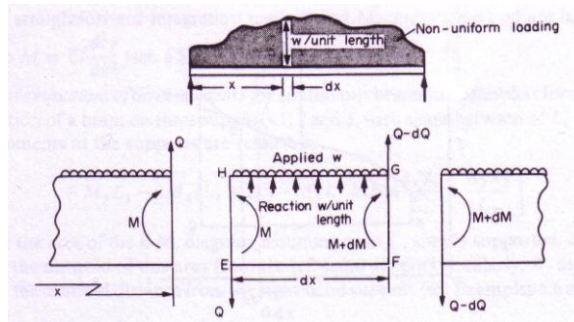


This is the basic differential equation for the deflection of beams. The term EI is called the **Flexural Stiffness**.

Now, consider a small element of a beam carrying a non-uniformly distributed load;



Considering the small element;

$$Q = wdx + Q - dQ$$

Therefore;

$$Q = \int w dx \quad [2]$$

Taking moments about F;

$$(M + dM) + wdx \frac{dx}{2} = M + Qdx$$

Therefore, neglecting the small terms;

$$M = \int Q dx \quad [3]$$

Equating equations [1] and [3], we can calculate our bending moment;

$$M = EI \frac{d^2 y}{dx^2} \quad [4]$$

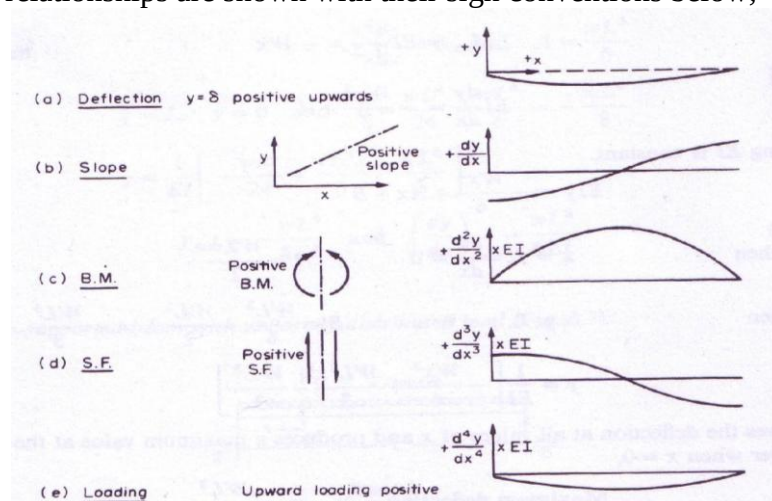
Equating equations [3] and [4] we can calculate the shear force;

$$Q = EI \frac{d^3 y}{dx^3} \quad [5]$$

Equating equations [2] and [5] we can calculate our distributed load;

$$w = EI \frac{d^4 y}{dx^4}$$

The above relationships are shown with their sign conventions below;



Direct integration Method

If the B.M. at any point on a beam is known in terms of x , the distance along the beam, and provided that the equation applies along the complete beam, then integration of equation [4] will yield slopes and deflections at any point;

$$M = EI \frac{d^2 y}{dx^2}$$

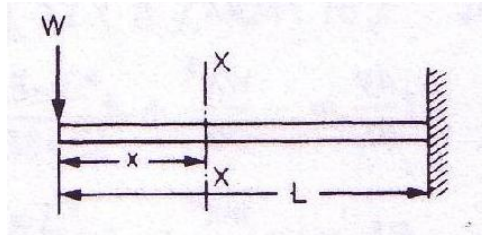
$$\frac{dy}{dx} = \int \frac{M}{EI} dx + A$$

$$y = \iint \left(\frac{M}{EI} dx \right) dx + Ax + B$$

Where A and B are constants of integration evaluated from known conditions of slope and deflection for particular values of x .

