



Artificial Intelligence and Machine Learning (ACCA6011)

Fuzzy Logic

Dr. Seena Joseph

Fuzzy Logic

Fuzzy Logic

It is a precise logic of imprecision

Human being – Don't require precise and numerical input to make decision.

So why should computers ?

Computers – Can accept noisy imperfect input to make decision ?

Lotfi A Zadeh (1965) – Father of Fuzzy Logic

Lotfi Zadeh proposed a new approach to explain the vagueness or fuzziness

Fuzzy Logic

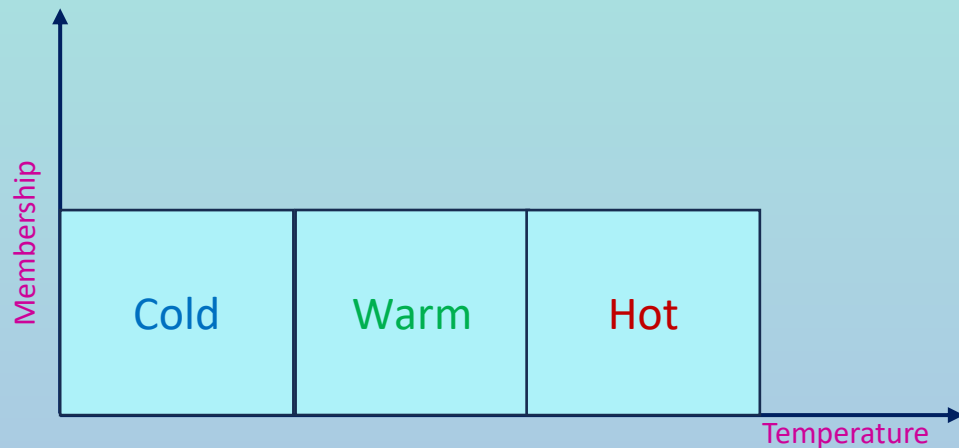
- Human Reasoning and Language
 - The classification used in language are often vague. What does the term like Large, Fast, Hot, or even very Hot mean ?
- Fuzzy logic – Allows this vagueness to be captured and applied in our solutions or problems.
- Fuzzy logic provides a way of capturing these Linguistic variables.
- In Artificial intelligence the action taken might depend on the value of a continuous variable
 - Traditional binary logic - Transition between actions may be abrupt
 - This may not be ideal for a situation where the action will be the result of the variables falling within specific range.

Fuzzy Logic

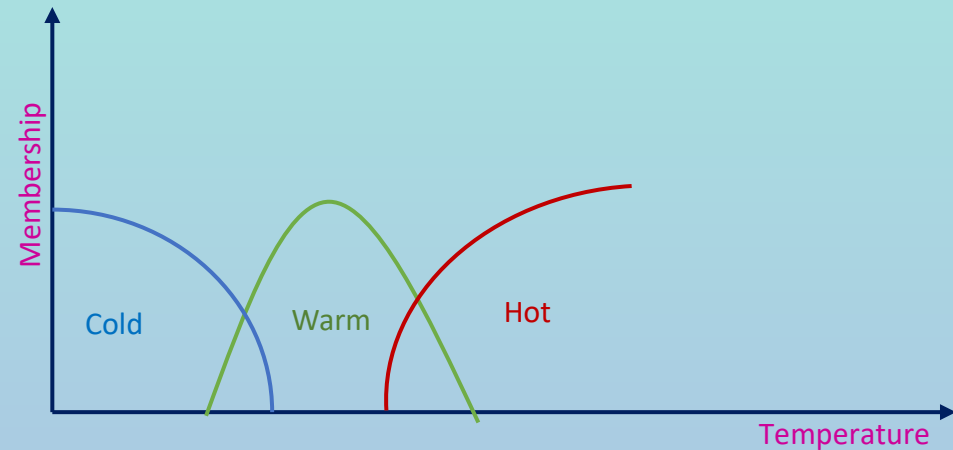
- Fuzzy – Things that are not clear or are vague
 - Handle the situation where we can't determine the statement is true or false
 - Provide a very valuable flexibility for reasoning
 - It is a precise logic of imprecision
 - It allows for a partial truth – a statement can be partially true or false rather than fully true or false
 - It resembles human decision making methodology – deals with vague and imprecise information

Fuzzy Logic

- Fuzzy sets
 - Fuzzy sets provide the mechanism to capture linguistic variables and record vagueness.
 - By allowing a non-binary membership of a set



