inferences. He showed that all these logical truths could be set out in an axiomatic system like Euclidean geometry: that is to say, given a handful of such truths as postulates or axioms and the single pattern of inference 'From "A", and "If A then B" infer "B"', one could derive an unlimited number of other logical truths as theorems. Frege himself in his *Begriffsschrift* chose half-a-dozen axioms including 'If p , then not not p ' and 'if p then q , then if not q , then not p '; none of the axioms contain any connectives except 'not' and 'if... then', and the others are defined in terms of these (e.g. 'either p or q ' is defined as 'if not p then q '). Many other sets of axioms are possible and have been studied by later logicians; one of the most famous was a set used by Russell and Whitehead in *Principia Mathematica* in 1910 which used 'or' and 'not' as primitive constants instead of 'if' and 'not'.

By itself the propositional calculus is not rich enough to symbolize the obviously valid argument

Therefore
All men are mortal
Socrates is a man
Socrates is mortal.

This can of course be symbolized as ' p, q , therefore r ', but this symbolism will not bring out the validity of the inference, since 'Grass is green, lavender is blue, so chalk is cheese' is of exactly the same form.

However, it had often been noticed that there was an analogy between the inference set out above and the following inference of a slightly different pattern

THE LEGACY OF FREGE AND RUSSELL

If Socrates is a man, then Socrates is mortal
Socrates is a man
Therefore Socrates is mortal.

This second inference is a valid inference in the propositional calculus, of the familiar form 'If p then q ; but p , therefore q '. But it does not yet seem to be quite a fair translation of the first inference, since it makes the first premiss look as if it states something about Socrates in particular, whereas if 'All men are mortal' is true then

if x is a man, then x is mortal

will be true no matter whose name is substituted for the variable ' x '. Indeed it will not be false even if we substitute the name of a non-man for x , since in that case the antecedent 'if x is a man' will not be fulfilled. Using Frege's terminology of argument and function, we might say that the function signified by

If x is a man, then x is mortal

is true for every argument, or more briefly, true for all x . Thus we can express the traditional subject-predicate proposition

All men are mortal

in a more or less Fregean notation as a universal proposition in this way:

For all x , if x is a man, x is mortal.

This means: 'Take anything you like: whatever thing you take, if it is a man, it is mortal.' The truth that if Socrates is a man then Socrates is mortal will then be an instance of this more general truth, and in a full formalization of the argument to prove his mortality would be set out as such.

Frege adopted for particular propositions (e.g. 'Some men are mortal') a device parallel to his treatment of universal

WITTGENSTEIN

propositions. If some men are mortal, then there will be some argument for which the function

x is a man and x is mortal

will take the truth-value *true*; so we may translate 'Some men are mortal'

For some x , x is a man and x is mortal.

The two expressions 'for some x ' and 'for all x ' are nowadays called *quantifiers*, and play an important part in logic.

In traditional logic it had long been observed that 'some' – understood as meaning 'at least one' – could be translated into 'all' and 'not': thus 'Some men are mortal' is equivalent to 'Not all men are not mortal.' Frege made use of this in his system, and took the universal quantifier 'for all x ' as being primitive, and the particular quantifier 'for some x ' as corresponding to the expression 'It is not the case that for all x it is not the case that . . .'. Thus, by adding to his propositional calculus the universal quantifier, with rules for its use, Frege was able to develop a complete system of logic covering all, and much more than, the ground covered by the traditional syllogistic logic.

Frege introduced into logic from mathematics the sign of equality ' $=$ ', and used it to mean 'is identical with'. In this connection he met with a philosophical problem. Consider one of Frege's examples, which has become a cliché.

