

Matrices

1.1 Definitions

A **matrix** is a rectangular array of numbers of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- We usually use *capital* letters (for example, A , B , C , ...) for the names of matrices, and we usually use *lowercase* letters (for example, a , b , c , ...) to represent the numbers inside of a matrix.
- The numbers inside a matrix are called the **elements** or **entries** of the matrix.
- The sequence of all entries on a horizontal line is called a **row**, and the sequence of all entries on a vertical line is called a **column**. We number the rows from top to bottom, and the columns from left to right.
- The entry in the k th row and l th column of a matrix A is denoted by a_{kl} .
- A matrix with m rows and n columns is called an $m \times n$ matrix, or a matrix of **order** $m \times n$.

Example 1.1.1.

(a) $A = \begin{bmatrix} 2 & -3 & 4 & 5 \\ 6 & 7 & 4 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix}$ is a 3×4 matrix since it has 3 rows and 4 columns.

The entry in the 2nd row and 4th column of A is 3, and so we write $a_{24} = 3$.

(b) $B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is a 4×1 matrix.

It can also be referred to as a **column matrix** or a **column vector**.

(c) $C = \begin{bmatrix} 1 & 2 & -3 & 4 & -1 \end{bmatrix}$ is a 1×5 matrix.

It can also be referred to as a **row matrix** or **row vector**.

Further Definitions:

- If a matrix has order $n \times n$ (that is, if the number of rows equals the number of columns) then the matrix is called **square**.
- The **main diagonal** of a square matrix consists of the entries on the diagonal from the top left corner of the matrix down to the bottom right corner of the matrix.
- A **diagonal** matrix is a square matrix in which all of the entries which are **not** on the main diagonal must equal zero.
- A **triangular** matrix is a square matrix in which
 - all the entries which lie **below** the main diagonal are zero, **or**
 - all the entries which lie **above** the main diagonal are zero.

Example 1.1.2.

(a) $D = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ is a 2×2 **square** matrix.

(b) $E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix}$ is a 3×3 **square** matrix.

We see that E is a 3×3 **diagonal** matrix.

(Note that E is also a 3×3 **triangular** matrix.)

(c) $F = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & -6 \end{bmatrix}$ and $G = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ -3 & 5 & -6 \end{bmatrix}$ are 3×3 **triangular** matrices.

1.2 Matrix Operations

Matrix equality. Two matrices are equal if they have the same order and their corresponding entries are equal.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \text{ if and only if } a = e, b = f, c = g \text{ and } d = h.$$

Matrix addition. Two matrices can be added if they have the same size. We add two such matrices by adding the corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

For example,

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 4 & \frac{1}{2} \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 5 & \frac{7}{2} \\ 1 & -4 \end{bmatrix}$$

Scalar multiplication. Any matrix can be multiplied by a single number (scalar). We do this by multiplying all the entries of the matrix by that number.

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

For example,

$$4 \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 12 \\ 8 & -4 \end{bmatrix}$$

Matrix multiplication. The product AB of two matrices A and B is defined if and only if the number of columns of A is equal to the number of rows of B .

$$\begin{array}{ccc} & n \text{ columns} & \\ \begin{bmatrix} \vdots & \vdots & & \vdots & \vdots \\ & & & & \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix} & & \begin{matrix} n \\ \text{rows} \end{matrix} \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \\ A & & B \end{array}$$

To calculate the entry in the (i, j) position (that is, in the i^{th} row and j^{th} column) of AB , we multiply the i^{th} row of A by the j^{th} column of B .

Example 1.2.1.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

If A has order $m \times n$ and B has order $n \times p$ then AB has order $m \times p$

Example 1.2.2.

$$\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 3a + 2d & 3b + 2e & 3c + 2f \\ 4a + 1d & 4b + 1e & 4c + 1f \end{bmatrix}$$

Example 1.2.3.