Cantilever Beam with concentrated load at the end

Consider the cantilever beam below;



The moment at XX is given by $-Wx$. Therefore;

$$-Wx = EI \frac{d^2 y}{dx^2}$$

$$EI \frac{dy}{dx} = -\frac{Wx^2}{2} + A \quad \text{and} \quad EIy = -\frac{Wx^3}{6} + Ax + B$$

Since we know that when x is L the slope is zero;

$$A = \frac{WL^2}{2}$$

We also know that when x is L the deflection is zero;

$$B = -\frac{WL^3}{3}$$

Therefore the deflection at all point on the beam is given by;

$$y = \frac{1}{EI} \left[-\frac{Wx^3}{6} + \frac{WL^2 x}{2} - \frac{WL^3}{3} \right]$$

Since the maximum deflection happens when x is zero;

$$\boxed{y_{\max} = -\frac{WL^3}{3EI}}$$

The slope is given by

$$EI \frac{dy}{dx} = -\frac{Wx^2}{2} + A$$

Therefore the slope at all point on the beam is given by;

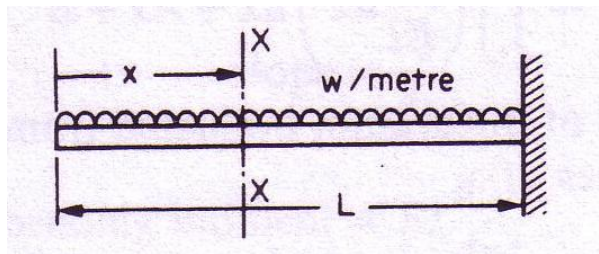
$$\frac{dy}{dx} = \frac{1}{EI} \left(-\frac{Wx^2}{2} + \frac{WL^2}{2} \right)$$

Since the maximum slope happens when x is zero;

$$\left(\frac{dy}{dx} \right)_{MAX} = \frac{WL^2}{2EI}$$

Cantilever Beam with uniformly distributed load

Consider the cantilever beam below;



The moment at XX is given by $-Wx \cdot \frac{x}{2}$ Therefore;

$$-Wx \cdot \frac{x}{2} = EI \frac{d^2y}{dx^2}$$

$$EI \frac{dy}{dx} = -\frac{Wx^3}{6} + A \quad \text{and} \quad EIy = -\frac{Wx^4}{24} + Ax + B$$

Since we know that when x is L the slope is zero;

$$A = \frac{WL^3}{6}$$

We also know that when x is L the deflection is zero;

$$B = -\frac{WL^4}{8}$$

Therefore the deflection at all point on the beam is given by;

$$y = \frac{1}{EI} \left[-\frac{Wx^4}{24} + \frac{WL^3x}{6} - \frac{WL^4}{8} \right]$$

Since the maximum deflection happens when x is zero;

$$y_{max} = \frac{-WL^4}{8EI}$$

The slope is given by

$$EI \frac{dy}{dx} = -\frac{Wx^2}{2} + A$$

Therefore the slope at all point on the beam is given by;

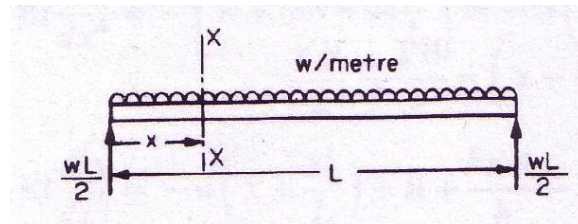
$$\frac{dy}{dx} = \frac{1}{EI} \left(-\frac{Wx^3}{6} + \frac{WL^3}{6} \right)$$

Since the maximum slope happens when x is zero;

$$\left(\frac{dy}{dx} \right)_{MAX} = \frac{WL^3}{6EI}$$

Simply Supported Beam with uniformly distributed load

Consider the beam below;



The moment at XX is given by $\frac{WL}{2}x - Wx \cdot \frac{x}{2}$ Therefore;

$$\frac{WLx}{2} - Wx \cdot \frac{x}{2} = EI \frac{d^2y}{dx^2}$$

$$EI \frac{dy}{dx} = \frac{WLx^2}{4} - \frac{Wx^3}{6} + A \quad \text{and} \quad EIy = \frac{WLx^3}{12} - \frac{Wx^4}{24} + Ax + B$$

Since we know that when x is 0 the deflection is zero;

$$B = 0$$

We also know that when x is L the deflection is zero;

$$A = -\frac{WL^3}{24}$$

Therefore the deflection at all point on the beam is given by;

$$y = \frac{1}{EI} \left[\frac{WLx^3}{12} + \frac{Wx^4}{24} - \frac{WL^3x}{24} \right]$$

