Deflection of Beams

Structural Mechanics

ACDA 6009

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Stuart Atkins 1809912

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Abstract

The purpose of the experiment was to measure beam deflection when load is applied. Two beams of different materials and properties (one of steel, the other brass) were used in the experiment and the load was increased at different positions on the beam. The measurements gathered in the practical experiment were then compared to the theoretical calculations for beam deflection.

Theoretical calculations using Macaulay's Method predicted a linear relationship between load and deflection, which was also reflected in the experimental results with a low degree of error.

There wasn't sufficient data gathered to establish the trend regarding the degree of error between the theory and practical results. It is recommended that more data is gathered in order to observe how this error is affected by load position and by load increases.

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Beam Deflection

1. Introduction

The aim of this report is to conduct a beam deflection experiment on beams of different materials and compare the practical results with theoretical results. The plan is to use Macaulay's Method, also known as the Double Integration method, to calculate the theoretic deflection results.

2. Background Theory

A load supporting element that is positioned in the horizontal plane and is subject to loads perpendicular to its length are known as beams. They are commonly used in the construction of buildings and can be classified in four main types. First, a simply supported beam which has supports at both its ends. Second, is an overhanging beam where a length of the beam is extended over its supports. Third, is an overhanging beam which has more than two supports and fourth, is a cantilever beam, which is only fixed at one end.

When load is applied on a beam, it produces a bending moment in the beam by applying shear forces across its cross section area. The beam reacts against this bending moment and shear force according to its Moment of Inertia. The Moment of inertia of a beam is dependent on its shape and shows the beam's ability to resist the bending moment applied to it.

Another factor which helps a beam to resist bending is its Modulus of Elasticity (also known as Young's Modulus). This is the material's property which defines the strain produced when a load is applied. The Moment of Inertia and Modulus of Elasticity combine to define the beam's reaction against the bending moment due to an applied load (also known as flexural rigidity). Another factor which affects this is the beam's length. The greater the length, the greater the bending moment produced due to an external load.

The deflection of a beam is calculated based on a number of factors, including the force applied, type of materials, the Moment of Inertia and the distance from support. There are a number of beam deflection equations that can be used to calculate a basic value for deflection in different types of beams.

Generally, deflection can be calculated by taking the double integral of the Bending Moment Equation, M(x), divided by EI (Young's Modulus x Moment of Inertia) (*SkyCiv, Nov 2021*).

Using Euler-Bernoulli Theory of Bending, at any point along a beam:

$$\frac{1}{R} = \frac{M}{EI}$$

Where:

- R is the radius of the shape of the curved beam at a distance x from the origin, normally taken at the left or right hand end of the beam, and 1/R is the curvature.

- M is the bending moment at the point.
- E is the Young's Modulus of the beam material.
- I is the Second Moment of Area of the beam's cross-section (also known as Moment of Inertia). The Moment of Inertia of a rectangle is given by the following expression:

$$I = \frac{bh^3}{12}$$

It is also known that:

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

Where:

- y is the deflection at the point.
- x is the distance of the point along the beam.

Hence, the fundamental equation in finding deflections is:

$$\frac{d^2y}{dx^2} = \frac{M_x}{EI_x}$$

(The subscript x shows that these are functions of x and may change along the beam's length)

Figure 1 shows a simply supported beam with a force applied at point W. Macaulay's method can be used to generate an equation to calculate beam deflection when a load is applied for different values of x.