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 4 & 5 \end{bmatrix} &= \begin{bmatrix} 1 \times 2 + 2 \times 1 + -3 \times 4 & 1 \times 3 + 2 \times 6 + -3 \times 5 \\ 4 \times 2 + 5 \times 1 + 6 \times 4 & 4 \times 3 + 5 \times 6 + 6 \times 5 \end{bmatrix} \\ &= \begin{bmatrix} -8 & 0 \\ 37 & 72 \end{bmatrix} \end{aligned}$$

Transpose. The transpose A^T of a matrix A is obtained by putting the i th row of A into the i th column of A^T . If A has order $m \times n$ then A^T has order $n \times m$.

Example 1.2.4. If $A = \begin{bmatrix} 2 & 4 & 1 \\ 9 & 9 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 2 & 9 \\ 4 & 9 \\ 1 & 6 \end{bmatrix}$

1.3 Properties of Matrix Operations

1.3.1 Addition Properties

$$\begin{aligned}A + B &= B + A \\A + (B + C) &= (A + B) + C \\A + O &= A = O + A \\A + -A &= O = -A + A\end{aligned}$$

The matrix O is called the **zero** matrix, and $-A$ is called the **negative** of A .

In the above properties, A , B , C , O and $-A$ all have the same order. Thus, for example,

$$(a) \text{ if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } -A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

$$(b) \text{ if } A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \text{ then } O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } -A = \begin{bmatrix} -a & -b \\ -c & -d \\ -e & -f \end{bmatrix}$$

1.3.2 Multiplication Properties

Provided that each of the following matrix products exists, we have

$$\begin{aligned}A(B + C) &= AB + AC \\(B + C)A &= BA + CA \\A(BC) &= (AB)C\end{aligned}$$

However, usually $AB \neq BA$

For a **square** matrix A , we have:

$$\begin{aligned}AO &= O = OA \\ \text{and } AI &= A = IA\end{aligned}$$

where I is a diagonal matrix with ones on the main diagonal and zeros elsewhere.

The matrix I is called the **identity matrix**, and must have the same order as A .

For example:

$$\text{If } A \text{ is } 2 \times 2 \text{ then } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{If } A \text{ is } 3 \times 3 \text{ then } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: It is important to realize that, if we choose any two matrices A and B , then usually

$$\boxed{AB \neq BA}$$

For example, consider the matrices A and B given below:

Example 1.3.1. Let $A = \begin{bmatrix} 3 & 6 \\ -4 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} -10 & -4 \\ 5 & 2 \end{bmatrix}$

Then

$$AB = \begin{bmatrix} 3 & 6 \\ -4 & -8 \end{bmatrix} \begin{bmatrix} -10 & -4 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -30 + 30 & -12 + 12 \\ 40 - 40 & 16 - 16 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -10 & -4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ -4 & -8 \end{bmatrix} = \begin{bmatrix} -30 + 16 & -60 + 32 \\ 15 - 8 & 30 - 16 \end{bmatrix} = \begin{bmatrix} -14 & -28 \\ -7 & 14 \end{bmatrix}$$

In this example,

$$\boxed{AB \neq BA}$$

1.3.3 Exercises

1. Find

$$(a) \begin{bmatrix} 2 & -2 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -1 & 0.6 \end{bmatrix}$$

$$(b) \begin{bmatrix} 4 & 2 \\ -\frac{2}{3} & -1 \end{bmatrix} + \begin{bmatrix} 3 & 43 \\ -4 & 3 \end{bmatrix}$$

2. Find

$$(a) -2 \begin{bmatrix} 2 & -\frac{1}{2} \\ 3 & 0 \end{bmatrix}$$

$$(b) 4 \begin{bmatrix} 3 & \frac{1}{4} \\ -1 & 0 \end{bmatrix}$$

$$(c) 3 \begin{bmatrix} 5 & 8 \\ 6 & -4 \end{bmatrix}$$

3. Find

$$(a) \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -1 & 4 \end{bmatrix}$$

$$(b) \begin{bmatrix} 2 & -1 \\ \frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & -3 \end{bmatrix}$$

4. Given that

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & x \\ 2 & 6 & 5 \end{bmatrix} \begin{bmatrix} 2 & 7 & 1 \\ 8 & 2 & 8 \\ 1 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 18 & 55 & 19 \\ 51 & 89 & 59 \\ 57 & 66 & 60 \end{bmatrix},$$

find x .