Since the maximum deflection happens when $x = \frac{L}{2}$;

$$y_{max} = \frac{-5WL^4}{384EI}$$

Also, the slope at all point on the beam is given by;

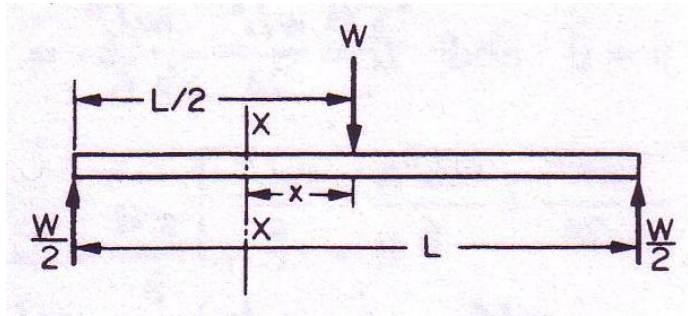
$$\frac{dy}{dx} = \frac{1}{EI} \left(\frac{WLx^2}{4} - \frac{Wx^3}{6} - \frac{WL^3}{24} \right)$$

Since the maximum slope happens when x is zero or x=L;

$$\left(\frac{dy}{dx} \right)_{MAX} = \pm \frac{WL^3}{24EI}$$

Simply Supported Beam with central concentrated load

Consider the beam below;



The moment at XX is given by $\frac{W}{2} \left(\frac{L}{2} - x \right)$ Therefore;

$$\frac{WL}{4} - \frac{Wx}{2} = EI \frac{d^2y}{dx^2}$$

$$EI \frac{dy}{dx} = \frac{WLx}{4} - \frac{Wx^2}{4} + A \quad \text{and} \quad EIy = \frac{WLx^2}{8} - \frac{Wx^3}{12} + Ax + B$$

Since we know that when x is 0 the slope is zero;

$$A = 0$$

We also know that when x is L/2 the deflection is zero;

$$B = -\frac{WL^3}{48}$$

Therefore the deflection at all point on the beam is given by;

$$y = \frac{1}{EI} \left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48} \right]$$

Since the maximum deflection happens when $x = 0$;

$$y_{max} = \frac{-WL^3}{48EI}$$

Also, the slope at all point on the beam is given by;

$$\frac{dy}{dx} = \frac{1}{EI} \left(\frac{WLx}{4} - \frac{Wx^2}{4} \right)$$

Since the maximum slope happens when $x=L/2$;

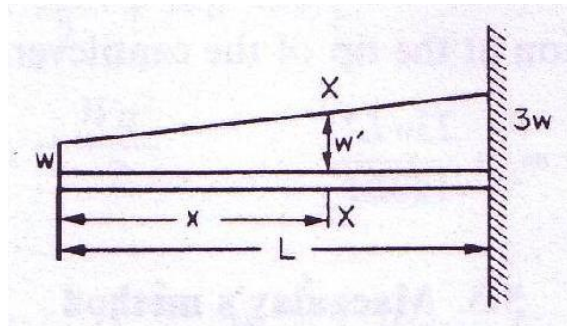
$$\left(\frac{dy}{dx} \right)_{MAX} = \pm \frac{WL^2}{16EI}$$

The theory of superposition for combined loads

This theory states that the slope and deflection of a beam at any point is the sum of the slopes and deflections which would be produced by each load acting on it's own. For beams with combinations of loads which are standard cases we only need to use the standard formulae.