(A) The Morning Star is identical with the Evening Star. If this is true, then it seems that what 'the Morning Star' means is the same as what 'the Evening Star' means. But if so, then it seems that in any sentence we can substitute one expression for the other without altering the meaning of the whole. Suppose we do this to the proposition set out above, we get:

(B) The Morning Star is identical with the Morning Star. But (B) does not seem to mean the same as (A) at all; for what

THE LEGACY OF FREGE AND RUSSELL

(A) enunciates was an astronomical discovery, while what (B) seems to say is an empty truism. Frege suggested, to solve this puzzle, a distinction between two sorts of meaning: sense (*Sinn*) and reference (*Bedeutung*). 'The Morning Star' means the same as 'the Evening Star' in that both expressions *refer* to the same thing, they both denote or stand for the planet Venus. But the two expressions mean different things in that they both have a different *sense*, e.g. they would be paraphrased in different ways. ' $A=B$ ' will be a truism if ' A ' and ' B ' have the same sense, but not necessarily so if they have the same reference. Frege seems to have believed that all expressions had both a sense and a reference: a proper name like 'Aristotle', for instance, though it might not have an agreed paraphrase like the common noun 'planet', would have a sense which might vary from user to user (e.g. for one man it might mean 'Plato's brightest pupil', for another 'Alexander's tutor').

Frege's logical work is constantly presupposed in Wittgenstein's writings. Like Frege, Wittgenstein allows a fundamental role in logic to the propositional connectives 'if . . . then', 'not', 'and', 'or', and to the quantifiers. Along with the sign of identity, he called these 'the logical constants' (an expression reserved by other writers for the propositional connectives) and much of his early work could be summed up as an investigation into the functioning of the logical constants. If he could get clear about their meaning, he felt, the nature of logic, and with it the nature of language, would become clear. Like Frege, Wittgenstein made technical use of the terms 'sense' and 'reference', though he disagreed in several respects with Frege's theory of meaning. Admiration for Frege's mingled with criticism of his views in many places in the *Tractatus*.

Wittgenstein invented one formal device which passed, like so many of Frege's inventions, into the logical textbooks (TLP 4.31-4.45, 5.101). It is possible to define the propositional connectives by setting out in a table the truth-conditions of propositions containing them (TLP 4.31). Thus the table

WITTGENSTEIN

p	q	p and q
T	T	T
F	T	F
T	F	F
F	F	F

represents that ' p and q ' is true in the case in which ' p ' and ' q ' are both true, and false if (a) ' p ' is false and ' q ' is true (b) ' p ' is true and ' q ' is false (c) ' p ' and ' q ' are both false. The truth-value of ' p and q ', as this table brings out, is determined unambiguously by the truth-values of the propositions ' p ' and ' q ' out of which it is compounded. Using once again the language of function, we can say that the truth-value of the compound proposition is a function of the truth-values of its constituents, or that the compound proposition is a *truth-function* of its constituents. Putting it the other way round, the possible combinations of the truth-values of the constituents are the *truth-conditions* of the truth and falsehood of the compound propositions (TLP 4.41).

Like ' p and q ', a proposition of the form ' p or q ' is a truth-function of its components; and this is so whether it is interpreted as an exclusive disjunction (i.e. as ' p or q but not both') or as an inclusive one (i.e. as ' p and/or q '). If it is interpreted in the latter way (as it was by Frege and Wittgenstein and most logicians) then it has the following truth-table.

p	q	p or q
T	T	T
F	T	T
T	F	T
F	F	F

Propositions of great length and complexity may be built up by repeated use of connectives such as 'and' and 'or'; but, however complex they are, their truth-value can always be determined from the truth-values of the simple propositions which make them up (TLP 5.31). This is done by repeated

THE LEGACY OF FREGE AND RUSSELL

applications of the tables belonging to the particular connectives. Consider, for instance, the following propositional formula:

Either p and q , or p and not- q .