1.4 The Determinant and Inverse of a 2×2 Matrix.

The matrix B is the **inverse** of A if $AB = I = BA$.

Not all matrices have inverses. When a matrix A does have an inverse B , then the inverse is *unique* (i.e., it has no other inverses), and we write A^{-1} instead of B . Thus

$$AA^{-1} = I = A^{-1}A$$

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we define the **determinant** of A by

$$\det(A) = ad - bc$$

We sometimes write $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ instead of $\det(A)$.

If $\det(A) \neq 0$ then we have

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Then A^{-1} satisfies

$$AA^{-1} = I = A^{-1}A$$

as required.

Note: If $\det(A) = 0$ then A^{-1} does not exist, and we say that A is **singular**.

Example 1.4.1. Find the determinant and the inverse of the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$.

Solution.

$$\text{We have } \det(A) = \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = 2 \times 0 - 3 \times 4 = -12$$

and

$$A^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}^{-1} = \frac{1}{-12} \begin{bmatrix} 0 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$$

1.4.1 Exercises

1. Evaluate

$$(a) \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$$

$$(b) \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$(c) \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix}$$

2. Find the inverse of the following matrices:

$$(a) \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.5 Solving Matrix Equations

Consider the matrix equation $AX = C$, where X is an unknown matrix.

We **cannot divide** by A . Instead, we **multiply** both sides of the equation by A^{-1} (if A^{-1} exists). In particular, if A^{-1} exists then we can solve the above equation for the matrix X , as follows:

$$\begin{aligned} AX = C &\Rightarrow A^{-1}AX = A^{-1}C \text{ (multiplying both sides of the equation on the **left** by } A^{-1}) \\ &\Rightarrow IX = A^{-1}C \\ &\Rightarrow X = A^{-1}C. \end{aligned}$$

Similarly,

$$\begin{aligned} XA = C &\Rightarrow XAA^{-1} = CA^{-1} \text{ (multiplying both sides of the equation on the **right** by } A^{-1}) \\ &\Rightarrow X = CA^{-1}. \end{aligned}$$

Note that

- Multiplying on the left is called **pre**multiplying.
- Multiplying on the right is called **post**multiplying.

Example 1.5.1. Find the matrix X such that

$$X \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix}.$$

Solution.

Since $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ is on the **right** of X , we multiply on the **right** by $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}^{-1}$.

Now

$$\begin{aligned} X \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} &= \begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix} \\ \Rightarrow X \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}^{-1} &= \begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}^{-1} \\ \Rightarrow XI &= \begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix} \left(\frac{1}{-12} \begin{bmatrix} 0 & -3 \\ -4 & 2 \end{bmatrix} \right) \\ \Rightarrow X &= \frac{1}{-12} \begin{bmatrix} 9 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ -4 & 2 \end{bmatrix} \\ &= \frac{1}{-12} \begin{bmatrix} 0 & -27 \\ -4 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{9}{4} \\ \frac{1}{3} & \frac{1}{12} \end{bmatrix} \end{aligned}$$

1.5.1 Uniqueness of Solutions

Sometimes there can be more than one matrix X such that $AX = C$, or there could be none! When there is only one X such that $AX = C$, we say that $AX = C$ has a *unique* solution.

Consider the matrix equation $AX = C$.

We have the following result:

- When $\det(A) \neq 0$ then $AX = C$ has a **unique** solution.

This unique solution is given by $X = A^{-1}C$.

- When $\det(A) = 0$ then $AX = C$ either has
 - **infinitely many** solutions,
 - or
 - **no** solutions.