Cantilever Beam with non-uniformly distributed load

In some cases if it is not convenient to commence the integration procedure with the B.M. equation since this may be difficult to obtain, it may be more convenient to commence with the equation for loading and integrate this four times getting four unknowns as shown in the below example;



By linear interpolation we can see that the load at XX is given by $- \left[W + (3W - W) \frac{x}{L} \right]$

Therefore;

$$-W \left(1 + \frac{2x}{L} \right) = EI \frac{d^4 y}{dx^4}$$

Hence we can obtain the following;

$$EI \frac{d^3 y}{dx^3} = -W \left(x + \frac{x^2}{L} \right) + A \quad [6]$$

$$EI \frac{d^2 y}{dx^2} = -W \left(\frac{x^2}{2} + \frac{x^3}{3L} \right) + Ax + B \quad [7]$$

$$EI \frac{d y}{dx} = -W \left(\frac{x^3}{6} + \frac{x^4}{12L} \right) + \frac{Ax^2}{2} + Bx + C \quad [8]$$

$$EIy = -W\left(\frac{x^4}{24} + \frac{x^5}{60L}\right) + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \quad [9]$$

Hence we have four unknowns we need to solve.

When $x=0$ the Shear Force $=0$, therefore from [6] $A=0$

When $x=0$ the Bending moment $=0$, therefore from [7] $B=0$

When $x=L$, the slope $=0$, thus from [8];

$$0 = -W\left(\frac{L^3}{6} + \frac{L^4}{12L}\right) + C$$

$$C = \frac{WL^3}{4}$$

Finally, when $x=L$ the deflection is zero, thus from [9];

$$0 = -W\left(\frac{L^4}{24} + \frac{L^5}{60L}\right) + \frac{WL^4}{4} + D$$

$$D = \frac{-23WL^4}{120}$$

So if we wanted to find the maximum deflection we could then put these constants back into equation [9];

$$y = \frac{1}{EI} \left[-W\left(\frac{x^4}{24} + \frac{x^5}{60L}\right) + \frac{WL^3x}{4} - \frac{23WL^4}{120} \right]$$

And since the maximum deflection happens when $x=0$;

$$y_{MAX} = \frac{-23WL^4}{120EI}$$

Examples

1)

(a) A uniform cantilever is 4 m long and carries a concentrated load of 40 kN at a point 3 m from the support. Determine the vertical deflection of the free end of the cantilever if $EI = 65 \text{ MN m}^2$.

(b) How would this value change if the same total load were applied but uniformly distributed over the portion of the cantilever 3 m from the support?

[-8.31mm, -3mm]

2)

A cantilever beam is 4 m long and has a point load of 5 kN at the free end. The flexural stiffness is 53.3 MNm^2 . Calculate the slope and deflection at the free end.

[-2mm, 750×10^{-6}]

3)

A cantilever beam is 4 m long and has a u.d.l. of 300 N/m. The flexural stiffness is 60 MNm^2 . Calculate the slope and deflection at the free end.

[-0.16mm, 53.3×10^{-6}]

4)

A simply supported beam is 8 m long with a load of 500 kN at the middle. The deflection at the middle is 2 mm downwards. Calculate the gradient at the ends.

[- 750×10^{-6} , 750×10^{-6}]

5)

A simply supported beam is 8 m long with a u.d.l. of 5000 N/m. Calculate the flexural stiffness which limits the deflection to 2 mm at the middle. Calculate the gradient at the ends.

[133.3 MNm^2 , -800×10^{-6} , 800×10^{-6}]

6)

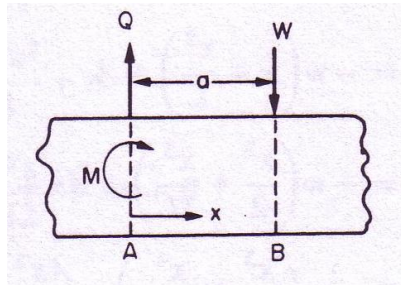
A cantilever beam is 4m long with a flexural stiffness of 20 MNm^2 . It has a point load of 1 kN at the free end and a u.d.l. of 300 N/m along its entire length. Calculate the slope and deflection at the free end.

[-1.54mm, 560×10^{-6}]

Macaulay's method

The double integration method previously shown only works if a single expression can be obtained for the Bending Moment along the complete length of the beam. Normally this is not possible and the method has to be adapted to cover all loading conditions.