This is a truth-function of ' p ' and ' q ' as is shown in the following table.

p	q	Either p and q ,	or	p and not- q
T	T	T	T	T
F	T	F	F	F
T	F	T	T	T
F	F	F	F	F

This table is constructed in the following manner. First the columns under the single propositional variables are filled in by copying out the values given in the two left-hand columns, which represent a conventional arrangement to ensure that all possible combinations of truth-values are covered (TLP 4.31). Then in the second column from the right the truth-value of 'not- q ' is filled in under the connective 'not' by reversing the truth-value of ' q ' on the principle that 'not- q ' is false whenever ' q ' is true, and true whenever ' q ' is false. Then the columns under the 'ands' are filled in by deriving the truth-value of the conjunctions from the truth-value of the propositions the 'ands' connect in accordance with the first table given above. Finally the 'or' column is completed, the truth-values being derived from those of the two complex propositions which the 'or' connects by means of the table for 'or'. The 'or' column shows the value of the whole formula 'Either p and q , or p and not- q ' for every possible combination of truth-values of ' p ' and ' q '. It turns out, not surprisingly, to be true when ' p ' and ' q ' are both true, or when ' p ' is true and ' q ' is false, and to be false in the other possible cases. It is an expression with the same truth-value as ' p ' in all cases. This result could have been foreseen by anyone who understood the first formula; but in the case of more complicated formulae it is often impossible to see the

WITTGENSTEIN

truth-conditions in this way without making a calculation by means of a table.

When we construct the truth-tables for complex propositions in this manner, we sometimes find that they take the same truth-value for every possible truth-value of their arguments. Thus the proposition ' p or not- p ' is true whether ' p ' is true or false, as we see thus:

p	p or not- p
T	T
F	T

On the other hand, the proposition ' p and not- p ' is false whatever ' p ' may be:

p	p and not- p
T	F
F	T

A proposition which is true for all truth-possibilities of its elementary propositions is called a *tautology*, a proposition which is false for all truth-possibilities is called a *contradiction* (TLP 4.46). It can be seen that the negation of a contradiction (since it will turn all the 'F's of the contradiction into 'T's) will be a tautology. Thus 'Not both p and not- p ', the negation of the contradiction 'Both p and not- p ', is a tautology, as can be seen in this table.

p	Not both p and not- p
T	T
F	T

The study of tautologies proves to be of great importance in logic. The tautology 'Either p or not- p ' corresponds to the traditional Law of Excluded Middle; the tautology 'Not both p and not- p ' corresponds to the traditional Law of Contradiction; and these are two of the three traditional Laws of Thought. Wittgenstein believed, as we shall see, that all the propositions of logic were tautologies. If that is true, or

THE LEGACY OF FREGE AND RUSSELL

even only an approximation to the truth, it is clearly of great advantage to have a mechanical method, such as Wittgenstein's truth-tables provide, of testing to see whether a proposition is a tautology.¹

It can be shown that all formulae of the propositional calculus which are tautologous by Wittgenstein's test are either axioms or theorems of the system of Frege's *Begriffsschrift*; and conversely, that anything which can be proved from Frege's axioms by his rules will be tautologous by Wittgenstein's test. Wittgenstein's truth-table method, therefore, and Frege's axiomatic system may be regarded as two different formal methods of handling the same material, namely, the logical truths of the propositional calculus. However, there are important differences between the methods, and when he wrote the *Tractatus* Wittgenstein thought his method superior in several ways.

First, in Frege's system a number of formulae are given a privileged status as axioms; but the choice of axioms is to some extent arbitrary, as is shown by the possibility of alternative equally consistent and powerful axiom-sets, such as that used by Russell and Whitehead. The method of truth-table testing is applied to all formulae alike, so that all logical truths are seen to be of equal rank, with none essentially primitive and undervivable (TLP 6.127).

Moreover, the primitive propositions are presented in Frege's system without justification as self-evident; Wittgenstein distrusted the appeal to self-evidence, and offered instead a method of calculation which was mechanical in the literal sense that it could be carried out by a machine (NB 3; TLP 6.1271).

Again, given a formula of the propositional calculus, we can always settle, by applying the truth-table method, whether or not it is a tautology. No similar method exists in Frege's system. By discovering a proof of the formula from Frege's axioms we can show it to be a theorem of his system; but if we

1. An alternative method is given by Wittgenstein at 6.1203.

WITTGENSTEIN

fail to discover a proof this does not show that it is not a theorem of the system; it may simply be that we have not been ingenious enough to find a proof. Thus if we are given a formula of the propositional calculus and asked 'Is this a tautology or not?' Wittgenstein's method offers us a foolproof way of settling the question either way, while Frege's method offers a way of answering 'yes' with certainty, but no way of giving a certain answer 'no'. To sum up in the customary technical expression, Wittgenstein offers, while Frege does not, a *decision procedure* for propositional logic.

