

An Improved Detonation Analogue to Reactive Euler Equations Using the Shallow Water Wave Model

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SIMULATION OF REACTIVE SHALLOW WATER WAVE MODEL

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ABSTRACT

Fluids can be classified according to their response toward change in pressure. An incompressible fluid is a fluid where the material density is constant, while a compressible fluid is a fluid for which its density varies significantly with changes of pressure. This properties is typically considered significant if the Mach number exceeds 0.3 or the fluid undergoes a large pressure change [1]. While studying supersonic flows, compressibility should be taken into consideration.

The rapid and violent form of combustion [is called detonation], in which energy release is coupled to a strong compression wave. The leading part of the detonation front is a strong shock wave propagating into the explosive. This shock heats the material by compressing it, thus triggering chemical reaction. A balance is attained such that the chemical reaction supports the shock [2] . The Burgers equation has been used as a one dimensional analogue, non-linear, hyperbolic system. It has also been used to capture the essential behavior of one-dimensional, detonation waves.

In this work we formulate a reactive 1D and 2D, shallow water wave model using shallow-water wave equations to include a height generation term. This model will be studied to determine the one dimensional behavior of a simplified, reactive, hyperbolic system with both forwards and backwards characteristics. With this knowledge we will evaluate the suitability of the model as a 2D analogue system.

INDEX WORDS: Detonation , Reactive equations, Euler , Shallow Water Wave , constant volume solution , ZND solution ,

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Chapter 1

Introduction and Literature Review

1.1 Introduction

Detonations are defined as a rapid and violent form of combustion that differs from other forms in that the energy transfer is by mass flow in strong compression waves. It is important to study this phenomena because it happens in real life and as scientist has the job to reveal true science it is important to study the detonation. Though it is important to study detonation but yet it is hard because detonation happens in a very fast and rapid way, in the brief instant of a high detonation, some remarkable events take place: the shock wave produces pressure up to 500,000 times that of Earth's atmosphere, the detonation wave travels as fast as 10 kilometers per second, temperatures can soar to 5,500 kelvins, and power approaches 20 billion watts per square centimeter. This whole events are very hard to be predicted and measured in a few seconds. Although there are equations that are governing these detonation which are known as the Euler equations but also it is difficult to evaluate those equations. For all these reasons an analogue is to be formed that enable engineers and scientist to study the detonation in a relatively simple but yet accurate method, for this purpose we evaluate the suitability of the so-called shallow water wave equations to approximate the Euler equation with.

1.2 Introduction to Fluid Mechanics

A Fluid is a substance that continually deforms under an applied stress [3]. Fluids are a subset of the phase of matter and include liquids, gases, plasmas and, to some extent, plastic solids. In this study, we are concerned about compressible fluids. A fluid can be classified as compressible or incompressible. While analyzing the fluid in motion, it is quite natural to classify the flow as compressible or

incompressible based on the density response to stress . This assumption is not absolutely correct because one cannot reach the conclusion about the compressibility of the fluid system without analysis the process involved in the system. Considering a flow of conditioned air in an air conditioning duct here the flow is taking place without much change in density this is achieved by allowing the gas to move with low velocities while its temperature is maintained more or less constant, even though the flowing fluid is compressible the flow cannot be considered as a compressible flow since it involves no density changes, this reveals that the classification as compressible and incompressible flow has to be made not on the basis of the type of fluid flowing but on the basis of the process the fluid is undergoing. [4]

1.3 Detonation Wave

Detonations are a rapid and violent form of combustion that differs from other forms by that the energy transfer is by mass flow in strong compression waves. The leading part of a detonation front is a strong shock wave propagating into the explosive thus triggering chemical reaction. The balance is attained by the chemical reaction supporting by the shock. The very rapid energy conversion in explosives is the property that makes them useful. The most easily measured characteristics property of a detonation is the velocity at which the wave propagates into the explosive. [2]

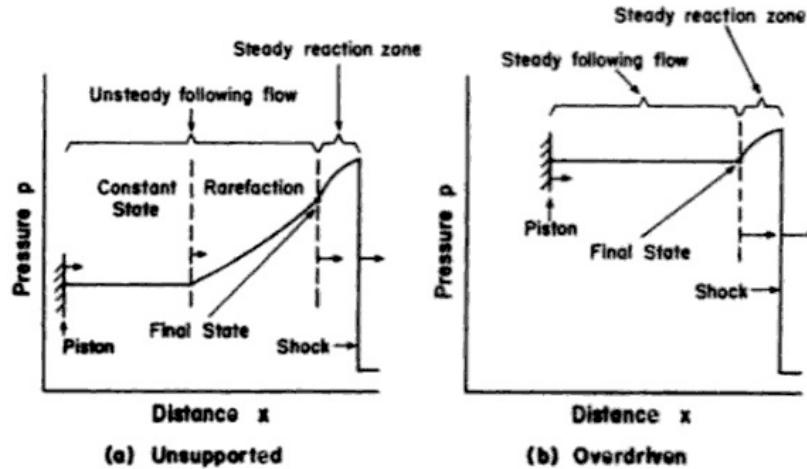


Figure 1.1: Steady Detonation [2]