Crisp Sets – Binary Membership

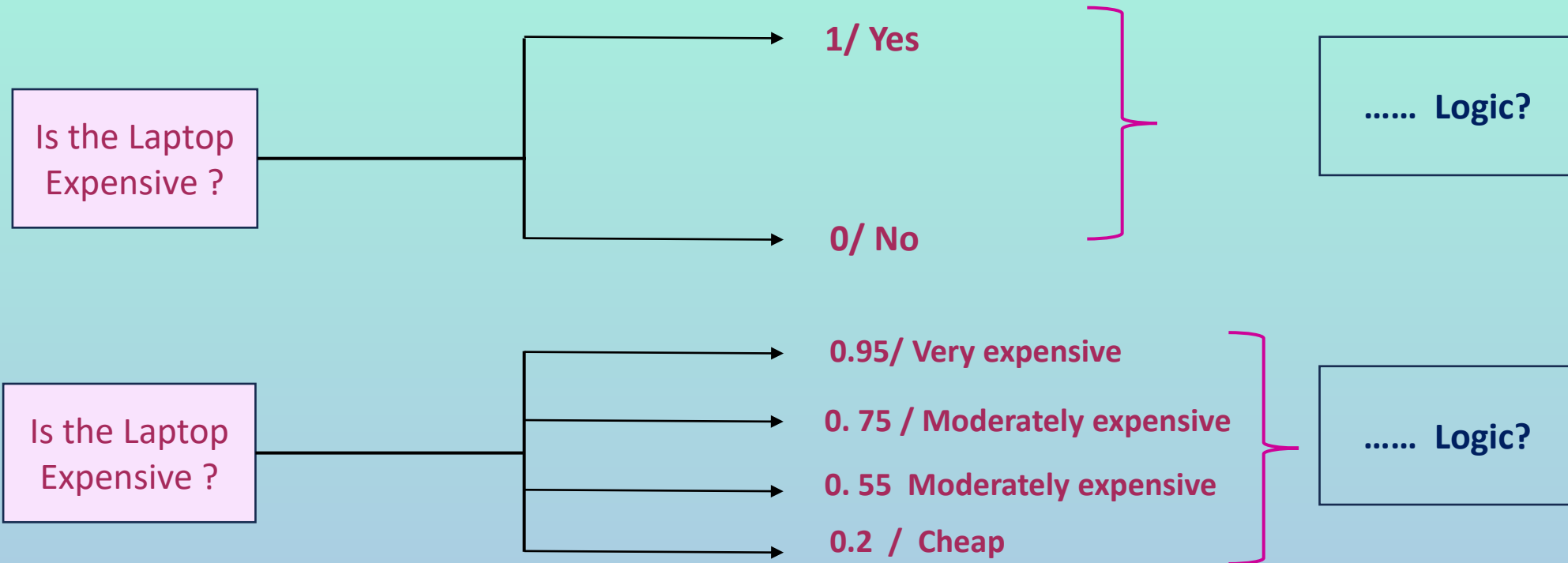


Fuzzy Sets – Continuous values Membership

Fuzzy Logic Vs Boolean Logic



Fuzzy Logic Vs Boolean Logic



FUZZY Membership Function

- Membership function
 - A function that specify the degree to which the given input value belongs to a set
- Degree of Membership or Membership Value – The output of a membership function; the value will be in the range of 0 to 1.
 - 0 indicates No Membership
 - 1 indicates Full Membership
 - Between 0 and 1 indicates Partial Membership
- Membership functions are used to describe the membership of each value for a variable for a fuzzy set.

Classical or Crisp sets

- Notion of Crisp sets : All individual in a universe set can be partitioned into two classes:
 - those who belongs to the set
 - OR
 - those who do not belongs to the set.
- A classical set is an unordered collection of different elements. It can be written explicitly by listing its elements using the set bracket.
- Example :
 - A set of Vowel in English language = { a, e, I, o , u }
 - A set of all positive integers less than 6 = { 1, 2, 3, 4, 5 }

Classical set Theory

- Sets are an unordered collection of objects
- Objects are called elements or members of the set.
- It can be written explicitly by listing its elements using the set bracket , { }.
- Example
 - Students in a class: { Alice, Bob, Chris }
 - Modules offered by a department: { DSTR101, INCP101, ... }
 - Colors of a rainbow: { red, orange, yellow, green, blue, purple }
 - States of matter { solid, liquid, gas }
 - Cities in Wales: { Swansea, Wales, Newport... }
 - Sets can contain non-related elements: { 3, a, red, Western Cape }