What is the relation between a formal script like Frege's and ordinary language? Frege himself compared it to the relationship between the microscope and the naked eye: by which he meant that it enabled one to discriminate between things which in ordinary language appear blurred and confused. Thus - to give a sample of Wittgenstein's choosing - the word 'is' in ordinary language signifies in three different ways, which correspond to three different variations of symbol in Frege's notation. Sometimes it appears as a copula linking subject and predicate - as in 'James is whistling' - in which case it will be absorbed, in Frege's notation, into the function-sign '... is whistling'. Sometimes it appears as the sign of identity, as in 'twice two is four', in which case it will be translated by the equals-sign, as in ' $2 \times 2 = 4$ '. Sometimes it is an expression for existence, as in 'There is a devil', in which case it will be translated by means of the particular quantifier, as 'For some x , x is a devil', where the 'is' is now simply part of the predicate '... is a devil' (TLP 3.323).

Failure to discriminate between such differences of significance can lead to confusion in our own minds about what we mean, and the drawing of false inferences. For the inferences which can be drawn from propositions of the form 'S is P' differ according to which 'is' comes in question. Can we, for instance, conclude from 'S is P' to 'P is S'? If the 'is' is the 'is' of identity, we can: four is indeed twice two. But if the 'is' is the copula we cannot: 'whistling is James' if not a poetic

THE LEGACY OF FREGE AND RUSSELL

inversion is a piece of nonsense. This shows the importance of distinguishing the copula from the sign of identity; and confusion of the copula with the sign for existence can lead to an equally nonsensical conclusion. Thus a philosopher may be struck by the fact that whatever exists can be said to be; and taking this verb as a copula, or predicate-marker, he may seek to investigate the nature of the attribute corresponding to this predicate which is applicable to everything in the world. He may be struck by the mystery of this *be-ing* which is not *being red* or *being a man*, but just pure *be-ing*. He may even deify his muddle by defining God as Pure Being.

In order to avoid such errors, Wittgenstein thought when he wrote the *Tractatus*, we need a language which does not use the same sign with different modes of signification: a language whose grammar is governed by logic, a language with a logical syntax instead of the superficial syntax of ordinary language (TLP 3.325). Frege's conceptual notation, Wittgenstein said, was only a preliminary attempt at such a language, and it failed to exclude all mistakes.

One of the distinctions which Frege failed to make was a distinction between names and descriptions. In Frege's system 'Socrates' and 'the teacher of Plato' are treated as the same kind of symbol, as being a name with a sense and a reference. Before Wittgenstein, Russell had argued that this was a mistake: a name like 'Socrates', if it was a genuine proper name, had meaning solely by having a reference; and an expression like 'the teacher of Plato' should not be called a name at all, if only because unlike a genuine name it had parts which were significant symbols in their own right. Russell's positive account of such expressions is called his *theory of definite descriptions*: it was much admired by Wittgenstein and had a considerable influence on his thought.

In *Principia Mathematica* Russell introduced the topic as follows:

Suppose we say 'The round square does not exist.' It seems plain that this is a true proposition, yet we cannot regard it as denying

the existence of a certain object called 'the round square'. For if there were such an object, it would exist: we cannot first assume that there is a certain object, and then proceed to deny that there is such an object. Whenever the grammatical subject of a proposition can be supposed not to exist without rendering the proposition meaningless, it is plain that the grammatical subject is not a proper name, i.e. not a name directly representing some object. Thus in all such cases the proposition must be capable of being so analysed that what was the grammatical subject shall have disappeared. Thus when we say 'The round square does not exist' we may, as a first attempt at such analysis, substitute 'It is false that there is an object x which is both round and square.' (*Principia Mathematica*, and ed., p. 66)

So far Russell's theory is an application of Frege's method of treating existence, as outlined above. But when it comes to sentences which consist of a definite description and a predicate, like 'The author of *Hamlet* was a genius', Russell parts company with Frege. For Frege such a sentence shares the basic form of 'Shakespeare was a genius'; for Russell it is of quite different form and has a hidden complexity. For the sentence to be true, it must be the case that one and only one individual wrote *Hamlet* (otherwise no one has the right to be described as 'the author of *Hamlet*'). So Russell proposes as an analysis of the sentence the following:

- For some x , (1) x wrote *Hamlet*
and (2) for all y , if y wrote *Hamlet*, y is identical
with x
and (3) x was a genius.