The configuration of the wave is shown in figure 1.1, (a) shows the unsupported detonation where the flow in the reaction zone is steady, the shock and the reaction zone propagates together at the constant detonation velocity.(b)

shows the overdriven case where there is a constant state only. The kinematic wave equation turns up in many places, but we are concerned with its role as the simplest analog for compressible fluid flow. A retain in the partial differential equation expressing the conservation of mass but replace everything else. In the nonreactive case shocks of all strengths are possible and in the limiting case of a zero-strength shock the shock velocity is equal to the sound velocity. With an exothermic reaction the conservation laws have no solution for shocks below a certain minimum velocity. Two main concepts showed be defined. First, the Rayleigh line which is used in the simplest theory (instantaneous reaction) and is derived from the conservations laws for more information on the derivation of the equation refer to Appendix A. It is expressed by the equation:

$$R = \rho_0^2 D^2 - (p - p_0)/(v_0 - v) = 0 \quad (1.1)$$

where v is the specific volume, the density and specific volume are often both used in an expression partly to avoid reciprocals. This p-v plane diagram shown

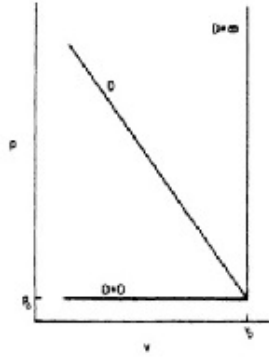


Figure 1.2: Curves of Constant Detonation Velocity, Rayleigh line [2]

in figure 1.2 is most convenient of the detonation process. A Rayleigh line passes through the point (p_0, v_0) and has a slope of $-\rho_0^2 D^2$, the limiting cases are the horizontal $D = 0$ and the vertical $D = \infty$. The vertical Rayleigh line, corresponding to an infinite propagation velocity, represents the limiting case of the constant-volume detonation in which all the material reacts at the same instant of time. Now with the second concept which is Hugoniot Curve this curve comes from the known Hugoniot relations which are the relations between the two sides of a shock wave, they establish the connection between eight eight quantities: four variables on the left (low pressure) side of the shock wave (p_0, h_0, ρ_0, v_0) and four variables on the right side which is high pressure side $(p_\infty, h_\infty, \rho_\infty, v_\infty)$. The Hugoniot curve equation can also be derived from the conservation of energy condition which will be discussed in later chapter, and the equation is given by :

$$H = E(p, v, \lambda = 1) + E(p_0, v_0, \lambda = 0) - 1/2(p + p_0)(v_0 - v) = 0 \quad (1.2)$$

The Hugoniot curve is shown in figure 1.3 The regions shown in the figure

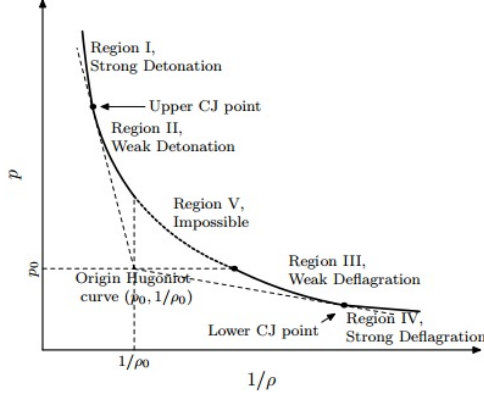


Figure 1.3: The Rankine-Hugoniot Diagram

represent the following:

- Region one, strong detonation regime.
- Region two, weak detonation regime.
- Region three, weak deflagration regime.
- Region four, strong deflagration regime.
- Region five, reflects imaginary values of v_0

The conservation condition requires that the final state point in the p - v plane lie on both the Rankine-Hugoniot Diagram and the Rayleigh line. the expression of the Rayleigh line contains the detonation velocity as a parameter so the whole family of possible Rayleigh lines must be considered. figure fig-both shows the combination of both to determine a sufficient solution. For a sufficiently small value D say D_2 the Rayleigh line and the Hugoniot curve have no intersection. so there is no solution that satisfies the assumptions. For a large value say D_1 , there will be two solutions which are marked S (strong solution) and W (weak solution). For a particular value D , called the Chapman-Jouguet velocity D_{cj} , there will be a unique solution to the two conditions, also it can be shown that the upper or strong point is subsonic with respect to the front while the lower or weak point the flow is supersonic with respect to the front. At the tangency point, called the CJ point, the flow is sonic. For more information see Appendix B

1.4 Euler Equations

Euler equations are a set of equations governing inviscid flow. They are named after Leonhard Euler. The equations represent conservation of mass (continuity), momentum, and energy, corresponding to the Navier-Stokes equations with zero viscosity and without heat conduction terms. Historically, only the continuity and momentum equations have been derived by Euler. However, fluid

