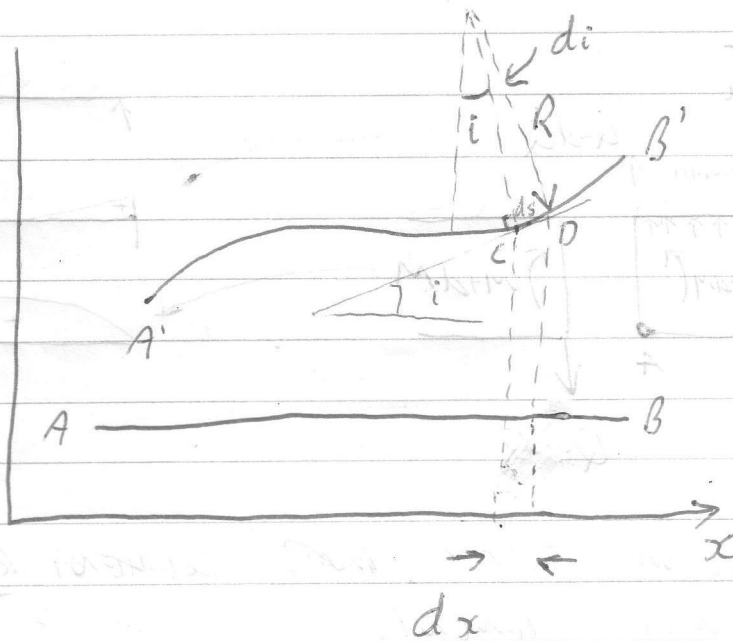


SLOPE & DEFLECTION OF BEAMS



① SLOPE AT
C

$$i = \frac{dy}{dx}$$

②

$$ds = dx = R di$$

$$dx = R di \quad \therefore \frac{di}{dx} = \frac{1}{R}$$

$$\frac{di}{dx} = \frac{1}{R}$$

(3) SINCE

$$i = \frac{dy}{dx}$$

$$\frac{1}{R} = \frac{di}{dx} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{d^2 y}{dx^2} = \frac{1}{R}$$

④

FROM

SIMPLE

BENDING

THEORY:

$$\frac{M}{I} = \frac{E}{R}$$

$$\therefore \frac{1}{R} = \frac{M}{EI}$$

$EI = \text{FLEXURAL STIFFNESS}$

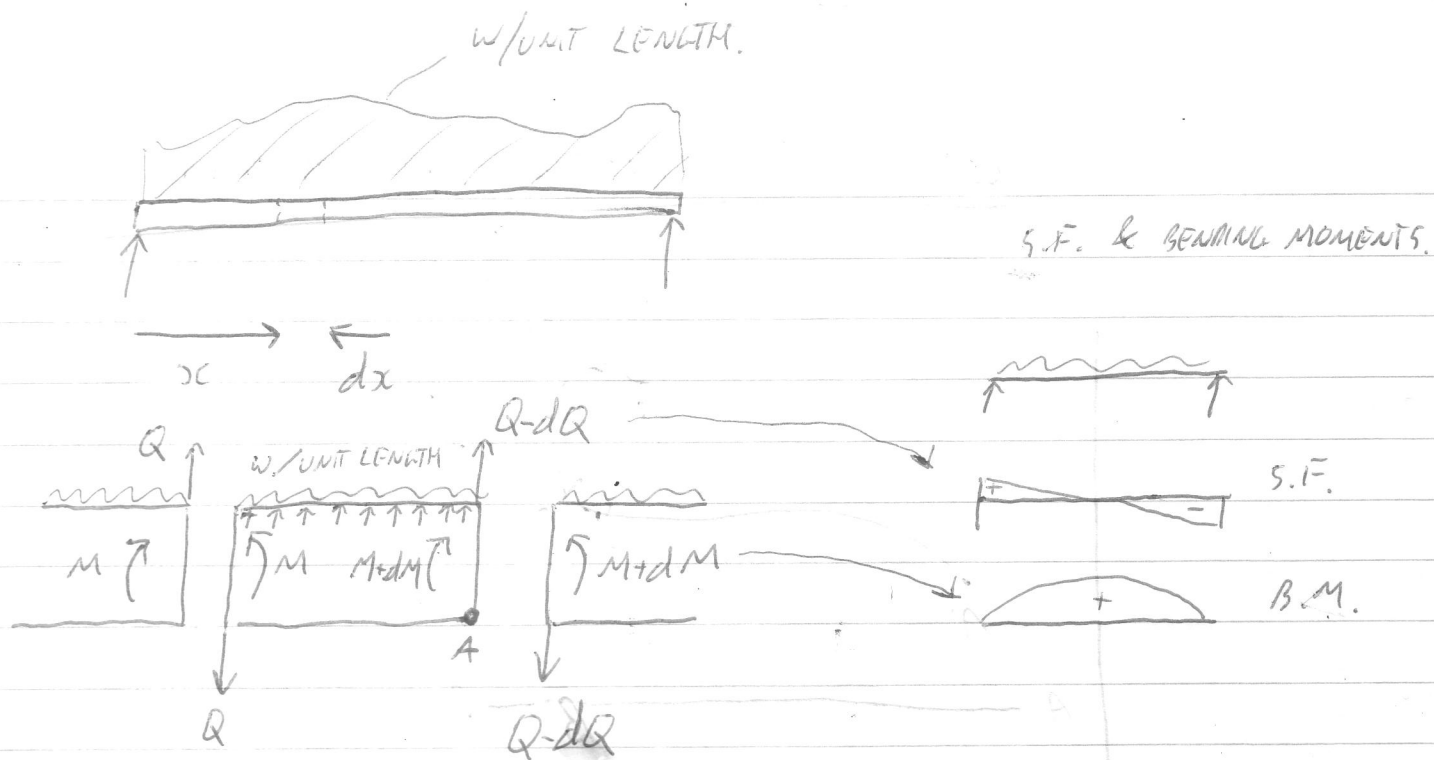
$$M = \frac{EI}{R}$$

$$EI \frac{d^2 y}{dx^2} = M$$

BASIC DIFFERENTIAL EQUATION FOR DEFLECTION OF BEAMS

①

(4)



BEAM CARRIES EXTERNAL FORCES. SMALL ELEMENT REACTIONS ARE SHOWN IN SMALL ELEMENT.

(5)

ON ELEMENT : $Q = w dx + Q - dQ$
(RESOLVING VERTICALLY) $\therefore dQ = w dx$

$$\therefore \boxed{Q = \int w dx} \quad (2)$$

(6)

MOMENTS ABOUT A:

$$(M + dM) + w dx \frac{dx}{2} = M + Q dx$$

$$\therefore dM = Q dx$$

$$\therefore \boxed{M = \int Q dx} \quad (3)$$

$$EI \frac{d^2 y}{dx^2} = M$$

$$M = EI \frac{d^2 y}{dx^2}$$

(7)

FROM

(1)

$$M = EI \frac{d^2 y}{dx^2}$$

FROM

(3)

$$M = \int Q dx$$

$$\therefore \int Q dx = EI \frac{d^2 y}{dx^2}$$

$$\therefore Q = EI \frac{d^3 y}{dx^3}$$

(DIFF w.r.t. x)

(8)

ALSO

FROM (2)

$$Q = \int w dx$$

$$\therefore \int w dx = EI \frac{d^3 y}{dx^3}$$

$$\therefore w = EI \frac{d^4 y}{dx^4}$$

(DIFF w.r.t. x)

(9)

$\Rightarrow$ DIRECT INTEGRATION METHOD

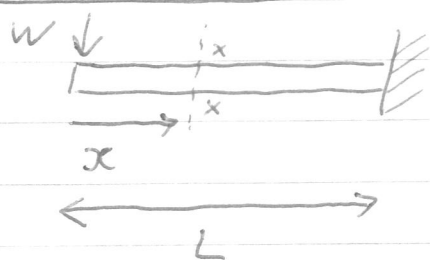
$$M = EI \frac{d^2 y}{dx^2}$$

SLOPE : $\frac{dy}{dx} = \int \frac{M}{EI} dx + A$

DEFLECTION : $y = \iint \left(\frac{M}{EI} dx \right) dx + Ax + B$

WHERE A & B ARE CALCULATED FROM KNOWN CONDITIONS OF SLOPE & DEFLECTION OF THE BEAM.

⑩ CANTILEVER WITH CONCENTRATED LOAD



$$M_{xx} = -Wx$$

$$\therefore EI \frac{d^2 y}{dx^2} = -Wx$$

$$EI \frac{dy}{dx} = -\frac{Wx^2}{2} + A \quad \text{① (INT. w.r.t. } x \text{)}$$

⑪

ASSUMING EI IS CONSTANT:

$$EI y = -\frac{Wx^3}{6} + Ax + B \quad \text{②}$$

⑫

WHEN $x = L \quad \frac{dy}{dx} = 0$

$\therefore$ FROM ①

$$0 = -\frac{WL^2}{2} + A$$

$$\therefore A = \frac{WL^2}{2}$$

⑬

WHEN $x = L \quad y = 0$

$\therefore$ FROM ②

$$0 = -\frac{WL^3}{6} + \frac{WL^2}{2} L + B$$

$$\therefore B = \frac{WL^3}{6} - \frac{WL^3}{2} = -\frac{WL^3}{3}$$

S & A (3)

FROM (2)

$$EIy = -\frac{wx^3}{6} + Ax + B$$

$$= -\frac{wx^3}{6} + \frac{wL^2x}{2} - \frac{wL^3}{3}$$

$$\therefore y = \frac{1}{EI} \left[-\frac{wx^3}{6} + \frac{wL^2x}{2} - \frac{wL^3}{3} \right]$$

THIS GIVES DEFLECTION AT ALL POINTS AT BEAM.
WHEN $x=0$ y IS MAX.

$$y_{\max} = -\frac{wL^3}{3EI}$$

THE DEFLECTION IS GIVEN BY (1)

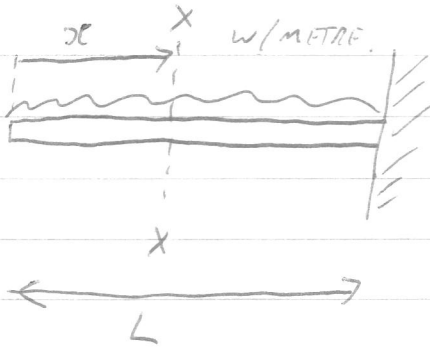
$$\begin{aligned} EI \frac{dy}{dx} &= -\frac{wx^2}{2} + A \\ &= -\frac{wx^2}{2} + \frac{wL^2}{2} \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{EI} \left(-\frac{wx^2}{2} + \frac{wL^2}{2} \right)$$

THIS GIVES SLOPE AT ALL POINTS ON BEAM.

WHEN $x=0$ $\frac{dy}{dx}$ IS MAX

$$\therefore \left(\frac{dy}{dx} \right)_{\max} = \frac{wL^2}{2EI}$$

CANTILEVER WITH U.D.L.

$$M_{xx} = -wx \cdot \frac{x}{2}$$

$$= -\frac{wx^2}{2}$$

$$\therefore EI \frac{d^2y}{dx^2} = -\frac{wx^2}{2}$$

$$\textcircled{1} \quad EI \frac{dy}{dx} = -\frac{wx^3}{6} + A \quad \textcircled{1}$$

$$\textcircled{2} \quad EI y = -\frac{wx^4}{24} + Ax + B \quad \textcircled{2}$$

$$\textcircled{3} \quad \text{WHEN } x=L \quad \frac{dy}{dx} = 0$$

$$\therefore \text{FROM } \textcircled{1} \quad 0 = -\frac{wL^3}{6} + A \quad \therefore A = \underline{\underline{+\frac{wL^3}{6}}}$$

$$\textcircled{4} \quad \text{WHEN } x=L \quad y = 0$$

$$\therefore \text{FROM } \textcircled{2} \quad 0 = -\frac{wL^4}{24} + \frac{wL^3L}{6} + B \quad \therefore B = \frac{wL^4}{24} - \frac{wL^4}{6}$$

$$= \underline{\underline{-\frac{wL^4}{8}}}$$

$$\textcircled{5} \quad \therefore \text{FROM } \textcircled{2} \quad y = \frac{1}{EI} \left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8} \right]$$

$$\therefore \frac{dy}{dx} = -\frac{wx^3}{6EI} + \frac{wL^3}{6EI}$$

$$\left(\frac{dy}{dx} \right)_{\max} = \frac{wL^3}{6EI}$$

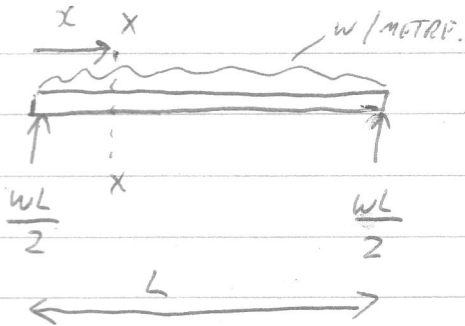
⑥ FROM ①

$$\frac{dy}{dx} = \frac{1}{EI} \left[\frac{-wx^3}{6} + \frac{wL^3}{6} \right]$$

WHEN $x=0$

$$y_{\max} = \frac{-wL^4}{8EI}$$

$$\left(\frac{dy}{dx} \right)_{\max} = \frac{wL^3}{6EI}$$

SIMPLY - SUPPORTED BEAM WITH UDL

$$\textcircled{1} \quad M_{xx} = \frac{wL}{2}x - \frac{wx \cdot x}{2} = \frac{wLx}{2} - \frac{wx^2}{2} = EI \frac{d^2y}{dx^2}$$

$$\textcircled{2} \quad \therefore EI \frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A \quad \textcircled{1}$$

$$\textcircled{3} \quad EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + Ax + B \quad \textcircled{2}$$

$$\text{WHEN } x=0 \quad y=0 \quad \therefore \underline{B=0}$$

~~WHEN~~

$$\text{WHEN } x=L \quad y=0 \quad \therefore \text{FROM } \textcircled{2} \quad 0 = \frac{wL^4}{12} - \frac{wL^4}{24} + AL$$

$$\therefore AL = -\frac{wL^4}{24}$$

$$\therefore A = -\frac{wL^3}{24}$$

S & D ⑥

FROM ③
$$y = \frac{1}{EI} \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} \right]$$

$y_{\max}$ IS WHEN $x = \frac{L}{2}$

$$\begin{aligned} y_{\max} &= \frac{1}{EI} \left[\frac{wL}{12} \left(\frac{L^3}{8} \right) - \frac{w}{24} \left(\frac{L^4}{16} \right) - \frac{wL^3}{24} \left(\frac{L}{2} \right) \right] \\ &= \frac{1}{EI} \left[\frac{wL^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48} \right] \end{aligned}$$

$$y_{\max} = \frac{-5wL^4}{384EI}$$

$\left(\frac{dy}{dx} \right)_{\max}$ IS WHEN $x=0$ OR $x=L$

FROM ②
$$\frac{dy}{dx} = \frac{1}{EI} \left[\frac{wLx^2}{4} - \frac{wx^3}{6} - \frac{wL^3}{24} \right]$$

WHEN $x=0$:

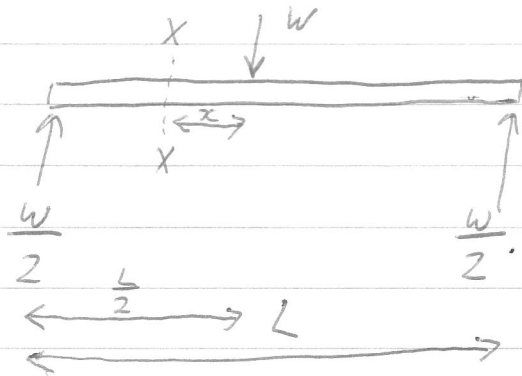
$$\left(\frac{dy}{dx} \right)_{\max} = -\frac{wL^3}{24EI}$$

WHEN $x=L$

$$\left(\frac{dy}{dx} \right)_{\max} = \frac{1}{EI} \left[\frac{wL^3}{4} - \frac{wL^3}{6} - \frac{wL^3}{24} \right]$$

$$= \frac{wL^3}{24EI}$$

$$\left(\frac{dy}{dx} \right)_{\max} = \pm \frac{wL^3}{24EI}$$

SIMPLY SUPPORTED WITH CENTRAL CONCENTRATED LOAD

$$\textcircled{1} \quad M_{xx} = \frac{W}{2} \left(\frac{L}{2} - x \right) = \frac{WL}{4} - \frac{Wx}{2} = EI \frac{d^2 y}{dx^2}$$

$$\textcircled{2} \quad \therefore EI \frac{dy}{dx} = \frac{WLx}{4} - \frac{Wx^2}{4} + A \quad \textcircled{1}$$

$$\textcircled{3} \quad EI y = \frac{WLx^2}{8} - \frac{Wx^3}{12} + Ax + B \quad \textcircled{2}$$

$$\textcircled{4} \quad \text{WHEN } x=0 \quad \frac{dy}{dx} = 0 \quad \therefore \underline{A=0} \quad (\text{FROM } \textcircled{1})$$

$$\textcircled{5} \quad \text{WHEN } x = \frac{L}{2} \quad y=0 \quad \therefore 0 = \frac{WL}{8} \left(\frac{L}{2} \right)^2 - \frac{W}{12} \left(\frac{L}{2} \right)^3 + B \quad (\text{FROM } \textcircled{2})$$

$$\therefore B = \frac{WL^3}{96} - \frac{WL^3}{32}$$

$$B = \underline{\underline{-\frac{WL^3}{48}}}$$

$$\textcircled{6} \quad \therefore y = \frac{1}{EI} \left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48} \right]$$

y_{max} IS WHEN $x=0$

$$\therefore y_{\text{max}} = \underline{\underline{-\frac{WL^3}{48EI}}}$$

$$7 \quad \frac{dy}{dx} = \frac{1}{EI} \left[\frac{WL}{4} x - \frac{Wx^2}{4} \right]$$

$\frac{dy}{dx}$ IS MAX WHEN $x = \frac{L}{2}$

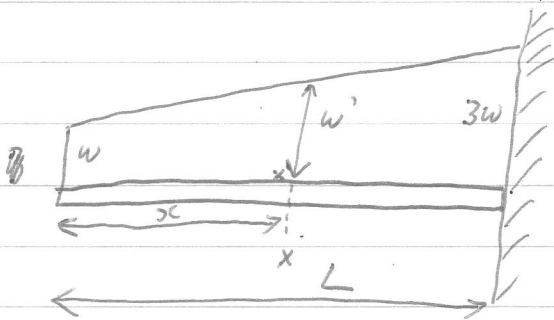
$$\left(\frac{dy}{dx} \right)_{\text{MAX}} = \frac{1}{EI} \left[\frac{WL}{4} \left(\frac{L}{2} \right) - \frac{W}{4} \left(\frac{L}{2} \right)^2 \right]$$

$$= \frac{1}{EI} \left(\frac{WL^2}{8} - \frac{WL^2}{16} \right)$$

$$\boxed{\left(\frac{dy}{dx} \right)_{\text{MAX}} = \frac{+WL^2}{16EI}}$$

CANTILEVER SUBJECTED TO NON-UNIFORM DISTRIBUTED LOAD

1 FOR NON STANDARD CASE IT MAY NOT BE CONVENIENT TO COMMENCE INTEGRATION WITH THE B.M. AS THIS MAY BE DIFFICULT TO OBTAIN.



2 LOADING AT XX IS GIVEN BY: LINEAR INTERPOLATION.

$$w' = - \left[w + (3w - w) \frac{x}{L} \right]$$

LOAD

$$\therefore w' = -w \left(1 + \frac{2x}{L} \right) = EI \frac{d^4 y}{dx^4}$$

$$\textcircled{3} \text{ SHEAR FORCE: } EI \frac{d^3 y}{dx^3} = -w \left(x + \frac{x^2}{L} \right) + A \quad \textcircled{1}$$

$$\text{BENDING MOMENT: } EI \frac{d^2 y}{dx^2} = -w \left(\frac{x^2}{2} + \frac{x^3}{3L} \right) + Ax + B \quad \textcircled{2}$$

$$\text{SLOPE: } EI \frac{dy}{dx} = -w \left(\frac{x^3}{6} + \frac{x^4}{12L} \right) + \frac{Ax^2}{2} + Bx + C \quad \textcircled{3}$$

$$\text{DEFLECTION: } EI y = -w \left(\frac{x^4}{24} + \frac{x^5}{60L} \right) + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \quad \textcircled{4}$$

$\textcircled{4}$ $\therefore$ FOUR UNKNOWN.

$$\text{WHEN } x=0 \text{ S.F.} = 0 \quad \therefore \text{ FROM } \textcircled{1} \quad \underline{A=0}$$

$$\text{WHEN } x=0 \text{ B.M.} = 0 \quad \therefore \text{ FROM } \textcircled{2} \quad \underline{B=0}$$

$$\text{WHEN } x=L \quad \frac{dy}{dx} = 0$$

$$\therefore \text{ FROM } \textcircled{3} \quad 0 = -w \left(\frac{L^3}{6} + \frac{L^4}{12L} \right) + C$$

$$\therefore C = \frac{wL^3}{6} + \frac{L^3}{12}$$

$$\underline{C = \frac{wL^3}{4}}$$

$$\text{WHEN } x=L, y=0$$

$$\therefore \text{ FROM } \textcircled{4} \quad 0 = -w \left(\frac{L^4}{24} + \frac{L^5}{60L} \right) + \frac{wL^4}{4} + D$$

$$\therefore D = \frac{L^4}{24} + \frac{L^4}{60} - \frac{wL^4}{4}$$

$$\underline{D = \frac{-23wL^4}{120}}$$

S & D (10)

$$\therefore y = \frac{1}{EI} \left[-w \left(\frac{x^4}{24} + \frac{x^5}{60L} \right) + \frac{wL^3 x}{4} - \frac{23wL^4}{120} \right] \quad (\text{from (4)})$$

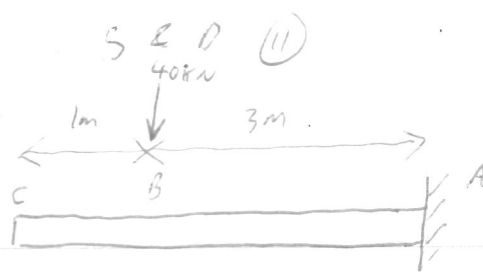
TO FIND

$y_{\max}$ WHEN $x=0$ FOR EXAMPLE.

$$y_{\max} = \frac{-23wL^4}{120EI}$$

~~SLACK AND BE FOUND WHEN~~

①



$$EI = 65 \text{ MNm}^2$$



② CANTILEVER IS 3m LONG WITH A POINT LOAD ON THE END - REMAINING 1m IS NOT BENDING.

$$\begin{aligned} A \rightarrow B: \quad y_{\max} &= -\frac{WL^3}{3EI} \\ &= -\frac{40 \times 10^3 \times 3^3}{3 \times 65 \times 10^6} \\ &= -5.538 \times 10^{-3} \text{ m} \end{aligned}$$

$$\delta_1 = \underline{\underline{5.54 \text{ mm}}}$$

$$\text{SLOPE } \left(\frac{dy}{dx} \right)_{\max} = \frac{WL^2}{2EI} = \frac{40 \times 10^3 \times 3^2}{2 \times 65 \times 10^6} = 2.769 \times 10^{-3} \text{ rad}$$

$$\therefore i = 2.769 \times 10^{-3} \text{ rad}$$

$$\begin{aligned} B \rightarrow C \text{ IS STRAIGHT. } \therefore \delta_2 &= -iL \\ &= -2.769 \times 10^{-3} \times 1 \\ &= -2.769 \times 10^{-3} \text{ m} \end{aligned}$$

$$\delta_2 = \underline{\underline{2.77 \text{ mm}}}$$

$$\begin{aligned} \text{At C: } \therefore \delta_{\text{TOTAL}} &= -5.538 \times 10^{-3} - 2.769 \times 10^{-3} \\ &= -8.31 \times 10^{-3} \text{ m} \\ &= \underline{\underline{-8.31 \text{ mm}}} \end{aligned}$$

(h)

$$w = \frac{40 \times 10^3}{3} = 13.33 \times 10^3 \text{ N/m}$$

$$\delta_1 = \frac{-wL^4}{8EI} = \frac{-13.33 \times 10^3 \times 3^4}{8 \times 65 \times 10^6} = \underline{\underline{-2.076 \times 10^{-3} \text{ m}}}$$

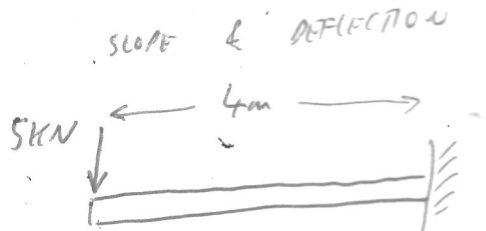
$$\text{SLOPE } \left(\frac{dy}{dx} \right)_{\max} = \frac{wL^3}{6EI} = \frac{13.33 \times 10^3 \times 3^3}{6 \times 65 \times 10^6} = \underline{\underline{0.923 \times 10^{-3} \text{ rad}}}$$

$$\begin{aligned} \delta_2 &= -0.923 \times 10^{-3} \times 1 \\ &= \underline{\underline{-0.923 \times 10^{-3} \text{ m}}} \end{aligned}$$

$$\begin{aligned} \text{C REFLECTION} &= -2.076 \times 10^{-3} - 0.923 \times 10^{-3} \\ &= \underline{\underline{-3 \text{ mm}}} \end{aligned}$$

63.9% reduction

②



$$EI = 53.3 \text{ MNm}^2$$

$$y_{\max} = \frac{-WL^3}{3EI} = \frac{-5000 \times 4^3}{3 \times 53.3 \times 10^6} = -0.002 \text{ m} = \underline{\underline{-2 \text{ mm}}}$$

$$\left(\frac{dy}{dx}\right)_{\max} = \frac{WL^2}{2EI} = \frac{5000 \times 4^2}{2 \times 53.3 \times 10^6} = 750 \times 10^{-6} \quad (\text{NO UNITS})$$

③

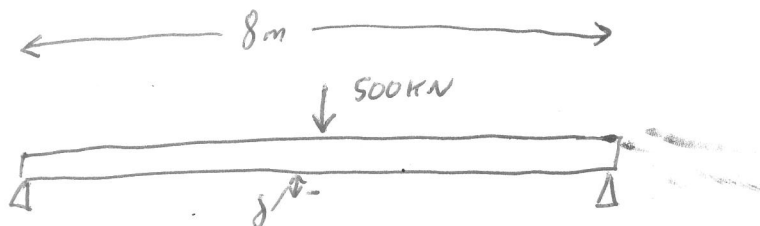


$$EI = 60 \text{ MNm}^2$$

$$y_{\max} = \frac{-WL^4}{8EI} = \frac{-300 \times 4^4}{8 \times 60 \times 10^6} = -0.00016 \text{ m} = \underline{\underline{-0.16 \text{ mm}}}$$

$$\left(\frac{dy}{dx}\right)_{\max} = \frac{WL^3}{6EI} = \frac{300 \times 4^3}{6 \times 60 \times 10^6} = 53.3 \times 10^{-6} \quad (\text{NO UNITS})$$

④

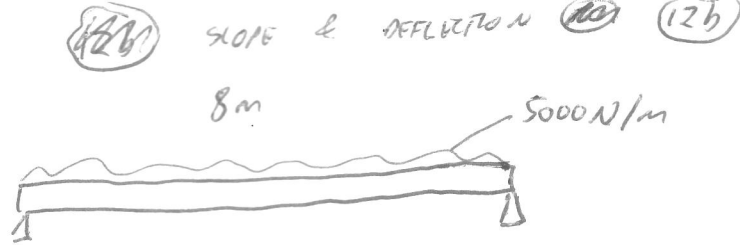


$$\delta = 2 \text{ mm}$$

$$y_{\max} = \frac{-WL^3}{48EI} \quad \therefore EI = \frac{-WL^3}{48y} = \frac{-500 \times 10^3 \times 8^3}{48 \times 2 \times 10^{-3}} = \underline{\underline{2.676 \text{ Nm}^2}}$$

$$\left(\frac{dy}{dx}\right)_{\max} = \pm \frac{WL^2}{16EI} = \pm \frac{500 \times 10^3 \times 8^2}{16 \times 2.67 \times 10^9} = \underline{\underline{\pm 750 \times 10^{-6}}} \quad (\text{NO UNITS})$$

-ve at LEFT
+ve at RIGHT



$EI = ?$

$\delta = 2 \text{ mm}$

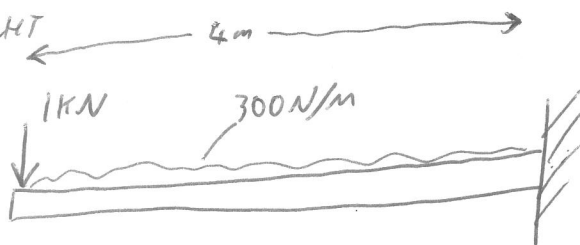
$$y_{\max} = \frac{-5WL^4}{384EI}$$

$$EI = \frac{-5WL^4}{384y_{\max}} = \frac{-5 \times 5000 \times 8^4}{384 \times 2 \times 10^{-3}} = 133.3 \text{ MNm}^2$$

$$\left(\frac{dy}{dx}\right)_{\max} = \pm \frac{WL^3}{24EI} = \pm \frac{5000 \times 8^3}{24 \times 133.3 \times 10^6} = \pm 800 \times 10^{-6} \text{ (NO UNITS)}$$

-ve AT LEFT.

+ve AT RIGHT



$EI = 20 \text{ MNm}^2$

POINT LOAD ONLY: $(y)_{\max} = \frac{-WL^3}{3EI} = \frac{-1000 \times 4^3}{3 \times 20 \times 10^6} = -0.00106 \text{ m} = \underline{\underline{-1.06 \text{ mm}}}$

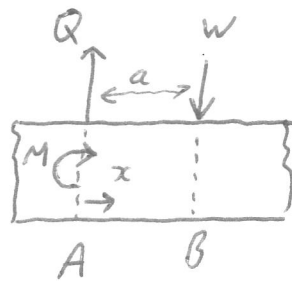
UDL ONLY: $y_{\max} = \frac{WL^4}{8EI} = \frac{-300 \times 4^4}{8 \times 20 \times 10^6} = -0.00048 \text{ m} = \underline{\underline{-0.48 \text{ mm}}}$

TOTAL DEFLECTION = $-1.06 \text{ mm} + -0.48 \text{ mm} = \underline{\underline{-1.54 \text{ mm}}}$

POINT LOAD ONLY: $\left(\frac{dy}{dx}\right)_{\max} = \frac{WL^2}{2EI} = \frac{1000 \times 4^2}{2 \times 20 \times 10^6} = \underline{\underline{400 \times 10^{-6}}}$

UDL ONLY: $\left(\frac{dy}{dx}\right)_{\max} = \frac{WL^3}{6EI} = \frac{300 \times 4^3}{6 \times 20 \times 10^6} = \underline{\underline{160 \times 10^{-6}}}$

TOTAL SLOPE = $\underline{\underline{560 \times 10^{-6}}}$



MACAULAY'S METHOD

BETWEEN A & B

$$M = EI \frac{d^2 y}{dx^2} = M + Qx \quad (1)$$

$$\therefore EI \frac{dy}{dx} = Mx + \frac{Qx^2}{2} + C_1 \quad (2)$$

$$EI y = \frac{Mx^2}{2} + \frac{Qx^3}{6} + C_1 x + C_2 \quad (3)$$

Beyond B

$$M = EI \frac{d^2 y}{dx^2} = M + Qx - W(x-a) \quad (4)$$

$$EI \frac{dy}{dx} = Mx + \frac{Qx^2}{2} - \frac{Wx^2}{2} + Wax + C_3 \quad (5)$$

$$EI y = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{Wx^3}{6} + \frac{Wax^2}{2} + C_3 x + C_4 \quad (6)$$

AT B THE SLOPE IS THE SAME $\therefore$ USING (2) & (5)

$$Mx + \frac{Qx^2}{2} + C_1 = Mx + \frac{Qx^2}{2} - \frac{Wx^2}{2} + Wax + C_3$$

AT B, $x=a$

$$\therefore \cancel{Ma} + \frac{Qa^2}{2} + C_1 = \cancel{Ma} + \frac{Qa^2}{2} - \frac{Wa^2}{2} + Wa^2 + C_3$$

$$\therefore C_1 = + \frac{Wa^2}{2} + C_3$$

$$\text{or } C_3 = C_1 - \frac{Wa^2}{2} \quad (6a)$$

Putting (6a) in (5)

$$\begin{aligned}
 EI \frac{dy}{dx} &= Mx + \frac{Qx^2}{2} - \frac{wx^2}{2} + Wax + C_3 \\
 &= Mx + \frac{Qx^2}{2} - \frac{wx^2}{2} + Wax + C_1 - \frac{Wa^2}{2} \\
 &= Mx + \frac{Qx^2}{2} - \frac{w}{2}(x^2 - 2ax + a^2) + C_1
 \end{aligned}$$

$$EI \frac{dy}{dx} = Mx + \frac{Qx^2}{2} - \frac{w}{2}(x-a)^2 + C_1 \quad (7)$$

ALSO THE DEFLECTION IS THE SAME AT (B). $\therefore$ USING (3) & (6)

$$\frac{Mx^2}{2} + \frac{Qx^3}{6} + C_1x + C_2 = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{wx^3}{6} + \frac{Wax^2}{2} + C_3x + C_4$$

$$x=a: \frac{Ma^2}{2} + \frac{Qa^3}{6} + C_1a + C_2 = \frac{Ma^2}{2} + \frac{Qa^3}{6} - \frac{wa^3}{6} + \frac{Wa^3}{2} + C_3a + C_4$$

$$C_1a + C_2 = -\frac{wa^3}{6} + \frac{Wa^3}{2} + C_3a + C_4$$

PUTTING C_3 (6a) in :

$$C_1a + C_2 = -\frac{wa^3}{6} + \frac{Wa^3}{2} + \left(C_1 - \frac{Wa^2}{2}\right)a + C_4$$

$$C_1a + C_2 = -\frac{wa^3}{6} + \frac{Wa^3}{2} + C_1a - \frac{Wa^3}{2} + C_4$$

$$\therefore C_2 = -\frac{wa^3}{6} + C_4$$

$$\therefore \boxed{C_4 = C_2 + \frac{wa^3}{6}} \quad (7a)$$

Putting (7a) in (6)

$$EIy = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{wx^3}{6} + \frac{Wax^2}{2} + C_3x + C_2 + \frac{wa^3}{6}$$

Putting (6a) in

$$EIy = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{wx^3}{6} + \frac{Wax^2}{2} + \left(C_1 - \frac{Wa^2}{2}\right)x + C_2 + \frac{wa^3}{6}$$

PUTTING (6) + (7) INTO (6)

$$EI_y = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{wx^3}{6} + \frac{wax^2}{2} + \left(C_1 - \frac{wa^2}{2}\right)x + C_2 + \frac{wa^3}{6}$$

$$= \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{wx^3}{6} + \frac{wax^2}{2} + C_1x - \frac{wa^2x}{2} + C_2 + \frac{wa^3}{6}$$

$$= \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{w}{6} \left[x^3 - 3ax^2 + 3a^2x - a^3 \right] + C_1x + C_2$$

$$EI_y = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{w}{6} (x-a)^3 + C_1x + C_2 \quad (8)$$

HENCE BEYOND B

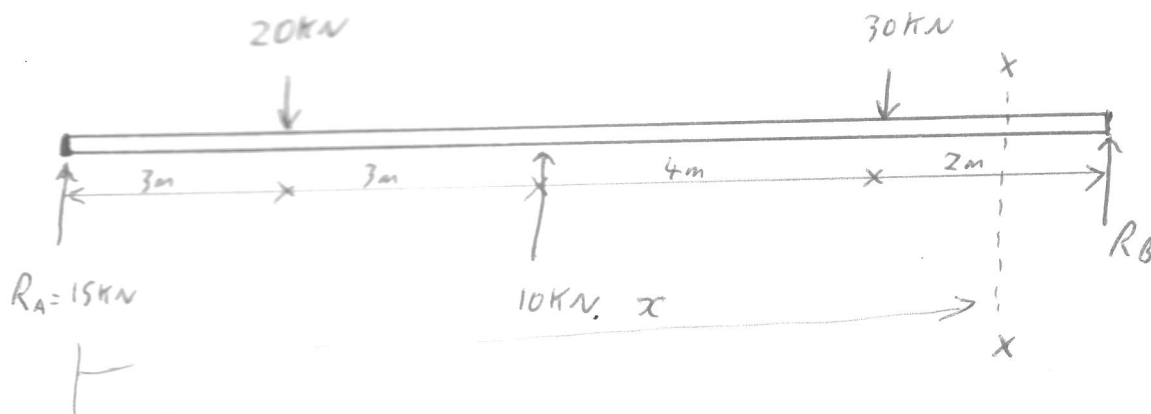
$$EI \frac{d^2y}{dx^2} = M + Qx - w(x-a)$$

$$EI \frac{dy}{dx} = Mx + \frac{Qx^2}{2} - \frac{w}{2} (x-a)^2 + C_1$$

$$EI_y = \frac{Mx^2}{2} + \frac{Qx^3}{6} - \frac{w}{6} (x-a)^3 + C_1x + C_2$$

HENCE WE INTEGRATE THE TERM $w(x-a)$ WITH RESPECT TO $(x-a)$ NOT x . $(x-a)$ IS THE MACAULAY TERM

$$[x-a]$$

EXAMPLE WITH POINT LOADS

$$G.M. \rightarrow \frac{EI \, dy}{dx} = 15x - 20[(x-3)] + 10[(x-6)] - 30[(x-10)]$$

NOTE THAT WE ARE FINDING THE MOMENT FURTHEST AWAY FROM ORIGIN TO
~~BEFORE~~ ~~WE CAN~~ FIND EQUATION THAT ACCOUNTS FOR
 EVERYTHING. WE CAN THEN SIMPLIFY IF " $x < 6$ ".

$$\frac{EI \, dy}{dx} = \frac{15x^2}{2} - 20 \left[\frac{(x-3)^2}{2} \right] + 10 \left[\frac{(x-6)^2}{2} \right] - 30 \left[\frac{(x-10)^2}{2} \right] + A$$

SINCE LOADS ARE IN kN TO KEEP THINGS CONSISTANT!

$$\frac{EI \, y}{10^3} = \frac{15x^3}{6} - 20 \left[\frac{(x-3)^3}{6} \right] + 10 \left[\frac{(x-6)^3}{6} \right] - 30 \left[\frac{(x-10)^3}{6} \right] + Ax + B$$

WHEN $x=0$, $y=0$, ~~4~~ ~~AND~~ ~~ANY~~ ~~NEGATIVE~~ ~~TERMS~~ ~~:~~ $B=0$

WHEN $x=12$, $y=0$

$$0 = \frac{15 \times 12^3}{6} - 20 \left(\frac{9^3}{6} \right) + 10 \left(\frac{6^3}{6} \right) - 30 \left(\frac{2^3}{6} \right) + 12A$$

$$= 4320 - 2430 + 360 - 40 + 12A$$

$$\therefore A = \underline{\underline{-184.2}}$$

IF WE WANTED TO FIND SLOPE AT $x=6$: SLOPE & DEFLECTION

$$\therefore \frac{EI}{1 \times 10^3} y = \frac{15x^3}{6} - 20 \left[\frac{(x-3)^3}{6} \right] + 10 \left[\frac{(x-6)^3}{6} \right] - 30 \left[\frac{(x-10)^3}{6} \right] - 184.2x$$

WHEN $x=6$

$$\therefore \frac{EI}{1 \times 10^3} y = \frac{15 \times 6^3}{6} - 20 \left(\frac{3^3}{6} \right) - 184.2 \times 6$$

$$\therefore y = \frac{-655.2 \times 10^3}{EI}$$

WITH TYPICAL VALUES OF

$$E = 2086 \text{ N/m}^2$$

$$I = 82 \times 10^{-6} \text{ m}^4$$

$$y = 38.4 \times 10^{-3} \text{ m}$$

$$y = \underline{\underline{38.4 \text{ mm}}}$$

EXPLAIN THIS FIRST

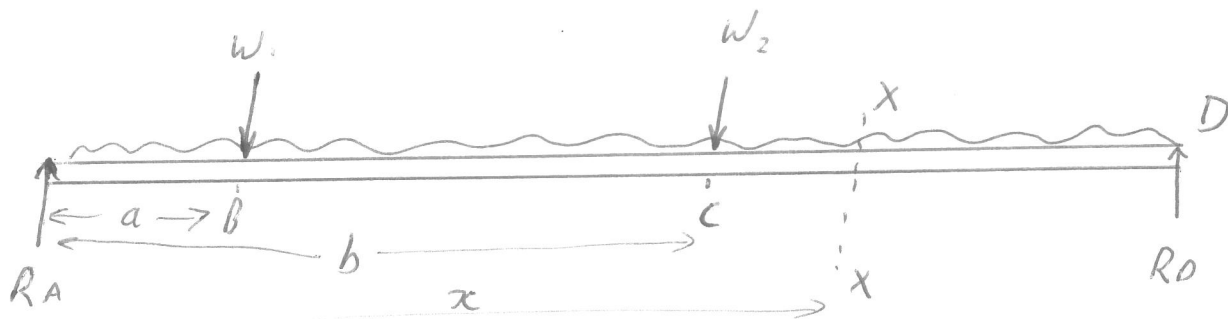
THEN THIS

GENERAL CASE

HENCE

$$\frac{EI}{1 \times 10^3} \frac{dy}{dx} = \frac{15x^2}{2} - 20 \left[\frac{(x-3)^2}{2} \right] + 10 \left[\frac{(x-6)^2}{2} \right] - 30 \left[\frac{(x-10)^2}{2} \right] - 184.2$$

$$\frac{EI}{1 \times 10^3} y = \frac{15x^3}{6} - 20 \left[\frac{(x-3)^3}{6} \right] + 10 \left[\frac{(x-6)^3}{6} \right] - 30 \left[\frac{(x-10)^3}{6} \right] - 184.2x$$



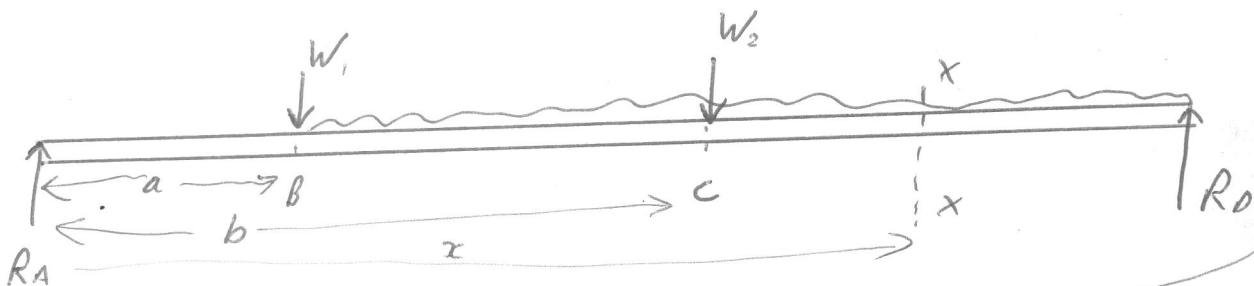
IF UDL IS OVER ENTIRE BEAM THEN

$$BM_{xx} = EI \frac{d^2y}{dx^2} = R_A x - \frac{wx^2}{2} - w_1[(x-a)] - w_2[(x-b)]$$

$\frac{wx^2}{2} = wx \cdot \frac{x}{2}$

IF THE U.D.L DOES NOT REQUIRE THE SPECIAL TREATMENT ASSOCIATED WITH THE MACAULAY TERMS.

IF HOWEVER THE U.D.L. STARTS AT B:



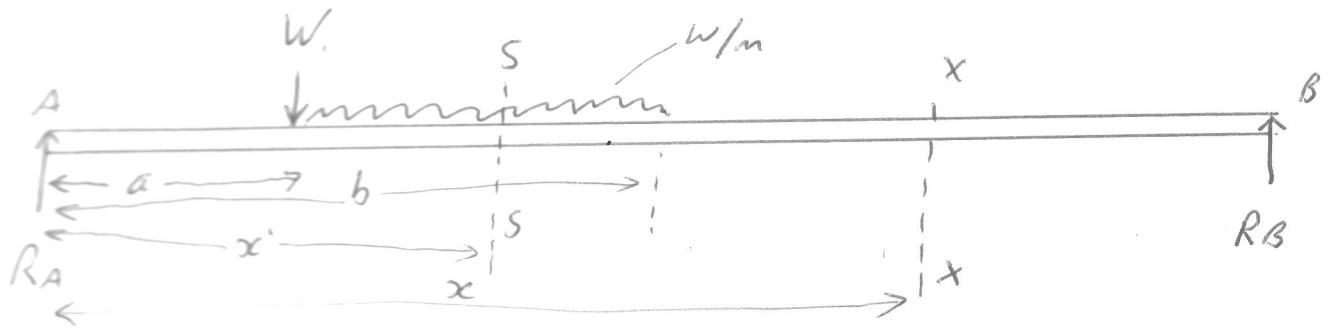
THEN THE U.D.L BECOMES A MACAULAY TERM

$$BM_{xx} = EI \frac{d^2y}{dx^2} = R_A x - w \left[\frac{(x-a)^2}{2} \right] - w_1[(x-a)] - w_2[(x-b)]$$

$$EI \frac{dy}{dx} = R_A \frac{x^2}{2} - w_1 \left[\frac{(x-a)^2}{2} \right] - w \left[\frac{(x-a)^3}{6} \right] - w_2 \left[\frac{(x-b)^3}{6} \right] + A$$

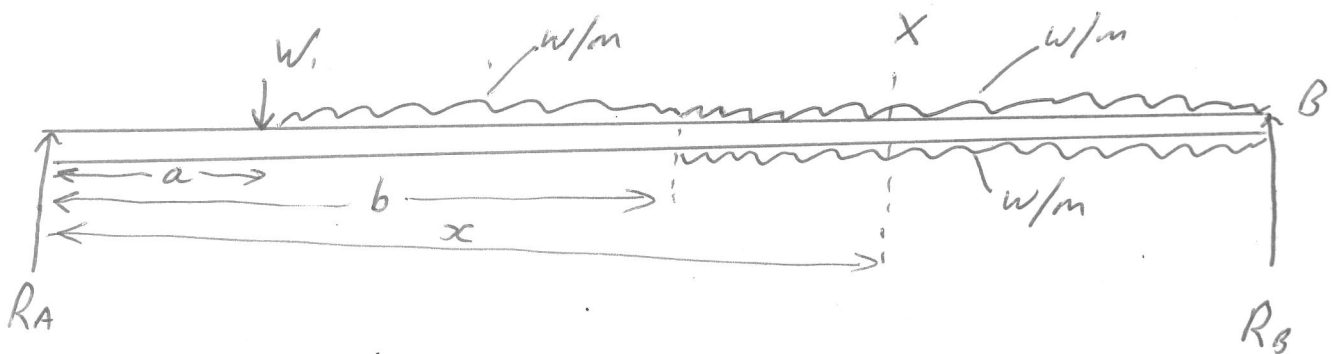
$$EI y = \frac{R_A x^3}{6} - w_1 \left[\frac{(x-a)^3}{6} \right] - w \left[\frac{(x-a)^4}{24} \right] - w_2 \left[\frac{(x-b)^3}{6} \right] + Ax + B$$

ONCE A ~~MAN~~ TERM IS INTRODUCED IN THE B.M. EXPRESSION IT WILL APPLY ACROSS THE REST OF THE BEAM. THUS IF THE UDL ACTUALLY STOPS A DUMMY ~~UOL~~ UDL (EQUAL AND OPPOSITE) IS APPLIED



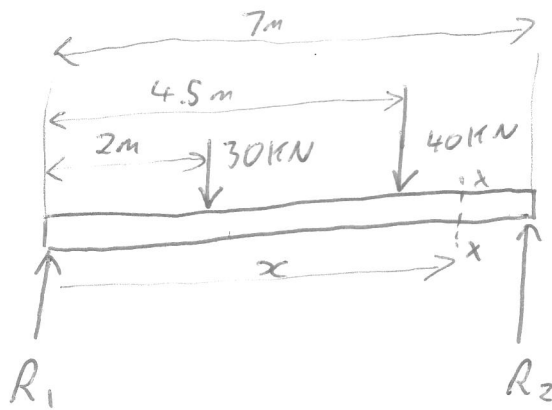
$$BM_{ss} = RA x' - W_1 [(x' - a)] - W \left[\frac{(x' - a)^2}{2} \right]$$

IF WE WANT TO FIND BM_{xx} THEN WE NEED TO CONTINUE THE U.D.L & APPLY AN EQUAL & OPPOSITE LOAD.



$$BM_{xx} = EI \frac{d^2 y}{dx^2} = RA x - W_1 [(x - a)] - W \left[\frac{(x - a)^2}{2} \right] + W \left[\frac{(x - b)^2}{2} \right]$$

①



$$EI = 200 \text{ MNm}^2$$

MOMENTS ABOUT
RIGHT END.

$$30 \times 5 + 40 \times 2.5 = 7R_1 \quad \therefore R_1 = 35.71 \text{ kN}$$

$$\therefore R_2 = 70 - 35.71 = 34.29 \text{ kN}$$

B.M.:

$$EI \frac{d^2y}{dx^2} = 35710x - 30000[x-2] - 40000[x-4.5]$$

$$\therefore EI \frac{dy}{dx} = 35710 \frac{x^2}{2} - 30000 \frac{[x-2]^2}{2} - 40000 \frac{[x-4.5]^2}{2} + A \quad (1)$$

$$EI y = 35710 \frac{x^3}{6} - 30000 \frac{[x-2]^3}{6} - 40000 \frac{[x-4.5]^3}{6} + Ax + B \quad (2)$$

WHEN $x=0$ & $y=0$

FROM (2)

$$EI y = 35710 \frac{x^3}{6} - 30000 \frac{[x-2]^3}{6} - 40000 \frac{[x-4.5]^3}{6} + Ax + B$$

$$0 = 0 + 0 + B$$

$$\therefore \underline{B=0}$$

WHEN $x=7$ $y=0$

$$\text{FROM (2)} \quad EI y = 35710 \frac{x^3}{6} - 30000 \frac{[x-2]^3}{6} - 40000 \frac{[x-4.5]^3}{6} + Ax$$

$$0 = 35710 \frac{7^3}{6} - 30000 \frac{5^3}{6} - 40000 \frac{2.5^3}{6} + 7A$$

$$\therefore \underline{A = -187400}$$

FROM EQUATION (1)

$$EI \frac{dy}{dx} = 35710 \frac{x^2}{2} - 30,000 \frac{[x-2]^2}{2} - 40,000 \frac{[x-4.5]^2}{2} - 187400$$

$$x = 3.5$$

$$200 \times 10^6 \frac{dy}{dx} = 35710 \frac{[3.5]^2}{2} - 30,000 \frac{[1.5]^2}{2} - 187400$$

$$1. \frac{dy}{dx} = \frac{-2426}{200 \times 10^6}$$

$$= -0.00001213 \quad : \text{SLOPE AT MIDDLE}$$

FROM (2)

$$EI y = 35710 \frac{x^3}{6} - 30000 \frac{[x-2]^3}{6} - 40000 \frac{[x-4.5]^3}{6} + 187400 x$$

$$x = 3.5: 200 \times 10^6 y = 35710 \frac{3.5^3}{6} - 30000 \frac{1.5^3}{6} - 187400 \cdot 3.5$$

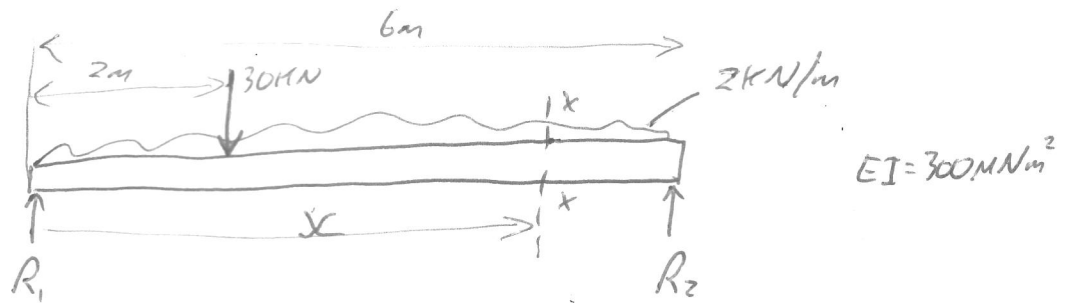
$$y = \frac{-417598}{200 \times 10^6} = -0.00209 \text{ m}$$

$$= \underline{\underline{-2.09 \text{ mm}}}$$

EI = FLEXURAL STIFFNESS

y = DEFLECTION

(2)



MOMENTS ABOUT RIGHT END.

$$30 \times 4 + 2 \times 6 \times \frac{6}{2} = 6R_1$$

$$\therefore R_1 = \underline{26 \text{ kN}}$$

$$R_2 = 42 - 26 = \underline{16 \text{ kN}}$$

$$\begin{aligned} \text{B.M. } EI \frac{d^2y}{dx^2} &= R_1 x - 30,000 [x-2] - wx \cdot \frac{x}{2} \\ &= 26,000x - 30,000 [x-2] - \frac{2000x^2}{2} \end{aligned}$$

$$EI \frac{dy}{dx} = \frac{26000x^2}{2} - \frac{30,000 [x-2]^2}{2} - \frac{2000x^3}{6} + A$$

$$EI y = \frac{26000x^3}{6} - \frac{30,000 [x-2]^3}{6} - \frac{2000x^4}{24} + Ax + B$$

WHEN $x=0, y=0$, IGNORING ANY NEGATIVE TERMS $\therefore \underline{B=0}$ WHEN $x=6, y=0$

$$0 = 26000 \cdot \frac{6^3}{6} - 30,000 \cdot \frac{4^3}{6} - 2000 \cdot \frac{6^4}{24} + 6A$$

$$\therefore 6A = -508,000$$

$$\therefore A = \underline{\underline{-84557}}$$

$$\therefore EI \frac{dy}{dx} = \frac{26000x^2}{2} - \frac{30,000 \cancel{[x-2]}^2}{2} - \frac{2000x^3}{6} - 84557$$

$$x=0 \quad EI \frac{dy}{dx} = -84557$$

$$\therefore \frac{dy}{dx} = \frac{-84557}{300 \times 10^6} = \underline{\underline{-0.000282}}$$

$$\text{Also } EIy = \frac{26000x^3}{6} - \frac{30000[x-2]^3}{6} - \frac{2000x^4}{24} - 84557x$$

$$\text{AT CENTRE } x=3$$

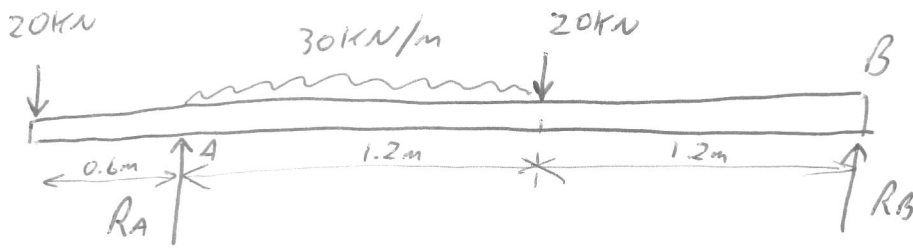
$$300 \times 10^6 y = \frac{26,000 \cdot 3^3}{6} - \frac{30,000 \cdot 1^3}{6} - \frac{2000 \cdot 3^4}{24} - 84557 \cdot 3$$

$$y = \frac{-160296}{300 \times 10^6}$$

$$= -0.000534 \text{ m}$$

$$\text{or } \underline{\underline{0.534 \text{ mm}}}$$

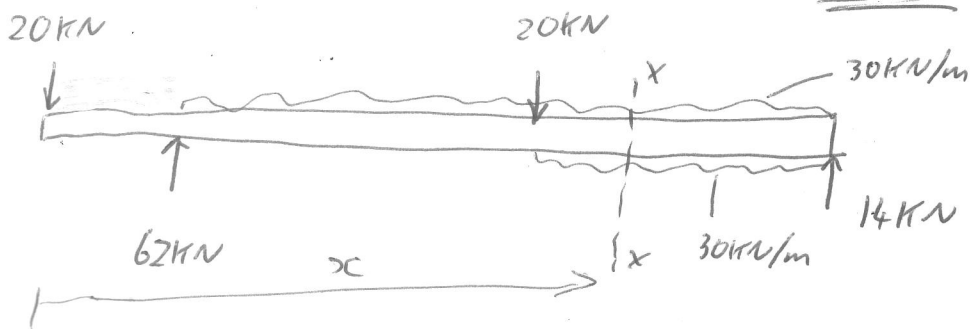
(3)



TAKING MOMENTS ABOUT B: $(3 \times 20) + (30 \times 1.2) \times 1.8 + 1.2 \times 20 = 2.4 \times R_A$

$\therefore R_A = 62 \text{ kN}$

$\therefore R_B = 20 + (30 \times 1.2) + 20 - 62$
 $= 14 \text{ kN}$



$M_{xx} = \frac{EI d^2 y}{1 \times 10^3 dx^2} = -20x + 62[x-0.6] - 20[x-1.8] - 30 \frac{[x-0.6]^2}{2} + 30 \frac{[x-1.8]^2}{2}$

$\frac{EI dy}{1 \times 10^3 dx} = \frac{-20x^2}{2} + 62 \frac{[x-0.6]^2}{2} - 20 \frac{[x-1.8]^2}{2} - 30 \frac{[x-0.6]^3}{6} + 30 \frac{[x-1.8]^3}{6} + A$

$\frac{EI y}{1 \times 10^3} = \frac{-20x^3}{6} + 62 \frac{[x-0.6]^3}{6} - 20 \frac{[x-1.8]^3}{6} - 30 \frac{[x-0.6]^4}{24} + 30 \frac{[x-1.8]^4}{24} + Ax + B$

WHEN $x=0.6$ $y=0$

$\therefore 0 = \frac{-20 \times 0.6^3}{6} + 0.6A + B$ (1)

WHEN $x=3$, $y=0$

$$0 = \frac{-20 \times 3^3}{6} + \frac{62 \times 2.4^3}{6} - \frac{30 \times 2.4^4}{24} + \frac{30 \times 1.2^4}{24} - \frac{20 \times 1.2^3}{6} + 3A + B$$

$$\therefore -8.208 = 3A + B \quad (2)$$

$$(2) - (1) \quad \text{Bkt. 8/20/18} \quad -8.928 - 2.4A \quad \therefore \underline{\underline{A = -3.72}}$$

FROM (1)

$$B = -0.72 - 0.6(-3.72)$$

$$\therefore \underline{\underline{B = 2.952}}$$

AT $x=1$

$$\therefore \frac{EIy}{1 \times 10^3} = \frac{-20x^3}{6} + \frac{62[x-0.6]^3}{6} - \frac{20[x-1.8]^3}{6} - \frac{30[x-0.6]^4}{24} + \frac{30[x-1.8]^4}{24} - 3.72x + 2.952$$

$$y = \frac{1 \times 10^3}{EI} \left[\frac{-20}{6} + \frac{62}{6} \times 0.4^3 - \frac{30}{24} \times 0.4^4 - 3.72 \times 1 + 2.952 \right]$$

$$= \frac{-10^3 \times 3.472}{0.65 \times 10^6}$$

$$= -5.34 \times 10^{-3} \text{ m}$$

$$= \underline{\underline{-5.34 \text{ mm}}}$$