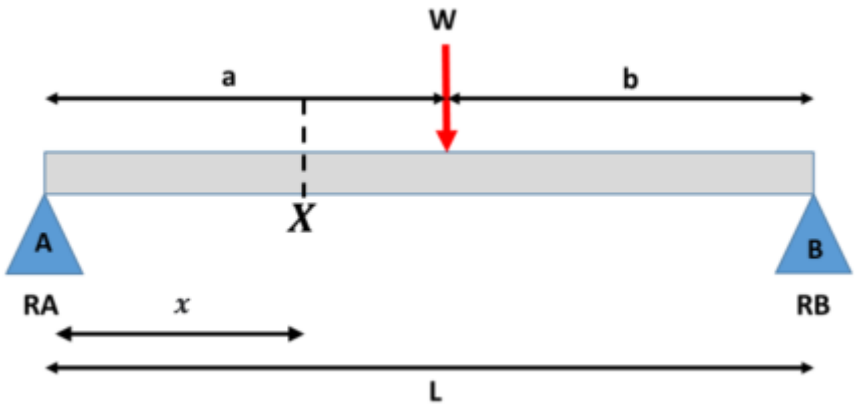

Figure 1 - Simply Supported Beam Diagram

The Bending Moment is a means of mathematically describing how much bending and deflection will occur in a beam when under a given load, and is defined as the sum of the moments of all forces to the left or right of the section under consideration.

Consider a point 'X' on the beam, which is at a distance x from the left hand support. The Bending Moment (M) can be calculated by taking the moments of all forces to the left of 'X', with the rule being that clockwise moments are taken as positive.

Therefore, the bending moment at X will be given as:

M = the clockwise moment of the support reaction – the anticlockwise moment of the load (*RAENG, no date)*

Using the terms in the *Figure 1* gives:

$$M = RA\,x - W(x - a)$$

It can also be seen in *Figure 1* that:

$$RA \times L = W \times b \quad \therefore \quad RA = \frac{Wb}{L}$$

Using the Euler-Bernoulli Theory of Bending, at any point along a beam it is known that:

$$\frac{1}{R} = \frac{M}{EI} \qquad \text{And} \qquad \frac{1}{R} = \frac{d^2y}{dx^2}$$

$$\therefore \quad \frac{1}{R} = \frac{d^2y}{dx^2} = \frac{1}{EI}\mathrm{M}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{EI}\left[RA\,x - \mathrm{W}(x - \mathrm{a})\right]$$

Equation 1

Equation 1 then needs to be integrated twice in order for the right hand side of the equation to equal y (deflection)

First order of integration;

$$\therefore \quad \frac{dy}{dx} = \frac{1}{EI}\left[\frac{RAx^2}{2} - \frac{W(x-a)^2}{2} + C\right]$$

Equation 2

Second order of integration;

$$\therefore \quad \mathrm{y} = \frac{1}{EI}\left[\frac{RAx^3}{6} - \frac{W(x-a)^3}{6} + Cx + D\right]$$

Equation 3

Boundary conditions need to be applied to work out the constants:

When x = 0, y = 0 and D = 0

When x = L, y = 0 and D = 0

This gives;

$$0 = \frac{1}{EI}\left[\frac{RAL^3}{6} - \frac{Wb^3}{6} + CL\right]$$

Equation 4

(Multiple both sides by EI)

$$0 = 1\left[\frac{RAL^3}{6} - \frac{Wb^3}{6} + CL\right]$$

(Rearrange to find CL)

$$\mathrm{CL} = \frac{Wb^3}{6} - \frac{RAL^3}{6}$$

Substituting RA = $\frac{Wb}{L}$ *which gives:*

$$\therefore \quad \mathrm{CL} = \frac{Wb^3}{6} - \frac{Wb}{L6}.L^3$$

$$\therefore \quad \mathrm{CL} = \frac{Wb^3}{6} - \frac{Wb}{6}.L^2$$

$$\therefore \quad \mathrm{C} = \frac{Wb}{6L}.(\mathrm{b}^2\text{-}\mathrm{L}^2)$$

Equation 5

Substituting $\mathrm{C} = \frac{Wb}{6L}.(\mathrm{b}^2\text{-}\mathrm{L}^2)$ *and* $RA = \frac{Wb}{L}$ *into Equation 3 gives:*

$$\mathrm{y} = \frac{1}{EI}\left[\frac{Wbx^3}{6L} - \frac{W(x-a)^3}{6} + \frac{Wb}{6L}.(\mathrm{b2} - \mathrm{L2})x\right]$$

