

Partial Fraction Decomposition (Summary)

Partial Fraction Decomposition is used when we have a fraction, $P(x)/Q(x)$, where P, Q are polynomials, and the degree of P is less than the degree of Q . **NOTE:** If the degree of the numerator is larger than the denominator, then perform long division first.

Assume Q is fully factored. We have 4 cases that we will consider.

Case I : Q has distinct linear factors,

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \dots (a_kx + b_k)$$

Then:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_k}{a_kx + b_k}$$

Example (be sure you could have found the constants):

$$\frac{3x}{(x-1)(x-2)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{6}{x-2} - \frac{3}{x-1}$$

Case II : Q has some repeated linear factors. Let $a_1x + b_1$ be repeated r times. Then, instead of the single term $A_1/(a_1x + b_1)$, we have one term for each successive power in the denominator:

$$\frac{B_1}{a_1x + b_1} + \frac{B_2}{(a_1x + b_1)^2} + \dots + \frac{B_r}{(a_1x + b_1)^r}$$

Example:

$$\frac{3x+1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} = -\frac{7}{x-1} - \frac{4}{(x-1)^2} + \frac{7}{x-2}$$

Case III : Q has some irreducible quadratic factors, not repeated. Let $ax^2 + bx + c$ be an irreducible quadratic factor for Q . Then the decomposition will have the term:

$$\frac{Ax + B}{ax^2 + bx + c}$$

Example:

$$\frac{3x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{3}{2} \frac{1}{x-1} + \frac{1}{2} \frac{-3x+3}{x^2+1}$$

Case IV : Q has some irreducible quadratic factors, some repeated. Suppose that $ax^2 + bx + c$ is a repeated quadratic factor (repeated r times). Then, instead of the single expression in Case III, we will have:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Example:

$$\frac{x+3}{(x-1)(x^2+1)^2} = \frac{1}{x-1} - \frac{x+1}{x^2+1} - \frac{2x+1}{(x^2+1)^2}$$

Worked Examples and Exercises

1. Worked Example:

$$\frac{6x^2 - 6x - 6}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$$

Clear fractions:

$$6x^2 - 6x - 6 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$$

And this statement must be true for all x . In particular, it must be true for $x = 1, x = -2$ and $x = 3$ (we chose these to zero out the others). Substituting, we get

$$A = 1 \quad B = 2 \quad C = 3$$

2. Worked Example:

$$\frac{x^2 - 2}{x(x^2 + 2)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2}$$

Clear fractions. In this case, it might be best to solve for the coefficients in a slightly different manner- Equate the coefficients to the polynomials on the left and right:

$$x^2 - 2 = A(x^2 + 2) + (Bx + C)x = (A + B)x^2 + Cx + 2A$$

so that:

$$1 = A + B, 0 = C, 2A = -2$$

so: $A = -1, B = 2$ and $C = 0$:

$$\frac{x^2 - 2}{x(x^2 + 2)} = \frac{-1}{x} + \frac{2x}{x^2 + 2}$$

Exercises

For each of the following, first give the general form for the Partial Fraction expansion, then solve for the constants.

1. $\frac{x^2 + 1}{x^2 + 3x + 2}$

2. $\frac{2x + 3}{(x + 1)^2}$

3. $\frac{4x^2 - 7x - 12}{x(x + 2)(x - 3)}$

4. $\frac{x^2 + 3}{x^3 + 2x}$

5. $\frac{x^2 - 2x - 1}{(x - 1)^2(x^2 + 1)}$

6. $\frac{3x^3 - x + 12}{x^2 - 1}$

Solutions

$$1. \frac{x^2 + 1}{x^2 + 3x + 2} = 1 - \frac{5}{x + 2} + \frac{2}{x + 1}$$

$$2. \frac{2x + 3}{(x + 1)^2} = \frac{2}{x + 1} + \frac{1}{(x + 1)^2}$$

$$3. \frac{4x^2 - 7x - 12}{x(x + 2)(x - 3)} = \frac{2}{x} + \frac{9}{5} \frac{1}{x + 2} + \frac{1}{5} \frac{1}{x - 3}$$

$$4. \frac{x^2 + 3}{x^3 + 2x} = \frac{3}{2} \frac{1}{x} - \frac{1}{2} \frac{x}{x^2 + 2}$$

$$5. \frac{x^2 - 2x - 1}{(x - 1)^2(x^2 + 1)} = \frac{1}{x - 1} - \frac{1}{(x - 1)^2} - \frac{x - 1}{x^2 + 1}$$

$$6. 3x + \frac{7}{x - 1} - \frac{5}{x + 1}$$