1.5.2 Exercises

1. Find the matrix X such that

$$(a) \quad X \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ -7 & 4 \end{bmatrix}$$

$$(b) \quad X \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ -13 & -1 \end{bmatrix}$$

$$(c) \quad X \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(d) \quad X \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

2. Find the matrix Y such that

$$(a) \quad \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} Y = \begin{bmatrix} 9 & 2 \\ -7 & 4 \end{bmatrix}$$

$$(b) \quad \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} Y = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$(c) \quad \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix} Y = \begin{bmatrix} 8 & 6 \\ -13 & -1 \end{bmatrix}$$

$$(d) \quad \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix} Y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

3. Find the matrix Z such that

$$(a) \quad \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} Z \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(b) \quad \begin{bmatrix} 2 & -1 \\ 1 & \frac{1}{2} \end{bmatrix} Z \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$$

1.6 Simultaneous Equations with 2 unknowns

An equation of the form $ax + by = p$ (where a , b and p are constants) represents a straight line in the x, y -plane.

The system of equations

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

represents two lines in the x, y -plane. When we solve these equations *simultaneously*, we are finding the point of intersection of the two lines. We can write the above system of equations in matrix form as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

Example 1.6.1. Solve the following system of equations:

$$\begin{array}{rcrcrcrcl} 2x & + & 3y & = & 8 \\ x & - & 4y & = & -7 \end{array}$$

Solution.

We need to find x and y such that **both** of the above equations are satisfied.

To do this, we write the above system in matrix form:

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \end{bmatrix}.$$

Then we premultiply the matrix equation by the inverse of $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ to get

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -7 \end{bmatrix} \\ &= \frac{1}{-11} \begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -7 \end{bmatrix} \\ &= \frac{1}{-11} \begin{bmatrix} -11 \\ -22 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \end{aligned}$$

Thus $x = 1$ and $y = 2$.

1.6.1 To Solve a System of Equations:

(a) Write the system of equations

$$\begin{array}{rcl} ax & + & by = p \\ cx & + & dy = q \end{array}$$

in matrix form:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

(b) Calculate the determinant

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(c) If $\Delta \neq 0$ then we have a **unique** solution given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix}.$$

(d) If $\Delta = 0$ then either

- there are no solutions.
 - In this case, the equations represent two parallel lines which don't intersect.

or

- there are infinitely many solutions.
 - In this case the equations represent the same line, and so any point on this line is a solution.

Example 1.6.2. Solve the following system of equations:

$$\begin{array}{rcl} 2x & + & 3y = 6 \\ 4x & + & 6y = -12 \end{array}$$

Solution.

In matrix form, we have

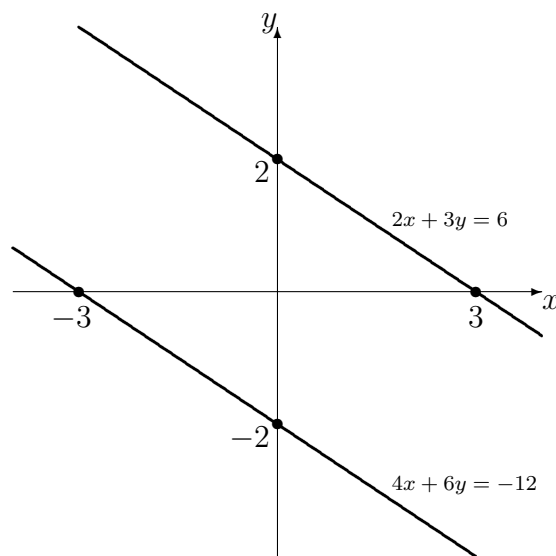
$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$$

Note that the determinant

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 2 \times 6 - 3 \times 4 = 12 - 12 = 0.$$

Therefore, there is no solution or there are infinitely many solutions.

Graphically, the lines $2x + 3y = 6$ and $4x + 6y = -12$ are non-intersecting parallel lines:



Therefore, there is no solution.