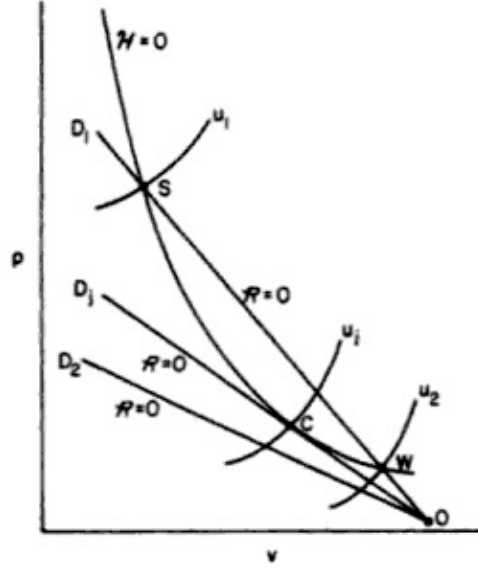


Figure 1.4: Diagram of the Hugoniot curve, Rayleigh lines, and curves of constant u for a detonation with instantaneous reaction

dynamics literature often refers to the full set including the energy equation together as "the Euler equations" [5]. Like the Navier-Stokes equations, the Euler equations are usually written in one of two forms: the "conservation form" and the "non-conservation form". The conservation form emphasizes the physical interpretation of the equations as conservation laws through a control volume fixed in space. The non-conservation form emphasizes changes to the state of a control volume as it moves with the fluid. The Euler equations can be applied to compressible as well as to incompressible flow using either an appropriate equation of state or assuming that the divergence of the flow velocity field is zero, respectively. The Euler Equations for an incompressible fluid is given by [6]

$$\nabla \cdot u = 0 \quad (1.3)$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho} \nabla p - g j \quad (1.4)$$

Also the Euler equation can be written for a compressible fluid in the form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad (1.5)$$

$$\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \otimes u) + \nabla p = 0 \quad (1.6)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (u(E + p)) = 0 \quad (1.7)$$

where $\otimes$ denotes the tensor product, ρ is the gas density, u is velocity, p is pressure, and E is the total energy density related to the internal molar energy density e by:

$$E = \rho e + \frac{1}{2}\rho|u|^2 \quad (1.8)$$

1.5 Equation Forms

Conservation form refers to an arrangement of an equation or system of equations, usually representing a physical system, that show that a property represented is conserved by making overall change equal to zero. On the other hand, the non-conservation equations are the result of approximation of the classical physics. The major difference between these two forms lies in the derivation and the selection of the control volume in the derivations of these equations. Fixed control volumes yield conservation forms and moving control volumes yields non-conservation forms. the conservative form consists of two parts which are the conserved quantity and the flux, in the following equation the first part $[U]$ is the conservative part which is as a function of time while the second part $[V]$ is the flux which is as a function of other variables it could be more than one term:

$$\int \frac{\partial}{\partial t}[U]\partial t + \int \frac{\partial}{\partial x}[V]\partial t = 0 \quad (1.9)$$

Chapter 2

Formulation of Analogue Model

In this chapter, it is required to formulate the basic equations of Euler and shallow water wave in the nonreactive case, and then assume a reactive element to propose a reactive analogue. At the end it is required to compare between then two reactive equations and propose a relation between them.

2.1 Derivation of Shallow Water Wave Equation

The shallow water wave equations named after Adhmar Jean Claude Barr de Saint-Venant are a set of hyperbolic partial differential equations which describes the flow below a pressure surface in a fluid. The shallow water equations can also be simplified to the commonly used 1-D Saint Venant Equation [7]. The equations are derived from depth-integrating the Euler equations, in the case where the horizontal length scale is much greater than the vertical length scale (taking only the horizontal component). Under this condition, conservation of mass gives the vertical velocity of the fluid is small. Horizontal pressure gradients are due to the displacement of the pressure surface, implying that the horizontal velocity field is constant throughout the depth of the fluid. Vertically integrating allows the vertical velocity to be removed from the equations. The shallow water equations are thus derived. While a vertical velocity term is not present in the shallow water equations, note that this velocity is not necessarily zero.[7] This is an important distinction because, for example, the vertical velocity cannot be zero when the floor changes depth, and thus if it were zero only flat floors would be usable with the shallow water equations. Shallow water equation models have only one vertical level, so they cannot directly encompass any factor that varies with height. However, in cases where the mean state is sufficiently simple, the vertical variations can be separated from the horizontal and several sets of shallow water equations can describe the state. The Euler

Equation for an incompressible fluid is given by [6]

$$\nabla \cdot \bar{u} = 0 \quad (2.1)$$

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla) \bar{u} = -\frac{1}{\rho} \nabla p - g \hat{j} \quad (2.2)$$

which can be written in the form:

$$\bar{u} = u_i \hat{e}_i + v \hat{j} \quad (2.3)$$

Taking the vertical component of equation 2.2 and assuming that v is not a function of y we get:

$$-\frac{1}{\rho} \frac{\partial p}{\partial y} - g = 0.$$

Integration leads to:

$$p - p_0 = g\rho(\eta - y). \quad (2.4)$$

The divergence term gives [8]:

$$(\bar{u} \cdot \nabla) \bar{u} = u_i \frac{\partial}{\partial x_i} \bar{u} + v \frac{\partial}{\partial y} \bar{u}. \quad (2.5)$$

Taking the horizontal components of equation 2.2, we have :

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + v \frac{\partial u_i}{\partial y} = -g \frac{\partial \eta}{\partial x_i}. \quad (2.6)$$