This formulation uses the first element to say that at least one individual wrote *Hamlet*, and the second to say that at most one individual wrote *Hamlet*; so between them the first two elements say that exactly one individual wrote *Hamlet*. The formulation uses the third element to go on to say that that unique individual was a genius. In the unanalysed sentence

the expression 'the author of *Hamlet*' looks like a complex name (and would have been so treated by Frege); in the analysed sentence no such expression appears, and instead we have a combination of predicates and quantifiers.

What is the point of this complicated analysis? To see this we have to consider a sentence which, unlike 'The author of *Hamlet* was a genius' is not true. Consider

- (A) The sovereign of Great Britain is male.
(B) The sovereign of the United States is male.

Neither of these sentences is true, but the reason differs in the two cases. The first sentence is clearly false because though there is a sovereign of Great Britain she is female; the second sentence is untrue because there is no such individual as the sovereign of the United States. The analysis brings out the two different ways in which such a sentence can be untrue. If (A) and (B) were analysed on the model of 'The author of *Hamlet* was a genius', then (A) would be untrue because of a falsehood affecting (3), whereas (B) would be untrue because of a falsehood affecting (1).

It will be seen that on Russell's view a sentence such as 'The sovereign of the United States is male' is not just untrue but positively false; and consequently 'It is not the case that the sovereign of the United States is male', which negates this falsehood, is true. (On the other hand 'The sovereign of the United States is not male', which asserts that there is a non-male individual who is sovereign of the United States, is, like 'The sovereign of the United States is male', positively false.) In this respect sentences containing vacuous definite descriptions differ sharply in Russell's system from sentences containing empty names, i.e. apparent names which name no objects. For Russell a would-be sentence such as 'Slawkenburgius was a genius' is not really a sentence at all, and therefore neither true nor false, since there was never anyone of whom 'Slawkenburgius' was the proper name.

Some philosophers have objected to Russell's treating a

WITTGENSTEIN

sentence such as 'The sovereign of the United States is male' as being false. Surely, they say, if there is no sovereign of the United States such a sentence is not so much false as misleading; the question its of truth-value does not arise. When I affirm 'The author of *Hamlet* was a genius' I do not assert, but rather presuppose, that one and only one individual wrote *Hamlet*.

Something of the kind is often the natural thing to say about our use of such definite descriptions in ordinary language; but Russell, like Frege, was interested in constructing a language which would be in some ways a more precise and scientific instrument than ordinary language for purposes of logic and mathematics. Both Frege and Russell regarded it as essential that such a language should contain only expressions which had a definite sense, by which they meant that all sentences in which the expressions could occur should have a truth-value. Neither therefore was prepared to let a proposition such as 'The sovereign of the United States is male' count as truth-valueless. Frege proposed to avoid such truth-value gaps by arbitrarily stipulating a reference for vacuous definite descriptions and empty names – e.g. that 'the sovereign of X' shall refer to the sovereign of X if X is a monarchy and otherwise shall refer to the number 0. Russell's analysis, according to which 'the sovereign of X' is not a referring expression at all, whether or not X is a monarchy, achieves the definiteness sought by Frege by far less artificial means.

Wittgenstein, when he wrote the *Tractatus*, accepted Frege's requirement of definiteness of sense (TLP 2.0211, 4.063, 5.4733) and Russell's method of securing this definiteness for propositions containing definite descriptions. He was interested in particular in applying and modifying Russell's theory to fit descriptions which described complex objects by enumerating their parts. 'Every statement about their constituents and into be resolved into a statement about their constituents and into the propositions that describe the complexes completely' (TLP 2.0201). Consider, for instance, the sentence

38

THE LEGACY OF FREGE AND RUSSELL

Austria-Hungary is allied to Russia.