Example 1.6.3. Consider the following simultaneous system of linear equations, where k is a constant:

$$\begin{aligned} kx + 3y &= 4 \\ 3x + ky &= 5. \end{aligned}$$

- (a) Write down the system of equations in matrix form

$$AX = B.$$

Solution.

$$\begin{bmatrix} k & 3 \\ 3 & k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

- (b) Calculate the determinant of the matrix A found in (a).

Solution.

$$A = \begin{bmatrix} k & 3 \\ 3 & k \end{bmatrix} \quad \text{and so} \quad \det(A) = \begin{vmatrix} k & 3 \\ 3 & k \end{vmatrix} = k^2 - 9.$$

- (c) Find the value(s) of k for which the system has a unique solution.

Solution.

There is a unique solution if and only if

$$\begin{aligned} \det(A) &\neq 0 \\ \Leftrightarrow k^2 - 9 &\neq 0 \\ \Leftrightarrow (k - 3)(k + 3) &\neq 0 \\ \Leftrightarrow k \neq 3 \quad \text{and} \quad k &\neq -3 \\ \Leftrightarrow k \in \mathbf{R} \setminus \{3, -3\}. \end{aligned}$$

1.6.2 Exercises

1. Solve the following systems of equations:

$$\begin{array}{rcl} \text{(a)} & 2x & + \quad 3y = 8 \\ & x & + \quad 4y = 9 \end{array}$$

$$\begin{array}{rcl} \text{(b)} & 2x & - \quad y = -5 \\ & x & + \quad 3y = 1 \end{array}$$

$$\begin{array}{rcl} \text{(c)} & 2x & + \quad 3y = 6 \\ & 4x & + \quad 6y = 3 \end{array}$$

$$\begin{array}{rcl} \text{(d)} & 2x & + \quad 3y = 6 \\ & 4x & + \quad 6y = 12 \end{array}$$

2. Consider the following simultaneous system of linear equations, where p is a constant:

$$\begin{array}{rcl} & 3x & + \quad py = 9 \\ (p+1)x & + \quad 2y = 9. \end{array}$$

(a) Show that the system has a unique solution if and only if $p \in \mathbf{R} \setminus \{-3, 2\}$.

(b) Find the value(s) of p for which the system has infinitely many solutions.

(c) Find the value(s) of p for which the system has no solution.

Hint for (b) and (c): Consider the cases $p = -3$ and $p = 2$ separately, i.e., substitute $p = -3$ into the system and then try to solve the system; do the same for $p = 2$.

Chapter 15

Answers

15.1 Answers for Chapter 1 Exercises

1.3.3: 1. (a) $\begin{bmatrix} 2 & 1 \\ 6 & 4.6 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 45 \\ -\frac{14}{3} & 2 \end{bmatrix}$

2. (a) $\begin{bmatrix} -4 & 1 \\ -6 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 12 & 1 \\ -4 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 15 & 24 \\ 18 & -12 \end{bmatrix}$

3. (a) $\begin{bmatrix} 2 & 1 \\ -4 & 37 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 11 \\ 13 & -10 \end{bmatrix}$

4. $x = 9$

1.4.1: 1. (a) 2 (b) 0 (c) 1

2. (a) $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$ (b) $\frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1.5.2: 1. (a) $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{26}{7} & \frac{2}{7} \\ \frac{-67}{14} & \frac{-12}{7} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$

2. (a) $\begin{bmatrix} \frac{23}{5} & \frac{-6}{5} \\ \frac{34}{5} & \frac{2}{5} \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ (d) $\frac{1}{14} \begin{bmatrix} 17 \\ 4 \end{bmatrix}$

3. (a) $\frac{1}{4} \begin{bmatrix} 38 & -15 \\ -42 & 17 \end{bmatrix}$ (b) $-\frac{1}{4} \begin{bmatrix} 3 & -\frac{3}{2} \\ -22 & 3 \end{bmatrix}$

1.6.2: 1. (a) $(1, 2)$ (b) $(-2, 1)$ (c) No solution

(d) More than one solution: any point on the line $2x + 3y = 6$.

2. (a) Omitted (b) $p = 2$ (c) $p = -3$.