Assuming that u_i is independent of y we get:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + g \frac{\partial \eta}{\partial x_i} = 0, \quad (2.7)$$

which is the momentum balance of the shallow water wave. Rewriting equation 2.1:

$$\frac{\partial u_i}{\partial x_i} + \frac{\partial v}{\partial y} = 0, \quad (2.8)$$

and integrating:

$$\int_{-h_0}^{\eta} \left(\frac{\partial u_i}{\partial x_i} + \frac{\partial v}{\partial y} \right) \partial y = 0, \quad (2.9)$$

$$\frac{\partial}{\partial x_i} \int_{-h_0}^{\eta} u_i \partial y + v \Big|_{-h_0}^{\eta} - u_i \Big|_{\eta} \frac{\partial \eta}{\partial x_i} - u_i \Big|_{-h_0} \frac{\partial h_0}{\partial x_i} = 0. \quad (2.10)$$

Using the boundary conditions which are [6], at the initial y give as :

$$y = -h_0(x_1, x + 2), \quad (2.11)$$

the velocity is given by:

$$v = \frac{\partial \eta}{\partial t} + u_i \frac{\partial \eta}{\partial x_i}, \quad (2.12)$$

to reduce equation 2.10:

$$\frac{\partial}{\partial x_i} \int_{-h_0}^{\eta} u_i \partial y + \frac{\partial \eta}{\partial t} + u_i|_{\eta} \frac{\partial \eta}{\partial x_i} + \frac{\partial h_0}{\partial t} + u_i|_{-h_0} \frac{\partial h_0}{\partial x_i} - u_i|_{\eta} \frac{\partial \eta}{\partial x_i} - u_i|_{-h_0} \frac{\partial h_0}{\partial x_i} = 0, \quad (2.13)$$

and finally:

$$\frac{\partial}{\partial x_i} \int_{-h_0}^{\eta} u_i \partial y + \frac{\partial \eta}{\partial t} + \frac{\partial h_0}{\partial t} = 0. \quad (2.14)$$

Since u_i is independent of y we have:

$$\int_{-h_0}^{\eta} u_i \partial y = u_i(\eta + h_0) = u_i h, \quad (2.15)$$

where:

$$h = \eta + h_0. \quad (2.16)$$

Then the second shallow water wave equation becomes:

$$\frac{\partial (hu_i)}{\partial x_i} + \frac{\partial h}{\partial t} = 0, \quad (2.17)$$

which is the mass balance of the shallow water wave equation. In the cases of interest here, $h_0 = 0$ and $h = \eta$. The shallow water wave equations are thus:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + g \frac{\partial h}{\partial x_i} = 0, \quad (2.18)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu_i)}{\partial x_i} = 0. \quad (2.19)$$

Now lets write the shallow water wave equation in a conservative form, meaning we seek to express the system of equations as:

$$\frac{\partial}{\partial t} [U] + \frac{\partial}{\partial x} [F] = 0 \quad (2.20)$$

Writing equation 2.18 in the conservative form we get:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} u_i^2 + gh \right) = 0. \quad (2.21)$$

Equation 2.19 is already written in the conservative form. Then the conservative shallow water wave equation becomes:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_i} \left(\frac{1}{2} u_i^2 + gh \right) = 0, \quad (2.22)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu_i)}{\partial x_i} = 0. \quad (2.23)$$

We will only concern ourselves with the 1D processes in this work. The Shallow Water Wave in 1D is:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{1}{2} u^2 + gh \right) = 0, \quad (2.24)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0. \quad (2.25)$$

2.2 Extension of Model to Reaction

The simplest chemical reaction considered is the single step reaction from A to B which we allow to proceed only in the forward direction. We specify the composition of the system by mass fraction of B which is called y . This dimensionless variable y is a progress variable for the reaction, progressing from one at no reaction to zero at complete reaction [9]. We have then y as a dependent variable. Now in the physical system, the addition of chemical reaction changes the governing equations in two ways: the pressure depends on the composition, as well as on other state variables, and there are one or more additional differential equations describing the rate of change of the composition with time.

2.2.1 Reactive Euler

The conservative form of the reactive Euler equation is given by [10] in 1D :

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \quad (2.26)$$

$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0 \quad (2.27)$$

$$\frac{\partial (\frac{1}{2}\rho u^2 + \rho E)}{\partial t} + \frac{\partial [(\frac{1}{2}\rho u^2 + \rho E + p)u]}{\partial x} = 0 \quad (2.28)$$

The reactive rate is what actually concerns us in this work it is what we actually seek to define, if we assume a small box that has potential energy and that can release it, the conservation of this quantity is actually given as that rate, which can be defined by the following equation:

$$\frac{D}{Dt}(\rho y) = 0, \quad (2.29)$$

$$\frac{\partial (\rho y)}{\partial t} + \frac{\partial (\rho u y)}{\partial x} = w. \quad (2.30)$$

Lets write the reactive rate in a differential form using equation 2.30:

$$\rho \frac{\partial y}{\partial t} + y \frac{\partial \rho}{\partial t} + y \frac{\partial (\rho u)}{\partial x} + \rho u \frac{\partial y}{\partial x} = w, \quad (2.31)$$

using equation 2.26 to simplify the previous form:

$$y[0] + \rho \frac{\partial y}{\partial t} + \rho u \frac{\partial y}{\partial x} = w, \quad (2.32)$$

$$\frac{\partial y}{\partial t} + u \frac{\partial y}{\partial x} = \frac{w}{\rho}. \quad (2.33)$$

Assuming the following:

$$E = e(p, v) + yq \quad (2.34)$$

Where q is a constant number.