This sentence was untrue when Wittgenstein wrote the *Tractatus*, because Austria-Hungary was at war with Russia. It is untrue now for the quite different reason that the political unit called 'Austria-Hungary' does not exist. Clearly the two possibilities of falsehood here parallel those for 'The sovereign of X is male'; and 'Austria-Hungary' can be looked on as a definite description, roughly, 'the union of Austria and Hungary'. We can adapt Russell's theory to this as follows

For some x and some y , $x = \text{Austria}$
and $y = \text{Hungary}$
and x is united to y
and x is allied to Russia
and y is allied to Russia.

Or more simply we can say that 'Austria-Hungary is allied to Russia' means: 'Austria is allied to Russia and Hungary is allied to Russia and Austria is united to Hungary' (NB 4; P1 1, 39, 60). (Here the use of the identity predicate, or of the proper name, renders superfluous the use of the uniqueness-securing clause.) As we shall see, Wittgenstein in the *Tractatus* built a great deal of metaphysics on the possibility of analysis of this kind. 'Russell's merit' he wrote 'is to have shown that the apparent logical form of a proposition need not be its real form' (TLP 4.0031).

What drove Russell and Frege to devise a more rigorous language, however, was not initially an interest in metaphysics but a desire to place mathematics on a firm logical basis. Both of them were dedicated to proving that mathematics, and especially arithmetic, was in fact a branch or extension of logic involving no special subject-matter of its own and derivable from purely logical axioms. This task was the object of Frege's major work *Die Grundgesetze der Arithmetik* (1893 and 1903) and of Russell and Whitehead's *Principia Mathematica* (1910–13). Both works define numbers as classes of classes with

39

WITTGENSTEIN

the same number of members: thus the number two is the class of pairs and the number three the class of trios. If this definition is not to be circular, it must be possible to say what is meant by two classes having the same number of members without making use of the notion of number.

This is not as difficult as it might appear. Frege observes that a waiter may know that there are as many knives as there are plates on a table without knowing how many of each there are, if he observes that there is just one knife to the right of each plate. He knows, that is to say, that the class of knives on the table has as many members as the class of plates on the table, though he does not know the number of members in either class. Two classes have as many members as each other if each member of one class can be correlated with a different member of the other class leaving none over. Two such classes are called equivalent classes. We can thus revise the definition of number given above, and say that a number is a class of classes with as many members as a given class. Thus we could define four as the class of all classes which have as many members as the class of gospel-makers. But such a definition would clearly be useless for the purposes of reducing arithmetic to logic, since the fact that there were four gospel-makers is a contingent fact which is certainly no part of logic. If Frege's and Russell's programme is to succeed, they must find, for each number, a class not only of the right size, but a class whose size is guaranteed by logic. What they did was to begin with the number zero, which they defined as the class of all classes with as many members as the class of objects which are not identical with themselves. Since there are no objects which are not identical with themselves, that class has no members; and since classes which have the same members are the same classes, there is in fact only one class which has no members. Zero is therefore the class whose only member is the class with no members, the null-class, as it is called. The fact that there is only one null-class is used in proceeding to the definition of the number one, which is defined as the class of classes with as many members as the class of null-

THE LEGACY OF FREGE AND RUSSELL

classes, or (what comes to the same thing) the class of classes with as many members as the class whose sole member is the number zero. Two can then be defined as the class of classes with as many members as the class whose members are zero and one, three as the class of classes with as many members as the class whose members are zero and one and two, and so on *ad infinitum*.

This is a very ingenious way of building up the series of natural numbers out of the merely logical notions of identity, class, class-membership, and class-equivalence. It contains, however, as Russell discovered, a fatal flaw. If we are to proceed in this way we must be able without restriction to form classes of classes, and classes of classes of classes, and so on; in other words, classes must be treated as classifiable, they must be capable of being members of other classes. Now can a class be a member of itself? Most classes are not (e.g. the class of men is not a man) but some apparently are (e.g. the class of classes is a class). It seems therefore that classes divide into two: the class of classes that are members of themselves, and the class of classes that are not members of themselves.