Equation 6

If x = a (the point of which the load is applied in this example) then the expression $\frac{W(x-a)^3}{6} = 0$

Therefore the beam deflection equation using Macaulay's Method is given as:

$$y = \frac{1}{EI}\left[\frac{Wbx^3}{6L} + \frac{Wb}{6L}\,.(b2 - L2)x\right]$$

Equation 7

3. Experiment Procedure and Equipment

3.1 Equipment

The equipment required for this experiment to be conducted is listed below;

- Steel Specimen Beam x 1

  (Breadth 9.5mm, depth 3.1mm)
- Brass Specimen Beam x 1

  (Breadth 9.5mm, depth 3.1mm)
- Fixing block with knife edge and fixings x 2
- Dial Caliper Gauge x 1
- Ruler
- Dial Indicator x 1
- Weight Hanger x 1
- 10g Disc weights x 50
- Experiment frame

Figure 2 - Experiment Equipment

3.2 Material Properties

The material properties for the beams used in this experiment can be found in *Figure 3*.

Material	Young's Modulus E (Pa)
Aluminium Alloy	69×10^9 (69 GPa)
Brass	105×10^9 (105 GPa)
Mild Steel	200×10^9 (200 GPa)

Figure 3 - Material Properties

3.2 Experiment Procedure

1. Set up the equipment (as shown in *Figure 2*).
2. Select the Steel beam and note down the beam materials Modulus of Elasticity and dimensions.
3. Position the beam upon the fixing blocks and measure the beam length between the supports.
4. Select the location of the load to be applied. Initially this will be 75mm from the left hand side of the beam.
5. Use a ruler to find the location and position and zero the Dial indicator.
6. Select the number of weights required to achieve a load of 100g and place them on the hanger (total weight 110g including hanger).
7. Read the measured deflection on the Dial Indicator.
8. Increase the load, in 100g intervals, and record the measured deflection displayed on the Dial Indicator.
9. Move the Dial Indicator and load position to a point 150mm from the left hand side of the beam.
10. Repeat steps 5-8.
11. Move the Dial Indicator and load position to a point 225mm from the left hand side of the beam.
12. Repeat steps 5-8.
13. Repeat the entire procedure for Brass Specimen Beam.

4. Results

4.1 Theoretical Calculation

Using Macaulay's Method of calculation to determine the theoretical deflection for a Steel beam, loaded with 110g at position 1 (75mm from left hand side).

The known values can be input into Equation 7 in order to calculate y.

E = $\boldsymbol{200X10^9}$ ***GPa***

$$I = \frac{bh^3}{12} = \frac{\mathbf{9.5\ x\ (3.1^3)}}{\mathbf{12}}$$ ***= 23.584541 mm⁴***

$$\boldsymbol{W = F = MA = 110\ x\ 9.81 = 1.079.1}\ \boldsymbol{N}$$

$$\boldsymbol{b = 225mm}$$

$$\boldsymbol{x = 75mm}$$

$$\boldsymbol{L = 320mm}$$

$$y = \frac{1}{EI}\left[\frac{Wbx^3}{6L} + \frac{Wb}{6L}\ .(b2 - L2)x\right]$$

Equation 7

$$y = \frac{1}{(200x10^9\ x\ 23.584541)}\left[\frac{1.0791\ x\ 225\ x\ 75^3}{6\ x\ 320} + \frac{1.0791\ x\ 225}{6\ x\ 320}\ x\ (225^2 - 320^2)\ x\ 75\right]$$

$$y = \frac{1}{(200x10^9\ x\ 23.584541)}\left[58091.19873\ -\ 437620.364\right]$$

$$y = -8.05\ x\ 10^{-2}\ mm\ \ or\ 0.081mm\ (to\ 3.s.f)$$

4.2 Steel Beam Results

Table 1 below shows the experiment results and the remaining calculated results for the Steel beam deflection.