Writing equation 2.27 in the differential form, using equation 2.26 to simplify, we get :

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0. \quad (2.35)$$

Similarly writing equation 2.28 in the differential form, we have:

$$\rho \frac{\partial e}{\partial t} + \rho u \frac{\partial e}{\partial x_j} + u \frac{\partial p}{\partial x} = -qw. \quad (2.36)$$

Then the differential reactive Euler equation becomes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \quad (2.37)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \quad (2.38)$$

$$\rho \frac{\partial e}{\partial t} + \rho u \frac{\partial e}{\partial x} + u \frac{\partial p}{\partial x} = -qw \quad (2.39)$$

$$\frac{\partial y}{\partial t} + u \frac{\partial y}{\partial x} = \frac{w}{\rho}. \quad (2.40)$$

2.2.2 Reactive Shallow Water Wave Equation

In this section we will work on both forms, conservative and non conservative, of the reactive shallow water wave equation and compare both results.

Non-Conservative Reactive Shallow Water Wave Equation

The shallow water wave equation as given in its non- conservative form [11] is ;

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial H}{\partial x} = 0, \quad (2.41)$$

$$\frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0. \quad (2.42)$$

The reactive rate can be defined as the rate that conserves the reaction, in the case of shallow water wave it is not the hy quantity that is conserved but it is only y because the height in this case is involved in the reaction and there is no mass to be conserved. Then the reactive rate is given by:

$$\frac{\partial y}{\partial t} + \frac{\partial (yu)}{\partial x} = w, \quad (2.43)$$

and that the total height H is defined by:

$$H = h + yq. \quad (2.44)$$

Expanding H in equation 2.23 gives:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = -qw, \quad (2.45)$$

Similarly equation 2.41 reduces to:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = -gq \frac{\partial y}{\partial x}. \quad (2.46)$$

The differential form of the reactive shallow water wave equation is thus:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = -gq \frac{\partial y}{\partial x}, \quad (2.47)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = -qw, \quad (2.48)$$

$$\frac{\partial y}{\partial t} + \frac{\partial (yu)}{\partial x} = w. \quad (2.49)$$

Conservative Reactive Shallow Water Wave Equation

The conservative form of the shallow water wave equation is given by:

$$\frac{\partial H}{\partial t} + \frac{\partial (Hu_i)}{\partial x_i} = 0, \quad (2.50)$$

$$\frac{\partial (Hu_i)}{\partial t} + \frac{\partial (Hu_i^2 + \frac{1}{2}gH^2)}{\partial x_i} + \frac{\partial (Hu_i u_j)}{\partial x_j} = 0. \quad (2.51)$$

This form is the two dimensional form lets assume a one dimensional equation:

$$\frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} = 0, \quad (2.52)$$

$$\frac{\partial (Hu)}{\partial t} + \frac{\partial (Hu^2 + \frac{1}{2}gH^2)}{\partial x} = 0. \quad (2.53)$$

The reactive rate is defined by the following equation:

$$\frac{\partial y}{\partial t} + \frac{\partial (yu)}{\partial x} = w, \quad (2.54)$$

and

$$H = h + yq, \quad (2.55)$$

where q is a constant value.

Writing equation 2.52 in the differential form and using equation 2.54 to reduce, we get

$$\frac{\partial (h + yq)}{\partial t} + \frac{\partial [(h + yq)u]}{\partial x} = 0, \quad (2.56)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = -qw. \quad (2.57)$$

Writing equation 2.53 in the same form we have

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + \frac{1}{2}gh^2)}{\partial x} = -quw - gqh \frac{\partial y}{\partial x}. \quad (2.58)$$

Then the reactive shallow water wave equation becomes:

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2)}{\partial x} + \frac{\partial (\frac{1}{2}gh^2)}{\partial x} = -quw - gqh \frac{\partial y}{\partial x}, \quad (2.59)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = -qw, \quad (2.60)$$

$$\frac{\partial y}{\partial t} + \frac{\partial (yu)}{\partial x} = w. \quad (2.61)$$

Notice that while equations 2.48 and 2.60 are the same we still have to prove that equations 2.47 and 2.59 are identical. Rearranging 2.59, we have

$$u \left[\frac{\partial h}{\partial t} + h \frac{\partial u}{\partial x} + u \frac{\partial h}{\partial x} \right] + hu \frac{\partial u}{\partial x} + h \frac{\partial u}{\partial t} + gh \frac{\partial h}{\partial x} + quw = -gqh \frac{\partial y}{\partial x}, \quad (2.62)$$

which can be simplified using 2.60

$$u[-qw] + hu \frac{\partial u}{\partial x} + h \frac{\partial u}{\partial t} + gh \frac{\partial h}{\partial x} + quw = -gqh \frac{\partial y}{\partial x}. \quad (2.63)$$

We then get:

$$u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} = -gq \frac{\partial y}{\partial x}, \quad (2.64)$$

which is identical to equation 2.47.

2.3 Sound Speed

In this section we derive the speed of sound for both the Euler and Shallow water wave system. The sound speed is the speed at which infinitesimal disturbances propagates and is important to determine the Mach number.

2.3.1 Euler

The Euler equations are known as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0, \quad (2.65)$$

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial x_i} = 0, \quad (2.66)$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho u_i^2 + \rho e \right) + \frac{\partial}{\partial x_j} \left\{ \left(\frac{1}{2} \rho u_i^2 + \rho e + p \right) u_j \right\} = 0. \quad (2.67)$$

A sound wave is an infinitesimal, isentropic, linear disturbance. In the Euler system, we start by the jump condition: The jump condition is conveniently defined as:

$$-U[f_i] + [g_i] = 0, \quad (2.68)$$

where F_i is the conserved quantity and g_i is the flux, U the shock speed. Writing the Euler equations in this form and assuming only one space dimension.

$$-U[\rho] + [\rho u] = 0, \quad (2.69)$$

$$-U[\rho u] + [\rho u^2 + p] = 0, \quad (2.70)$$

$$-U \left[\frac{1}{2} \rho u^2 + \rho e \right] + \left[\left(\frac{1}{2} \rho u^2 + \rho e \right) u + pu \right] = 0. \quad (2.71)$$

Let's define a relative velocity v which is $v = U - u$ and the enthalpy h which is equal to $h = e + p/\rho$, equation 2.69 becomes:

$$[\rho v] = 0, \quad (2.72)$$

equation 2.70 :

$$[-\rho U v + \rho v^2 + p] = 0, \quad (2.73)$$

and in the same manner equation 2.71 :

$$\left[u (\rho v^2 + p) + \rho v \left(\frac{1}{2} \rho v^2 + h \right) - \rho U^2 v \right] = 0. \quad (2.74)$$

Assuming a linear combination meaning that all higher than one terms are considered negligible the set of equations becomes:

$$[\rho v] = 0, \quad (2.75)$$

$$[p + \rho v^2] = 0, \quad (2.76)$$

$$\left[\rho v \left(h + \frac{1}{2} v^2 \right) \right] = 0. \quad (2.77)$$

from the assumption of small infinitesimal volume we can find the relation of the sound speed figure 2.1 shows the relation between initial and final states of the disturbance.

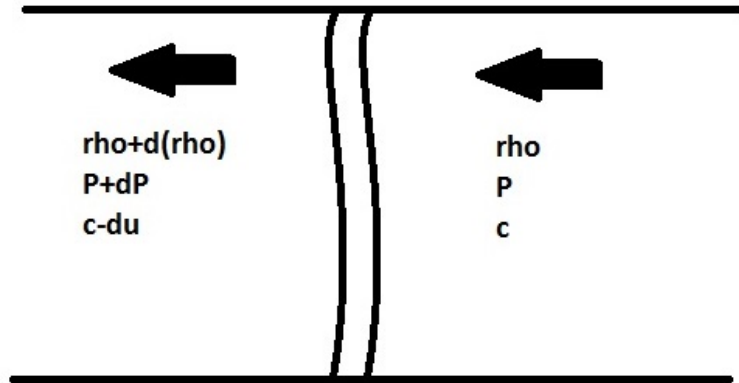


Figure 2.1: The disturbance of an infinitesimal element

The relation from this figure are:

$$\rho c = (\rho + d\rho)(c - du), \quad (2.78)$$

$$\rho c = \rho c - \rho du + c d\rho - d u d\rho, \quad (2.79)$$

$$\rho du = c d\rho. \quad (2.80)$$

In the same manner we have:

$$dp = \rho c du, \quad (2.81)$$

eliminating du :

$$dp = \frac{\rho c^2 d\rho}{\rho}, \quad (2.82)$$

$$dp = c^2 d\rho, \quad (2.83)$$

$$\left(\frac{dp}{d\rho}\right) = c^2. \quad (2.84)$$

For an isentropic process in ideal gas, it is well known that:

$$\frac{p}{\rho^\gamma} = \text{constant}, \quad (2.85)$$

hence:

$$\left(\frac{dp}{d\rho}\right)_s = \frac{p\gamma}{\rho} = c^2 \quad (2.86)$$

which is the sound speed.

2.3.2 Shallow water wave

The non-Reactive Shallow Water Wave equations we now are:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0, \quad (2.87)$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + \frac{1}{2}gh^2)}{\partial x} = 0. \quad (2.88)$$

Just as for an Euler sound wave, we examine on infinitesimal jump, as shown in figure 2.2

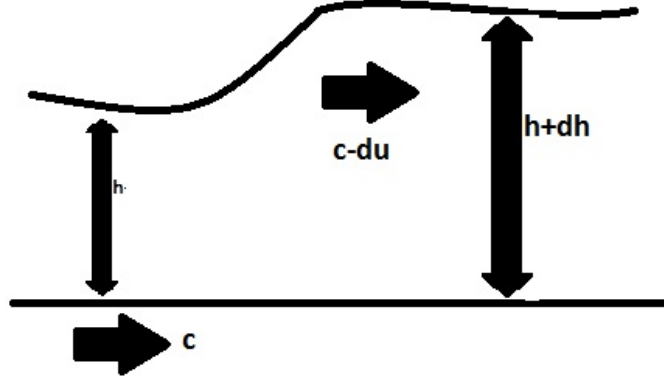


Figure 2.2: The jump in the shallow water wave

$$-U[f_i] + [g_i] = 0$$

we get:

$$-U[h] + [hu] = 0 \quad (2.89)$$

$$-U[hu] + \left[hu^2 + \frac{1}{2}gh^2 \right] = 0 \quad (2.90)$$

$$hc = (h + dh)(c - du), \quad (2.91)$$

$$hc = hc - hdu + cdh - dudh, \quad (2.92)$$

$$cdh = hdu. \quad (2.93)$$

In the other hand:

$$hc^2 + \frac{1}{2}gh^2 = (h + dh)(c - du)^2 + \frac{1}{2}g(h + dh)^2, \quad (2.94)$$

$$hc^2 + \frac{1}{2}gh^2 = hc^2 + dhc^2 - 2cduh + \frac{1}{2}(h^2 + 2hdh), \quad (2.95)$$

$$c^2dh - 2c^2dh + gh dh = 0, \quad (2.96)$$

$$gh dh = c^2 dh, \quad (2.97)$$

$$c^2 = gh. \quad (2.98)$$

2.4 Shock Relations

we derive here the shock relations meaning the conditions relating the state before a finite jump to the state after. These conditions will be used to determine the Hugoniot curve and Rayleigh line.

2.4.1 Euler

Non-Reactive Euler

Lets find the equation of the Hugoniot curve for the Euler equation, starting with conservation law :

$$h_2 + \frac{1}{2}u_2^2 = h_1 + \frac{1}{2}u_1^2, \quad (2.99)$$

$$h_2 + \frac{1}{2}u_2^2 = h_1 + \frac{1}{2}u_1^2, \quad (2.100)$$

it is known that :

$$h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \quad (2.101)$$

then we get:

$$\frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} - \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} = \frac{1}{2} (u_1^2 - u_2^2) \quad (2.102)$$

$$\frac{\gamma}{\gamma - 1} \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) = \frac{1}{2} \left(u_1^2 - \frac{p_1 - p_2 + \rho_1 u_1^2}{\rho_2} \right) \quad (2.103)$$

$$\frac{\gamma}{\gamma - 1} \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) = \frac{1}{2} \left(u_1^2 \left(1 - \frac{\rho_1}{\rho_2} \right) - \frac{p_1}{\rho_2} + \frac{p_2}{\rho_2} \right). \quad (2.104)$$

$$(2.105)$$

Let's try to find a relation between the velocity and pressure from the shock relations:

$$[\rho v] = 0, \quad (2.106)$$

$$\rho_1 u_1 = \rho_2 u_2, \quad (2.107)$$

$$\rho_1 U = \rho_2 (U - u_2) \quad (2.108)$$

$$\rho_2 U - \rho_1 U = \rho_2 u_2, \quad (2.109)$$

$$U - \frac{\rho_1}{\rho_2} U = u_2, \quad (2.110)$$

$$U \left(1 - \frac{\rho_1}{\rho_2} \right) = u_2, \quad (2.111)$$

now lets find the second relation from equation 2.70

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2, \quad (2.112)$$

$$p_1 + \rho_1 U^2 = p_2 + \rho_2 (U - u_2)^2, \quad (2.113)$$

$$p_1 + \rho_1 U^2 = p_2 + \rho_2 (U^2 - 2Uu_2 + u_2^2), \quad (2.114)$$

$$(p_1 - p_2) + (\rho_1 - \rho_2)U^2 = -2\rho_2 U^2 \left(1 - \frac{\rho_1}{\rho_2}\right) + \rho_2 U^2 \left(1 - \frac{\rho_1}{\rho_2}\right)^2 \quad (2.115)$$

$$(p_1 - p_2) + (\rho_1 - \rho_2)U^2 = \rho_2 U^2 \left(1 - \frac{\rho_1}{\rho_2}\right) \left(-2 + 1 - \frac{\rho_1}{\rho_2}\right), \quad (2.116)$$

$$(p_1 - p_2) + (\rho_1 - \rho_2)U^2 = \rho_2 U^2 \left(1 - \frac{\rho_1}{\rho_2}\right) \left(-1 - \frac{\rho_1}{\rho_2}\right), \quad (2.117)$$

$$(p_1 - p_2) + (\rho_1 - \rho_2)U^2 = -\rho_2 U^2 \left(1 - \left(\frac{\rho_1}{\rho_2}\right)^2\right), \quad (2.118)$$

$$(p_1 - p_2) = -\rho_2 U^2 \left(1 - \left(\frac{\rho_1}{\rho_2}\right)^2 + \frac{\rho_1}{\rho_2} - 1\right), \quad (2.119)$$

$$(p_1 - p_2) = -\rho_1 U^2 \left(-\frac{\rho_1}{\rho_2} + 1\right), \quad (2.120)$$

$$U^2 = \frac{p_2 - p_1}{\rho_1 \left(1 - \frac{\rho_1}{\rho_2}\right)}. \quad (2.121)$$