Consider the below small section of a beam that has a shearing force Q , and a moment M at a particular point, A. A small distance, a , along, at point B, a concentrated load W is applied which will change the B.M. for points beyond B;



Beyond B the moment is given by;

$$EI \frac{d^2y}{dx^2} = M + Qx - W(x-a)$$

It can be shown that

$$EI \frac{dy}{dx} = Mx + \frac{Qx^2}{2} - \frac{W}{2}(x-a)^2 + C_1$$

And,

$$EIy = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{W}{6}(x-a)^3 + C_1x + C_2$$

Hence the general method of obtaining slopes and deflections will still apply provided that the term $W(x-a)$ is integrated with respect to $(x-a)$ and not x . Thus, when integrated, the term becomes

$$W \frac{(x-a)^2}{2} \quad \text{and} \quad W \frac{(x-a)^3}{6} \quad \text{etc}$$

In addition, since the term $W(x-a)$ applied only after the discontinuity, i.e. when $x > a$, it should only be considered when $x > a$ or when $(x-a)$ is positive. For these reasons such terms are conventionally put into square or curly brackets and called Macaulay terms.

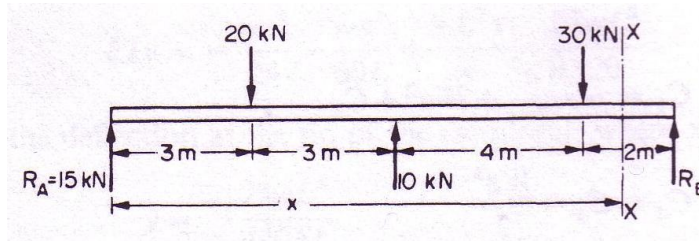
Thus Macaulay terms must be

- Integrated with respect to themselves
- Neglected when negative

For the whole beam therefore,

$$EI \frac{d^2y}{dx^2} = M + Qx - W[(x-a)]$$

Example with point loads



$$B.M._{xx} = 15x - 20[(x-3)] + 10[(x-6)] - 30[(x-10)]$$

$$\frac{EI}{1 \times 10^3} \frac{d^2 y}{dx^2} = \frac{15x^2}{2} - 20 \left[\frac{(x-3)^2}{2} \right] + 10 \left[\frac{(x-6)^2}{2} \right] - 30 \left[\frac{(x-10)^2}{2} \right] + A$$

$$\frac{EI}{1 \times 10^3} y = \frac{15x^3}{6} - 20 \left[\frac{(x-3)^3}{6} \right] + 10 \left[\frac{(x-6)^3}{6} \right] - 30 \left[\frac{(x-10)^3}{6} \right] + Ax + B$$

When $x=0$, $y=0$ therefore, ignoring any negative terms, $B=0$

When $x=12$, $y=0$, therefore;

$$0 = \frac{15 \times 12^3}{6} - 20 \left[\frac{9^3}{6} \right] + 10 \left[\frac{6^3}{6} \right] - 30 \left[\frac{2^3}{6} \right] + 12A$$

Therefore $A = -184.2$

At $x=6$

$$\frac{EI}{1 \times 10^3} y = \frac{15x^3}{6} - 20 \left[\frac{(x-3)^3}{6} \right] + 10 \left[\frac{(x-6)^3}{6} \right] - 30 \left[\frac{(x-10)^3}{6} \right] + Ax$$

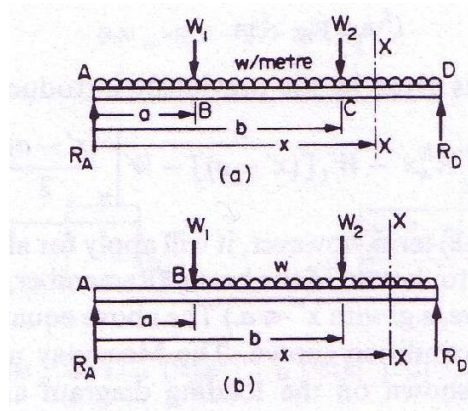
$x-10$ is negative

$$\frac{EI}{1 \times 10^3} y = \frac{15 \times 6^3}{6} - 20 \left[\frac{3^3}{6} \right] - 184 \times 6$$

$$y = \frac{-655.2 \times 10^3}{EI}$$

Thus for typical values of E (208 GN/m^2) and I (82×10^{-6}) we would get a central deflection of 38.4 mm