Classical set Theory

- A set can contain (almost) anything, we will most often use sets of numbers
 - All positive numbers less than or equal to 5: $\{1, 2, 3, 4, 5\}$
 - A few selected real numbers: $\{2.1, \pi, 0, -6.32\}$

Classical set Theory

- Sets are usually represented by a capital letter (A, B, S, etc.)
- Elements are usually represented by lower-case letter (a, x, y, etc.)
- Set elements can be enumerated by **Roaster method** or **Set builders**
- **Roaster method** : easiest way to represent set is by listing all elements
 - $A = \{1, 2, 3, 4, 5\}$ (Always not feasible for large sets)
 - Can use an ellipsis (...): $B = \{0, 1, 2, 3, \dots\}$
 - Can cause confusion. Consider the set $C = \{3, 5, 7, \dots\}$. What comes next?
 - If the set is all odd integers greater than 2, it is 9
 - If the set is all prime numbers greater than 2, it is 11

Classical set Theory

Set builders

- $S = \{x \mid x \text{ is prime and } x > 2\}$
- $E = \{x \mid x \text{ is odd and } x > 2\}$
 - The vertical bar means “such that”
 - Thus, set S is read (in English) as: “all elements x such that x is prime and x is greater than 2”
- A set is said to “contain” the various “members” or “elements” that make up the set
- If an element a is a member of (or an element of) a set S, we use then notation, $a \in S$
 - $4 \in \{1, 2, 3, 4\}$
- If an element is not a member of (or an element of) a set S, we use the notation $a \notin S$
 - $7 \notin \{1, 2, 3, 4\}$
 - $P = \{1, 2, 3, 4\}, \text{ Red} \notin P$

Classical set Theory

U is the **universal set** – the set of all of elements (or the “universe”) from which given any set is drawn.

- For the set $\{-2, 0.4, 2\}$, U would be the real numbers
- For the set $\{0, 1, 2\}$, U could be the natural numbers (zero and up), the integers, the rational numbers, or the real numbers, depending on the context
- For the set of the students in this class, U would be all the students in the University (or perhaps all the people in the world)
- For the set of the vowels of the alphabet, U would be all the letters of the alphabet
- To differentiate U from U (which is a set operation), the universal set is written in a different font (and in bold and italics)

Classical set Theory

Empty set

If a set has zero elements, it is called the empty (or null) set

Written using the symbol $\emptyset$

Thus, $\emptyset = \{ \}$ VERY IMPORTANT

If you get confused about the empty set in a problem, try replacing $\emptyset$ by $\{ \}$

As the empty set is a set, it can be an element of other sets

- $\{\emptyset, 1, 2, 3, x\}$ is a valid set

Classical set Theory

Empty set

Note that $\emptyset \neq \{\emptyset\}$

The first is a set of zero elements

The second is a set of 1 element (that one element being the empty set)

Replace $\emptyset$ by $\{\}$, and you get: $\{\} \neq \{\{\}\}$

It's easier to see that they are not equal that way

Classical set Theory

Set Equality

- Two sets are equal if they have the same elements
 - $\{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\}$
 - Remember that order does not matter!
 - $\{1, 2, 3, 2, 4, 3, 2, 1\} = \{4, 3, 2, 1\}$
 - Remember that duplicate elements do not matter!
- Two sets are not equal if they do not have the same elements
 - $\{1, 2, 3, 4, 5\} \neq \{1, 2, 3, 4\}$

Classical set Theory

Subset

- If all elements of a set S are also elements of a set T , then S is a subset of T
- For example, if $S = \{2, 4, 6\}$ and $T = \{1, 2, 3, 4, 5, 6, 7\}$, then S is a subset of T
 - This is specified by $S \subseteq T$
 - Or by $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$
- If S is not a subset of T , it is written as such:
 $S \not\subseteq T$
 - For example, $\{1, 2, 8\} \not\subseteq \{1, 2, 3, 4, 5, 6, 7\}$
- Note that any set is a subset of itself!
 - Given set $S = \{2, 4, 6\}$, since all the elements of S are elements of S , S is a subset of itself
 - This is kind of like saying 5 is less than or equal to 5
 - Thus, for any set S , $S \subseteq S$

Classical set Theory

Subset

- The empty set is a subset of all sets (including itself!)
 - Recall that all sets are subsets of themselves
- All sets are subsets of the universal set
- To define a subset:
 - $\forall x (x \in A \rightarrow x \in B)$
 - English translation: for all possible values of x , (meaning for all possible elements of a set), if x is an element of A , then x is an element of B

Classical set Theory

Proper Subset

- If S is a subset of T , and S is not equal to T , then S is a proper subset of T
 - Let $T = \{0, 1, 2, 3, 4, 5\}$
 - If $S = \{1, 2, 3\}$, S is not equal to T , and S is a proper subset of T
 - A proper subset is written as $S \subset T$
 - Let $R = \{0, 1, 2, 3, 4, 5\}$. R is equal to T , and thus is a subset (but not a proper subset) or T
 - Can be written as: $R \not\subset T$ and $R \subseteq T$ (or just $R = T$)
 - Let $Q = \{4, 5, 6\}$. Q is neither a subset of T nor a proper subset of T
 - $Q \not\subset T$; $Q \not\subseteq T$
- The difference between “subset” and “proper subset” is like the difference between “less than or equal to” and “less than” for numbers
- The empty set is a proper subset of all sets other than the empty set (as it is equal to the empty set)

Classical set Theory

Set cardinality

The cardinality of a set is the number of elements in a set

- Written as $|A|$

Examples

- Let $R = \{1, 2, 3, 4, 5\}$. Then $|R| = 5$
- $|\emptyset| = 0$
- Let $S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $|S| = 4$

Classical set Theory

Power set

- Given the set $S = \{0, 1\}$. What are all the possible subsets of S ?
 - They are: $\emptyset$ (as it is a subset of all sets), $\{0\}$, $\{1\}$, and $\{0, 1\}$
 - The power set of S (written as $P(S)$) is the set of all the subsets of S
 - $P(S) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$
 - Note that $|S| = 2$ and $|P(S)| = 4$
- Let $T = \{0, 1, 2\}$. The $P(T) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
 - Note that $|T| = 3$ and $|P(T)| = 8$
- $P(\emptyset) = \{\emptyset\}$
 - Note that $|\emptyset| = 0$ and $|P(\emptyset)| = 1$
- If a set has n elements, then the power set will have 2^n elements

Classical set Theory

- Classical set operation ?
 - Union
 - Intersection
 - Difference
 - Complement sets

Classical set Theory – Finite and Infinite sets

- Finite Sets
- A set which contains a definite number of elements is called a **finite set**.
 - Example – $S = \{x | x \in \mathbb{N} \text{ and } 100 < x < 150\}$
- Infinite sets
 - A set with infinite number of elements
 - Example – $S = \{x | x \in \mathbb{N} \text{ and } x > 70\}$

Fuzzy sets

- In real life scenario where inclusion and non-inclusion in a set are not clearly defined.
 - Example : classes of short people; beautiful garden; expensive cell phones; sunny days
- The boundaries of these sets are vague and the transition from inclusion to non-inclusion or member to non-member is gradual rather than abrupt.
- In fuzzy set, an element can belong to a set to a degree, k ($0 \leq k \leq 1$).
- This in contrast to crisp set theory where an element must definitely belongs to a set or not to a set.

Fuzzy sets

- Fuzzy set representation

- $X = \{x_1, x_2, x_3, x_4, \dots, x_n\}$

- Corresponding membership function is

$$\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \mu_A(x_4), \dots, \mu_A(x_n) \rightarrow [0,1]$$

- Fuzzy A set can be represented as :

- $A = \{(x_1, \mu_A(x_1)), (x_2, \mu_A(x_2)), (x_3, \mu_A(x_3)), (x_4, \mu_A(x_4)), \dots, (x_n, \mu_A(x_n))\}$

- If X is a collection of objects, then a **fuzzy set A** in X is a set of ordered pairs:

$$A = \{ (x, \mu_A(x)) : x \in X, \mu_A(x) : X \rightarrow [0, 1] \}$$

where $\mu_A(x)$ is the membership function of A and is defined as the function from X into [0, 1].

The membership function gives a grade of membership or value of membership

Fuzzy sets

Fuzzy set representation

For Example,

- $X = \{ P, Q, R, S, T \}$ -> Set of satisfied ACCA6011 students

- A vague or uncertain

$$A = \{(P, 0.5), (Q, 0.3), (R, 0.8), (S, 0.6), (T, 0.7)\}$$

Most satisfied

R

Least satisfied

Q

What are the other interpretation ?