Table 1 - Steel Beam Results

	Position 1 (75mm from LHS)			Position 2 (150mm from LHS)			Position 3 (225mm from LHS)		
	Deflection (mm)			Deflection (mm)			Deflection (mm)		
Load (N)	Experimental	Theoretical	Difference	Experimental	Theoretical	Difference	Experimental	Theoretical	Difference
110	0.100	0.081	0.019	0.141	0.155	-0.014	0.090	0.109	-0.019
210	0.166	0.154	0.012	0.263	0.296	-0.033	0.190	0.208	-0.018
310	0.235	0.227	0.008	0.420	0.437	-0.017	0.272	0.307	-0.035
410	0.310	0.300	0.010	0.531	0.578	-0.047	0.370	0.406	-0.036
510	0.371	0.373	-0.002	0.649	0.718	-0.069	0.455	0.505	-0.050

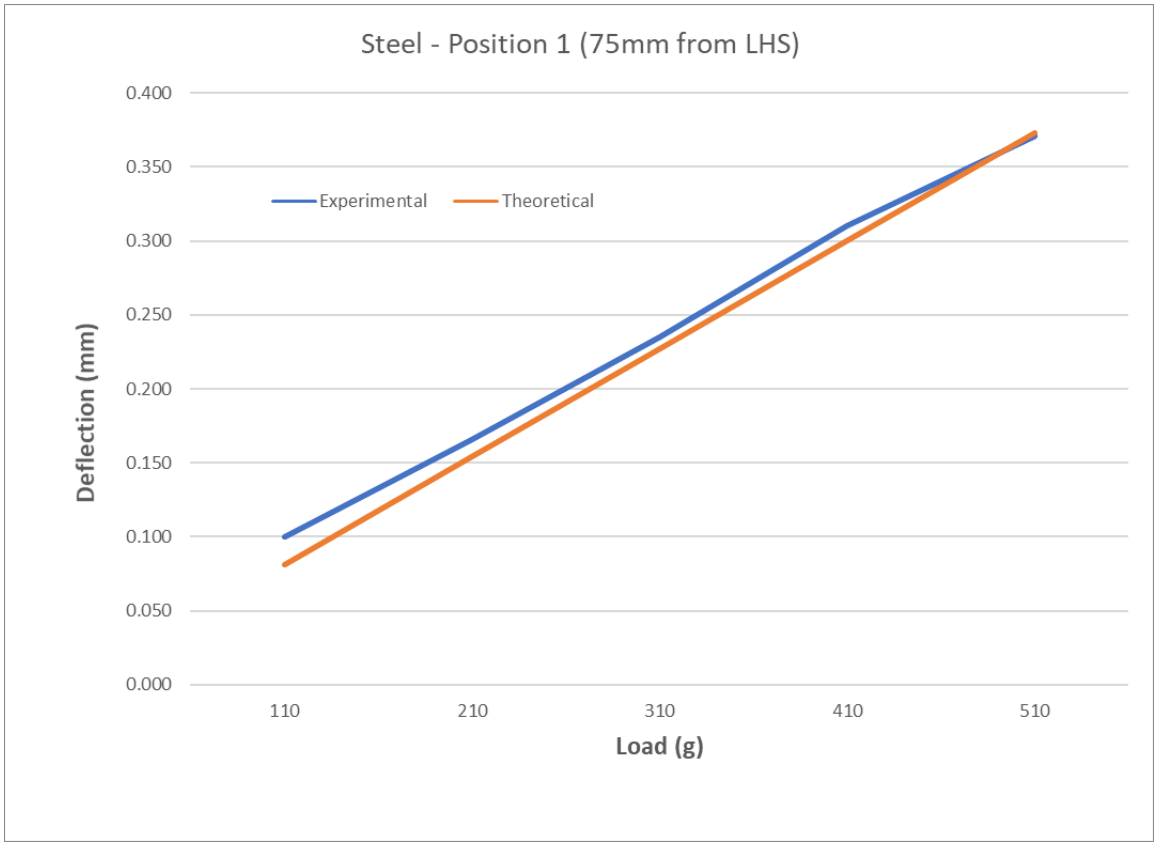

Figure 4 - Graph showing Load vs Deflection for Steel beam in position 1

Figure 5 - Graph showing Load vs Deflection for Steel beam in position 2

Figure 6 - Graph showing Load vs Deflection for Steel beam in position 3

4.3 Brass Beam Results

Table 2 below shows the experiment results and the calculated results for the Brass beam deflection.

Table 2 - Brass Beam Results

	Position 1 (75mm from LHS)			Position 2 (150mm from LHS)			Position 3 (225mm from LHS)		
	Deflection (mm)			Deflection (mm)			Deflection (mm)		
Load (N)	Experimental	Theoretical	Difference	Experimental	Theoretical	Difference	Experimental	Theoretical	Difference
110	0.160	0.153	0.007	0.280	0.295	-0.015	0.229	0.207	0.022
210	0.311	0.293	0.018	0.588	0.563	0.025	0.422	0.396	0.026
310	0.456	0.432	0.024	0.879	0.832	0.047	0.615	0.584	0.031
410	0.619	0.571	0.048	1.155	1.100	0.055	0.821	0.773	0.048
510	0.751	0.711	0.040	1.431	1.370	0.061	1.040	0.962	0.078