Examples with u.d.l.'s



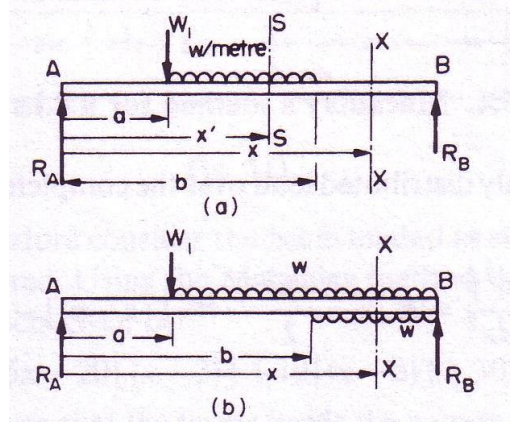
If the u.d.l. is over the entire length of the beam, as in (a), then it requires no special treatment associated with the Macaulay Method;

$$B.M._{xx} = \frac{EId^2y}{dx^2} = R_Ax - \frac{wx^2}{2} - W_1[(x-a)] - W_2[(x-b)]$$

If however the u.d.l. starts a distance (a) along the beam then the u.d.l. becomes a Macaulay term and would be integrated as such;

$$B.M._{xx} = \frac{EId^2y}{dx^2} = R_Ax - w\left[\frac{(x-a)^2}{2}\right] - W_1[(x-a)] - W_2[(x-b)]$$

Once a term has been introduced in the B.M. expression it will apply across the rest of the beam. Thus if a u.d.l. actually stops a 'dummy' u.d.l. (equal and opposite), is applied.



Thus if the beam is loaded as shown in (a), the B.M. term would represent a beam loaded as (b);

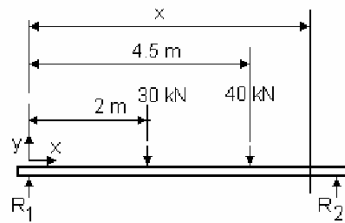
$$B.M._{xx} = R_Ax - W_1[(x-a)] - w\left[\frac{(x-a)^2}{2}\right] + w\left[\frac{(x-b)^2}{2}\right]$$

Thus Macaulay's Method can be summarised as follows;

1. Take the origin at the left hand end of the beam (usually)
2. Write down the bending moment expression at a section at the extreme right-hand end of the beam in terms of all loads to the left of that section
3. Do not simplify the bending moment expressions by expanding any of the terms involving the distance x which are contained within brackets (Macaulay brackets)
4. Integrate, keeping all bracketed terms within brackets and integrating the bracket as if it was a single variable
5. Apply the resulting equations to any part of the beam but neglect all terms within the brackets wherever they are negative or zero.
6. Where a distributed load does not extend to the right-hand end of the beam, then introduce two equal but opposite dummy loads. These two loads will have no net effect on the behaviour of the beam.

Examples

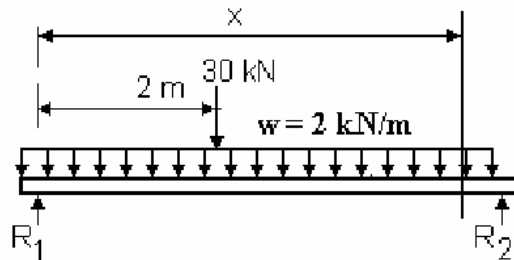
1)



The beam shown is 7 m long with an $E I$ value of 200 MNm^2 . Determine the slope and deflection at the middle.

[-2.09mm, -0.00001213]

2)

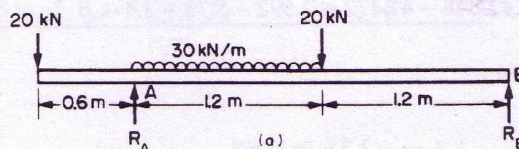


The beam shown is 6 m long with an $E I$ value of 300 MNm^2 . Determine the slope at the left end and the deflection at the middle.

[-0.534mm, -0.000282]

3)

Determine the deflection at a point 1 m from the left-hand end of the beam loaded as shown in Fig. 5.39a using Macaulay's method. $EI = 0.65 \text{ MNm}^2$.



[-5.34mm]

Notes adapted from/further reading: Mechanics of materials 1, E J Hearn