Consider now this second class: is it a member of itself or not? If it is a member of itself, then since it is precisely the class of classes that are *not* members of themselves, it must be not a member of itself. But, if it is not a member of itself, then it qualifies for membership of the class of classes that are not members of themselves, and therefore it is a member of itself. It seems that it must either be a member of itself or not; but whichever alternative we choose we get into trouble. This frustrating discovery is known as Russell's paradox, and it showed that something must be wrong with the permissive procedure of forming classes *ad lib*.

Russell's remedy for the trouble is known as the theory of types. The root error, he thought, lay in treating classes as randomly classifiable objects. Classes and individuals, he now thought, were of different logical types, and what can be true or false of one cannot be significantly asserted of the other.

'The class of men is a man' should be regarded, not as false, but as meaningless. What can meaningfully be said of individuals cannot meaningfully be said of classes, and what can meaningfully be said of classes cannot meaningfully be said of classes of classes, and so on through the hierarchy of logical types. If the difference of type between the different levels of the hierarchy is observed, the paradox cannot be meaningfully formulated.

But now another difficulty met Russell. Once there were prohibitions on forming classes of classes, how was he to define the series of natural numbers? He retained the definition of zero as the class whose only member is the null-class, but he now treated one as the class of all classes similar to the class whose members are (a) the members of the null-class, plus (b) any object not a member of that class. Two was treated in turn as the class of all classes similar to the class whose members are (a) the members of the class used to define one, plus (b) any object not a member of that defining class. In this way the numbers can be defined one after the other, and each number is a class of classes of individuals. But the natural-number series can be continued thus *ad infinitum* only if there is an infinite number of objects in the universe; for if there are only n individuals, then there will be no classes with $n + 1$ members, and so no cardinal number $n + 1$. Russell accepted this and therefore added to his axioms an axiom of infinity, i.e. the hypothesis that the number of objects in the universe is not finite. This hypothesis may be, as Russell thought it was, highly probable; but on the face of it it is far from being a logical truth; and the need to postulate it is therefore a sully of the purity of the original programme of deriving arithmetic from logic alone.

Wittgenstein was profoundly dissatisfied with the theory of types and the axiom of infinity, and a great part of his early work on logic, reflected in his *Notebooks*, arose from this dissatisfaction.

CHAPTER 3

The Criticism of *Principia*

'All theories of types' Wittgenstein wrote to Russell in 1913 'must be done away with by a theory of symbolism showing that what seem to be *different kinds of things* are symbolized by different kinds of symbols which *cannot* possibly be substituted in one another's places' (NB 121). During 1913 and 1914 Wittgenstein was working out such a theory of symbolism, and his earliest attempts can be seen in the 'Notes on Logic' (September 1913), the notes dictated to G. E. Moore in Norway (April 1914) and the notebook dating from the second half of 1914.

Wittgenstein came to think that what was wrong with the theory of types was that it was an attempt to say something which was unsayable. The theory of types says that certain types of symbols cannot sensibly be combined: e.g. 'The class of men is a man' is a piece of nonsense. But this itself is a piece of nonsense; or rather, it does not achieve what it tries to. For what is meant by the subject of the sentence below?

(A) 'The class of men is a man' is a piece of nonsense. If what is in quotation marks is meant to be just the sounds, or marks on the paper, then the whole sentence states at best a trivial empirical fact about arbitrary linguistic conventions, for there is nothing in that set of sounds which disqualifies it from being given a meaning. Surely there could be a language which was exactly like English, except that the sign 'class' was used as English uses the sign 'tallest'. In that case the set of signs in the quotation marks would have perfectly good sense and express a truth.

All this, however, is clearly irrelevant to Russell's purpose. What (A) is meant to be about is not the sounds, but their meaning. So shall we say rather the following: