

FACTORISING

A factor is a common part of two or more terms which makes up an algebraic expression. Thus the expression $3x + 3y$ has two terms which have the number 3 common to both terms. Thus:

$$3x + 3y = 3(x + y)$$

We say that 3 and $(x + y)$ are the factors of $3x + 3y$.

To factorise algebraic expressions of this kind, we first find the Highest Common Factor (HCF) of all terms making up the expression. The HCF then appears outside the brackets. To find the terms inside the brackets divide each of the terms making up the expression by the HCF.

I. HIGHEST COMMON FACTOR (HCF)

The HCF of a set of algebraic expressions is the highest expression which is a factor of each of the given expressions. To find the HCF we therefore select the lowest power of each of the quantities which occur in **all** of the expressions and multiply them together. The method is shown in the following examples:

- a. Find the factors of: $ax + bx$

The HCF of ax and bx is x , therefore:

$$ax + bx = x(a + b)$$

- b. Find the factors of: $m^2n - 2mn^2$

The HCF of m^2n and $2mn^2$ is mn , therefore:

$$m^2n - 2mn^2 = mn(m - 2n)$$

- c. Find the factors of: $ab^2c^2 + a^2b^3c^3 + a^2b^4c^4$

The HCF of ab^2c^2 , $a^2b^3c^3$ and $a^2b^4c^4$ is ab^2c^2 , therefore:

$$ab^2c^2 + a^2b^3c^3 + a^2b^4c^4 = ab^2c^2(1 + abc + ab^2c^2)$$

- d. Find the factors of: $x^2y^3 + y^2x^3z^3 + xy^2z^2$

The HCF of x^2y^3 , $y^2x^3z^3$ and xy^2z^2 is xy^2 , therefore:

$$x^2y^3 + y^2x^3z^3 + xy^2z^2 = xy^2(xy + x^2z^3 + z^2)$$

- e. Find the factors of: $3m^2np^3 + 6m^3n^2p^2 + 24m^3p^4$

The HCF of $3m^2np^3$, $6m^3n^2p^2$ and $24m^3p^4$ is $3m^2p^2$, therefore:

$$3m^2np^3 + 6m^3n^2p^2 + 24m^3p^4 = 3m^2p^2(np + 2mn^2p^4 + 8mp^2)$$

- f. Find the factors of:

$$\frac{ac}{x} + \frac{bc}{x^2} - \frac{cd}{x^3}$$

The HCF of $\frac{ac}{x}$, $\frac{bc}{x^2}$ and $\frac{cd}{x^3}$ is $\frac{c}{x}$, therefore:

$$\frac{ac}{x} + \frac{bc}{x^2} - \frac{cd}{x^3} = \frac{c}{x} \left(a + \frac{b}{x} + \frac{d}{x^2} \right)$$

II. FACTORISING BY GROUPING

Four termed expressions can sometimes be factorised by grouping into two binomial expressions and extracting common factors from each.

For example:

- a. Factorise $ax + ay + bx + by$

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \\ &= (x + y)(a + b) \end{aligned}$$

- b. Factorise $mp + nq + np + mq$

$$\begin{aligned} mp + nq + np + mq &= (mp + mq) + (nq + np) \\ &= m(p + q) + n(p + q) \\ &= (p + q)(m + n) \end{aligned}$$

- c. Factorise $4ax - 6by - 4bx - 6ay$

$$\begin{aligned} 4ax + 3by - 2bx - 6ay &= (4ax - 2bx) - (6ay - 3by) \\ &= 2x(2a - b) - 3y(2a - b) \\ &= (2a - b)(2x - 3y) \end{aligned}$$

- d. Factorise $x^3 - 4x^2y + xy^2 - 4y^3$

$$\begin{aligned} x^3 - 4x^2y + xy^2 - 4y^3 &= (x^3 + xy^2) - (4x^2y + 4y^3) \\ &= x(x^2 + y^2) - 4y(x^2 + y^2) \\ &= (x^2 + y^2)(x - 4y) \end{aligned}$$

III. DIFFERENCE OF SQUARES

$$a^2 - b^2 = (a + b)(a - b)$$

IV. SUM OF SQUARES

$$a^2 + b^2 = (a + b + \sqrt{2ab})(a + b - \sqrt{2ab})$$

This identity is always true, but it's useful for factoring only when $2ab$ is a perfect square.

Explanation:

Notice that the factors that have the form of $(p + q)(p - q)$, multiplies to $p^2 - q^2$. This suggests that, for factoring $a^2 + b^2$, it might be fruitful to look at $(a + b)^2$ minus something. That's all well and good, but minus *what*?

The key is that $(a + b)^2 = a^2 + 2ab + b^2$. Comparing that to $a^2 + b^2$, you see that there's an extra term of $2ab$. So you have

$$a^2 + b^2 = (a + b)^2 - 2ab$$

That right-hand side is factorable as a difference of squares, if $2ab$ is a perfect square. And that's our factorization:

$$a^2 + b^2 = (a + b + \sqrt{2ab})(a + b - \sqrt{2ab})$$

V. FORM $a^2 \pm 2ab + b^2$

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

VI. SUM AND DIFFERENCE OF CUBES

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Examples:

- a. Factorise $16x^2 + 40xy + 25y^2$

$$\begin{aligned} 16x^2 + 40xy + 25y^2 &= (4x)^2 + 2 \times 4x \times 5y + (5y)^2 \\ &= (4x + 5y)^2 \end{aligned}$$

- b. Factorise $9x^2 - 12xy + 4y^2$

$$\begin{aligned} 9x^2 - 12xy + 4y^2 &= (3x)^2 - 2 \times 3x \times 2y + (2y)^2 \\ &= (3x - 2y)^2 \end{aligned}$$

- c. Factorise $(2x + 3y)^2 - (x - 4y)^2$

$$(2x + 3y)^2 - (x - 4y)^2 \text{ is of the form } a^2 - b^2 \text{ therefore}$$

$$\begin{aligned} (2x + 3y)^2 - (x - 4y)^2 &= [(2x + 3y) + (x - 4y)].[(2x + 3y) - (x - 4y)] \\ &= (3x - y).(x + 7y) \end{aligned}$$

- d. Factorise $x^3 - 8$

$$\begin{aligned} x^3 - 8 &= x^3 - 2^3 \\ &= (x - 2)(x^2 + 2x + 4) \end{aligned}$$

- e. Factorise $27x^3 + 1$

$$\begin{aligned} 27x^3 + 1 &= (3x)^3 + 1^3 \\ &= (3x + 1)(9x^2 - 3x + 1) \end{aligned}$$

f. Factorise $x^3y^6 - 64$

$$\begin{aligned}x^3y^6 - 64 &= (xy^2)^3 - 4 \\&= (xy^2 - 4)(x^2y^4 + 4xy^2 + 16)\end{aligned}$$

g. Factorise $4x^4 - 625y^4$

$$\begin{aligned}4x^4 - 625y^4 &= (2x^2 + 25y^2 + \sqrt{2 \times 2x^2 \times 25y^2})(2x^2 + 25y^2 - \sqrt{2 \times 2x^2 \times 25y^2}) \\&= (2x^2 + 25y^2 + \sqrt{100x^2y^2})(2x^2 + 25y^2 - \sqrt{100x^2y^2}) \\&= (2x^2 + 25y^2 + 10xy)(2x^2 + 25y^2 - 10xy)\end{aligned}$$