Figure 7 - Graph showing Load vs Deflection for Brass beam in position 1

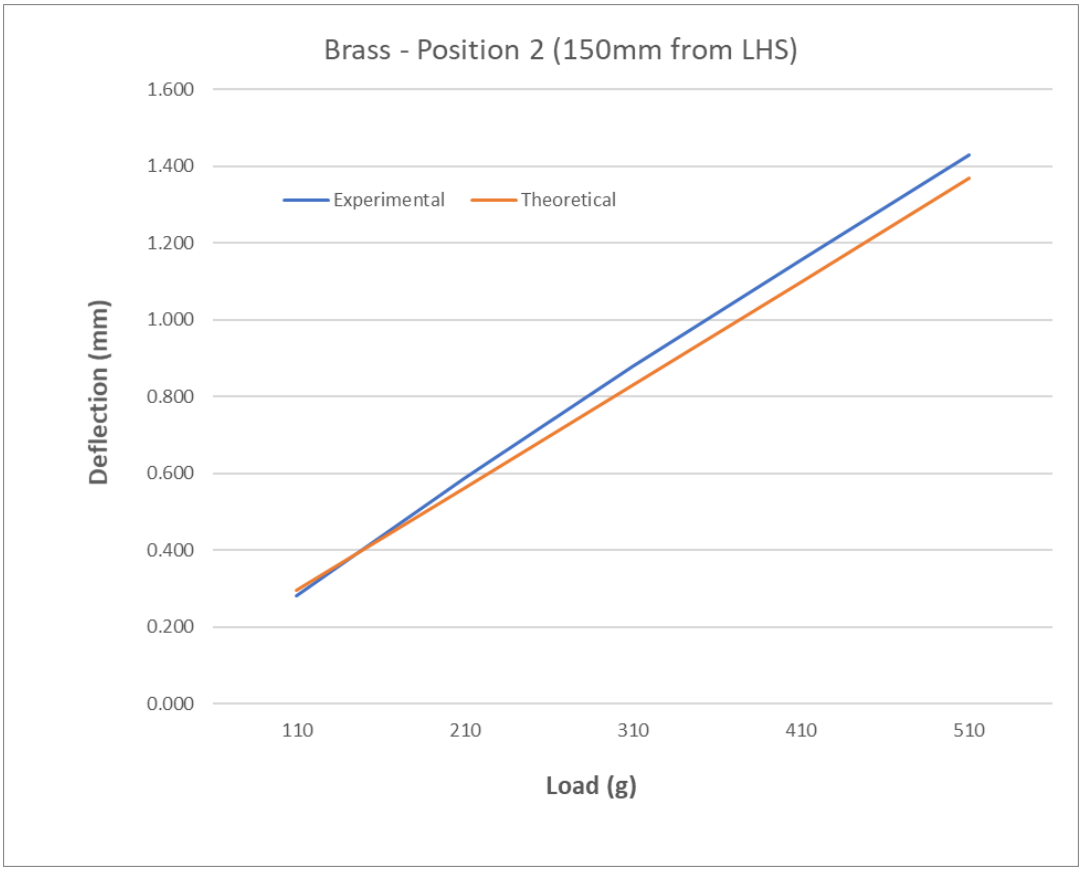

Figure 8 - Graph showing Load vs Deflection for Brass beam in position 2

Figure 9 - Graph showing Load vs Deflection for Brass beam in position 3

5. Discussion/Analysis of Results

The theoretical results produced a linear relationship between the applied load and deflection. This is also reflected in the experimental results with a low degree of error. Using the difference between the theoretical and experimental results the degree of error was found to range between 0.5 - 23.5%, however, there wasn't sufficient data to determine a trend.

Theoretic deflections assume the following to be true; the beam is made from a material that is homogeneous, isotropic and elastic, deflections that occur are small, the beam has a solid cross section and that Euler-Bernoulli beam theory applies. In this experiment the ratio of beam length to cross section dimensions is high, the cross section of the beam is solid, and the selected beam materials (steel and brass) are close to homogeneous and isotropic. In addition to this, the load applied resulted in small deflections and the material remained linear-elastic which meant that as a result the experimental results closely matched the theoretical results.

The results of the deflection, for both the calculated and measured methods, showed that the Brass beam had a higher amount of deflection than the Steel beam, for a given load value. This is to be expected as steel has a higher value for its Modulus of Elasticity (E). With regards to the deflection of beams, the Modulus of Elasticity when multiplied with the Moment of Inertia (I) equates to the flexural rigidity of the beam, which is the beam's ability to resist deflection. Therefore the higher the Modulus of Elasticity, the higher the flexural rigidity and therefore deflection would decrease as the modulus of elasticity increased.

From the Load vs Deflection graphs is can be observed that the deflection was higher in the experimental results when compared to the theoretical results, for the brass beam. Whereas the results for the steel beam show that the theoretical deflection was higher than the experimental results, with the results for load position 1 being the exception.