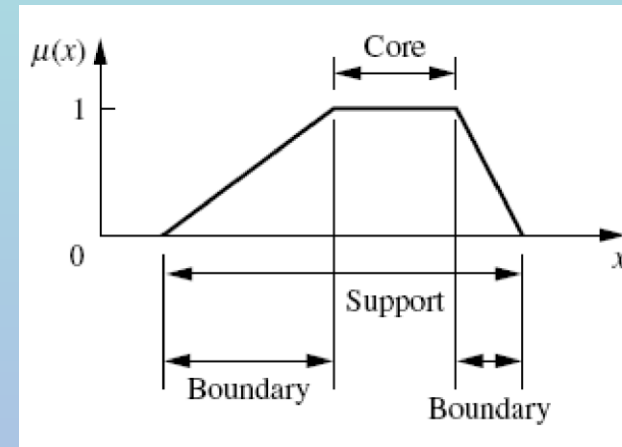
Feature of the membership function

Core : It is defined as the region of the universe that is characterized by complete and full membership in the Set A. **Core** comprises those elements x of the universe such that $\mu_A(x)=1$.

where the membership is completely true.

Support : Region of the universe that is characterized by the nonzero membership in the set A. **Support** comprises those elements x of the universe such that $\mu_A(x) > 0$.

Boundaries : Region of the universe containing elements that have a nonzero membership but not complete membership. **Boundaries** comprise those elements x of the universe such that $0 < \mu_A(x) < 1$. These elements are with some degree of fuzziness.

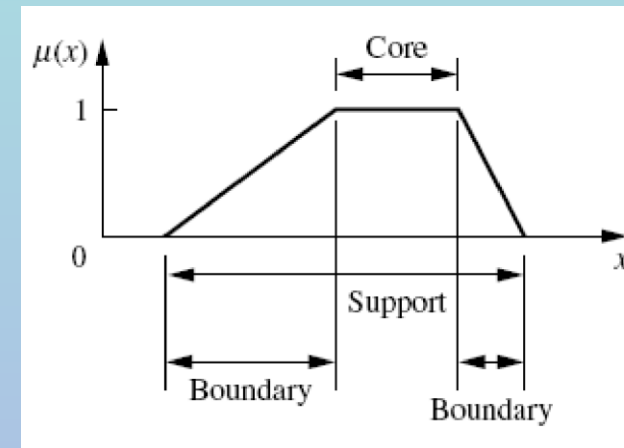


Feature of the membership function

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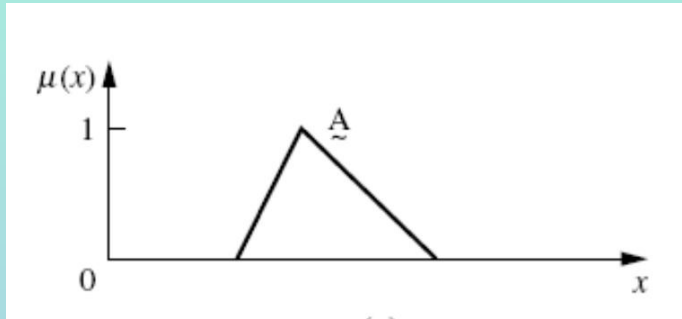
Support : Region of the universe that is characterized by the nonzero membership in the set A. **Support** comprises those elements x of the universe such that $\mu_A(x) > 0$.

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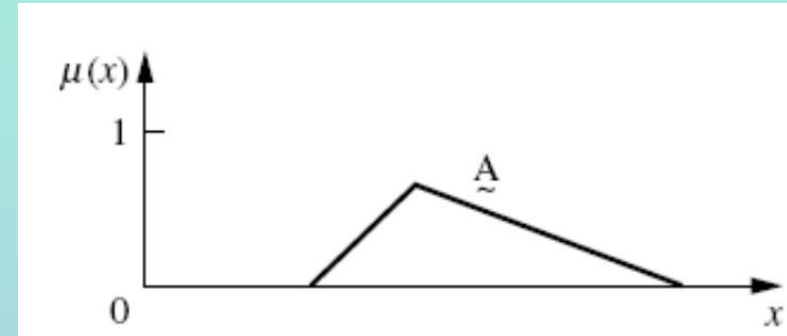


Features of membership function

- A **normal** fuzzy set is one whose membership function has at least one element, x in the universe whose membership value is unity.