In the practical experiment it is difficult to reproduce the theoretical boundary conditions. The theory does not take into account friction from the fixing blocks, for example.

Another factor which may have contributed to the difference between the theory and practical results was the method of measuring the position of the Dial Indicator (i.e. the position of x). Whilst the Dial Caliper gauge (Vernier caliper) was used to measure the beam dimensions breadth and height in order to calculate the Moment of Inertia and had a high degree of precision, the location of the Dial Indicator was measured using a standard ruler instead. Clearly, this is much less precise, with a ruler accurate to only 1 mm when used correctly.

Furthermore, in the experiment repeat readings for each of these dimensions and the deflection measurements were not taken, which makes it hard to rule out the possibility of random error. The experimental results are also at risk of human error when it comes to setting up the equipment, measuring dimensions and reading dial gauges.

6. Conclusion and Recommendations

The theoretical and experimental results both produced a linear relationship between the applied load and deflection and in general the experimental deflection results are greater than that of the theoretical deflection results.

The results obtained in this case could be easily improved through the use of more precise measuring equipment, or by taking repeat readings from different members of the experiment team to rule out human error.

There wasn't sufficient data gathered to establish the trend regarding the degree of error between the theory and practical results. It is recommended that more data is gathered in order to observe how this error is affected by load position and by load increases. In real world applications of beams the loads involved are much higher and understanding the degree of error between this and the theory is crucial.

7. Reference List

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