- (a) Normal



- (b) Subnormal

Fuzzy number

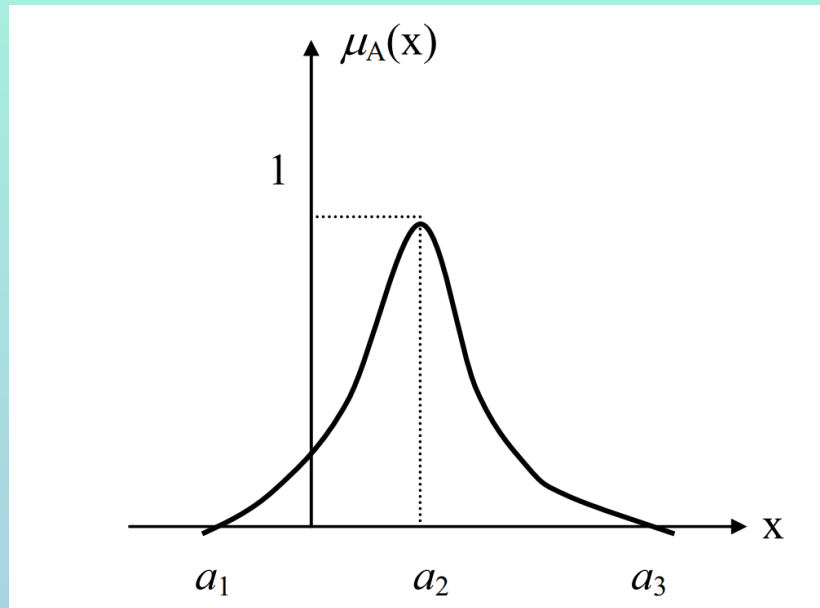
- Fuzzy number
 - A type of fuzzy set, that represents an imprecise or uncertain numerical quantities.
 - It allows for a range of possible values with variable degree of membership reflecting uncertainty or vagueness.
 - It defines a fuzzy interval in the real number.
 - Generally, a fuzzy interval is represented by two end points, a_1 and a_3 and a peak point a_2 as $[a_1, a_2, a_3]$

Fuzzy number

- Fuzzy number
 - Characteristics of Fuzzy number
 - Convex - The membership function of a fuzzy number increases to a peak and then decreases symmetrically
 - Normality - A fuzzy number has at least one point where its membership degree is 1.
 - Membership function is piecewise continuous
 - The function is defined by different expressions over different parts (interval) of its domain
 - Example: one expression for x in the range $[a, b]$ and a different expression for x in the range $[c, d]$
 - Continuous in each segment, the function changes smoothly without any abrupt jumps.
 - It is defined in the real number

Fuzzy number

- Fuzzy number



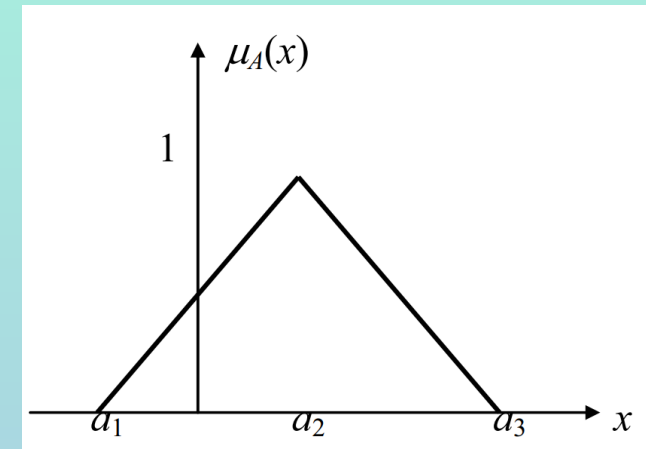
Fuzzy number, $A = [a_1, a_2, a_3]$

Fuzzy number

- Fuzzy number – Triangular
 - It is a fuzzy number represented as three points

$$A = (a_1, a_2, a_3)$$

$$\bullet \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\ 0, & x > a_3 \end{cases}$$



A Triangular Fuzzy number, $A = [a_1, a_2, a_3]$

Fuzzy number

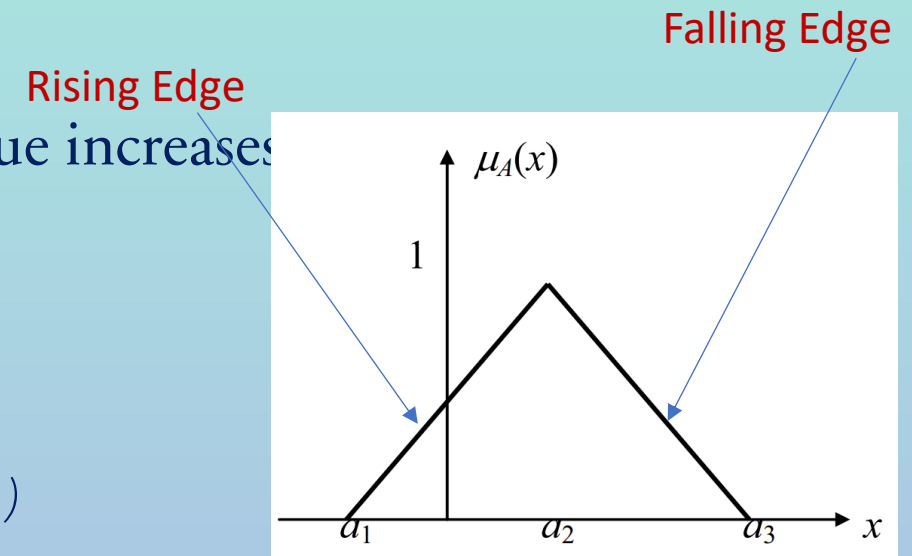
Rising Edge $a_1 \leq x \leq a_2$

- This is the left side of the triangle
- As x increases from a_1 to a_2 the membership value increases from 0 to 1
- $\mu(x) = \frac{x-a_1}{a_2-a_1}, \quad a_1 \leq x \leq a_2$

Falling Edge $a_2 \leq x \leq a_3$

- This is the right side of the triangle
- As x increases from a_2 to a_3 the membership value decreases from 1 to 0
- $\mu(x) = \frac{a_3-x}{a_3-a_2}, \quad a_2 \leq x \leq a_3$

- *Rising edge: how much has x gone up from a_1*
- *Falling edge: how far has x moved from the start of the fall (a_2)*



A Triangular Fuzzy number, $A = [a_1, a_2, a_3]$

Fuzzy number

- Calculation

Fuzzy number

- Fuzzy number – Triangular

- **Activity 1 :**

- Calculate the membership function for the fuzzy number $A = (-5, -1, 1)$

Solution = ?

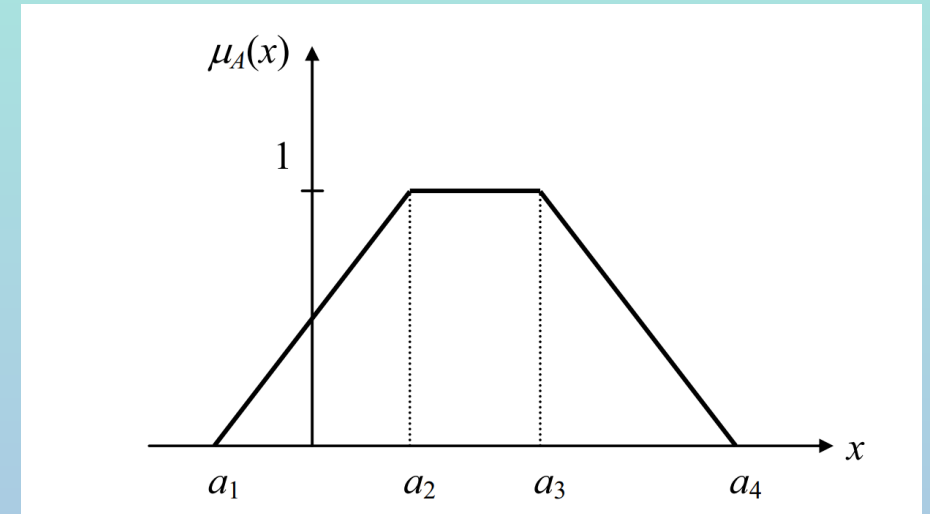
- **Activity 2 :**

- You are building a fuzzy logic system to estimate whether a person's income can be classified as “Low”, “Medium”, “High”. The fuzzy set of Medium Income is represented as a triangular fuzzy number with parameters ($a=30, b=50, c=70$) (In thousands of pounds)
 - Define the membership function for the fuzzy set, Medium Income.
 - Calculate the membership value for the following income levels : £30,000, £40,000, £50,000, £60,000, and £70,00

Fuzzy number

- Fuzzy number – Trapezoidal
 - It is a fuzzy number represented as four points
 $A = (a_1, a_2, a_3, a_4)$
 - There are several points whose membership degree is maximum ($\alpha = 1$)

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 0, & x > a_4 \end{cases}$$



A Trapezoidal Fuzzy number, $A = [a_1, a_2, a_3, a_4]$

Fuzzy number

- Calculation

Fuzzy number

- Fuzzy number – Trapezoidal
 - **Activity 1** : You are developing a fuzzy system for a car dealership that categorizes car prices as "Affordable." The fuzzy set for "Affordable" is represented as a trapezoidal fuzzy number with parameters ($a=10$, $b=20$, $c=30$, $d=40$) (in thousands of pounds).
 - Define the membership function for the fuzzy set Affordable
 - Calculate the membership values for the following car prices: £10,000, £15,000, £25,000, £35,000, and £40,000.
 - Explain the significance of the trapezoidal shape in this context

Fuzzy number

- Fuzzy number – Gaussian ?

Fuzzification and defuzzification

Fuzzification : It is the method of transforming a crisp quantity (set) into a fuzzy quantity (set).

Converts crisp inputs into fuzzy sets with membership values

For example, The temperature 45 Degree Celsius crisp input value into linguistic variables like hot , warm, or cold.

Fuzzification Methods

- Triangular Membership functions
- Trapezoidal Membership function
- Gaussian Membership function
- ?

Fuzzification and defuzzification

Defuzzification : It is the process of converting fuzzy output values back into a single crisp value, which can be used to make actionable decisions.

Defuzzification methods

- Mean of maximum method (MOM)
- Center of Area method (COA)
- Bisector of Area (BOA)

Fuzzy sets operation

Union

Combines two fuzzy sets into a single set.

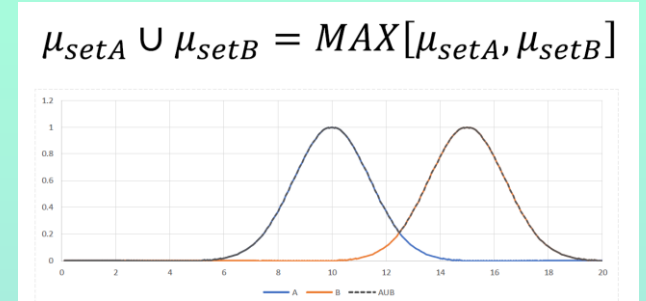
Given two fuzzy sets A and B defined on the same universe of discourse X , the fuzzy union of A and B is denoted by $A \cup B$ and is defined by the membership function:

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

$\mu_A(x)$, is the membership function of fuzzy set A .

$\mu_B(x)$ is the membership function of fuzzy set B .

Max indicates that the membership value in the fuzzy union is the higher of the two membership values for each element x .



Fuzzy sets operation

Union

Let's consider two fuzzy sets defined for the concept of "Tall" and "Short" in a group of people: Fuzzy Set A (Tall):

$$\mu_A(160)=0.2$$

$$\mu_A(170)=0.5$$

$$\mu_A(180)=0.9$$

Fuzzy Set B (Short):

$$\mu_B(160)=0.8$$

$$\mu_B(170)=0.4$$

$$\mu_B(180)=0.1$$

Now, to find the fuzzy union of these two sets, we will apply the maximum operation:

$$\mu_{A \cup B}(160)=\max(0.2,0.8)=0.8$$

$$\mu_{A \cup B}(170)=\max(0.5,0.4)=0.5$$

$$\mu_{A \cup B}(180)=\max(0.9,0.1)=0.9$$

Fuzzy sets operation

Intersection

Combines two fuzzy sets into a single set.

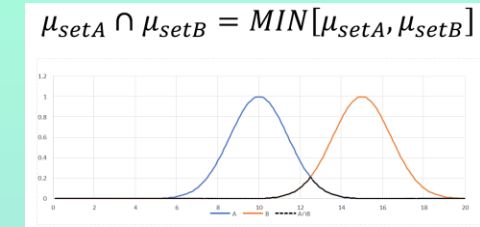
Given two fuzzy sets A and B defined on the same universe of discourse X , the fuzzy intersection of A and B is denoted by $A \cap B$ and is defined by the membership function:

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

$\mu_A(x)$, is the membership function of fuzzy set A .

$\mu_B(x)$ is the membership function of fuzzy set B .

Max indicates that the membership value in the fuzzy union is the higher of the two membership values for each element x .



Fuzzy sets operation

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$$\mu_B(180)=0.1$$

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$$\mu_{A \cup B}(170)=\max(0.5,0.4)=0.5$$

$$\mu_{A \cup B}(180)=\max(0.9,0.1)=0.9$$

Fuzzy sets operation

Complement

The fuzzy complement is a fundamental operation in fuzzy set theory that allows you to express the degree to which an element does not belong to a given fuzzy set.

Given a fuzzy set A defined on a universe of discourse X , the fuzzy complement of A is denoted as A^c (or sometimes $\sim A$). The membership function for the fuzzy complement is defined as: $\mu_{A^c}(x) = 1 - \mu_A(x)$

Example ?