Using equation 2.121 and knowing that $u_1 = U$ then the relation becomes:

$$\frac{1}{\gamma - 1} \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) = \frac{1}{2} \left(\frac{p_2 - p_1}{\rho_1 \left(1 - \frac{\rho_1}{\rho_2}\right)} \left(1 - \frac{\rho_1}{\rho_2}\right) - \frac{p_1}{\rho_2} + \frac{p_2}{\rho_2}\right), \quad (2.122)$$

$$\frac{1}{\gamma - 1} p_1 v_1 (xy - 1) = \frac{1}{2} \left(\frac{p_2 - p_1}{\rho_1} - p_1 v_2 + p_2 v_2\right), \quad (2.123)$$

$$\frac{1}{\gamma - 1} p_1 v_1 (xy - 1) = \frac{1}{2} (p_2 v_1 - p_1 v_1 - p_1 v_2 + p_2 v_2) \quad (2.124)$$

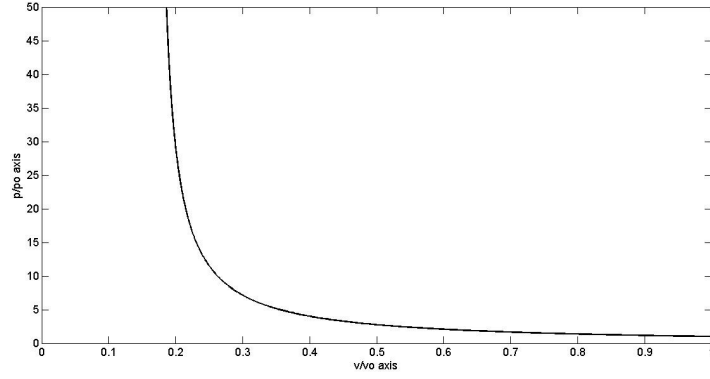
$$\frac{1}{\gamma - 1} p_1 v_1 (xy - 1) = \frac{1}{2} [p_1 (v_2 + v_1) - p_2 (v_2 + v_1)], \quad (2.125)$$

$$\frac{1}{\gamma - 1} p_1 v_1 (xy - 1) = \frac{1}{2} (v_2 + v_1) (p_1 - p_2), \quad (2.126)$$

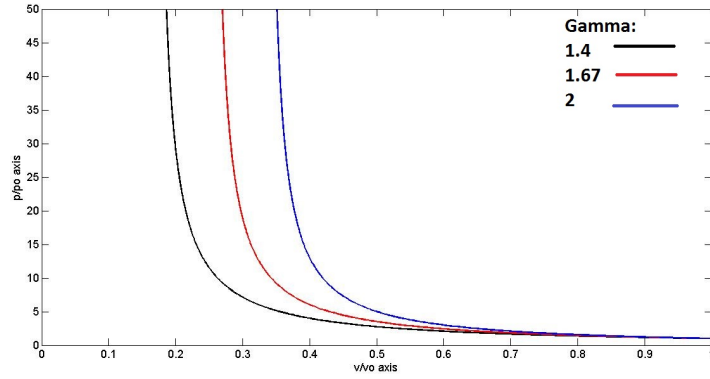
$$\frac{1}{\gamma - 1} p_1 v_1 (xy - 1) = \frac{1}{2} v_1 p_1 (1 - x) (1 + y), \quad (2.127)$$

$$\frac{1}{\gamma - 1} (xy - 1) = \frac{1}{2} (1 - x) (1 + y). \quad (2.128)$$

this is the new hugoniot curve equation in its implicit form, where x is the specific volume ratio v_2/v_1 or v/v_0 , and y is the pressure ratio p_2/p_1 or p/p_0 . Plotting this curve give a shape similar to the one shown in figure 2.3.

Figure 2.3: Hugoniot curve for non-reactive Euler equations for $\gamma = 1.4$

let's plot the Hugoniot curve for different values of γ , this plot is shown in figure 2.4 for a set of γ values of 1.4, 1.67, 2.

Figure 2.4: Hugoniot curve for non-reactive Euler for different values of γ

Other relations such as velocity ratio to Mach number is stated in the relation :

$$\frac{u_2}{c} = \frac{2(M^2 - 1)}{(\gamma + 1)M} \quad (2.129)$$

this relation can be plotted in the figure 2.5.

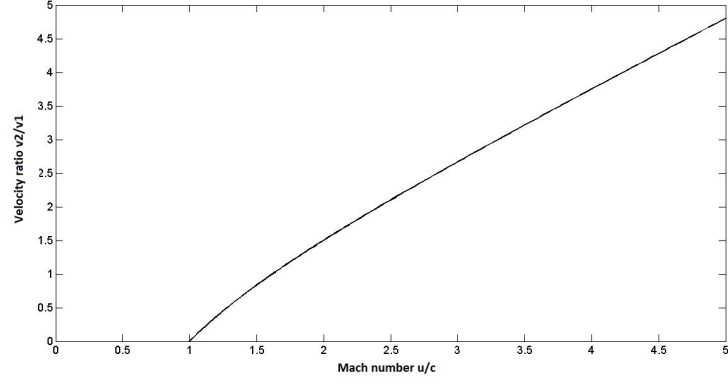


Figure 2.5: Variation of velocity ratio with respect to Mach number

also the relation between the density ratio and the Mach number is given by the relation:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M^2}{(\gamma - 1)M^2 + 2} \quad (2.130)$$

this relation is also plotted in figure 2.6.

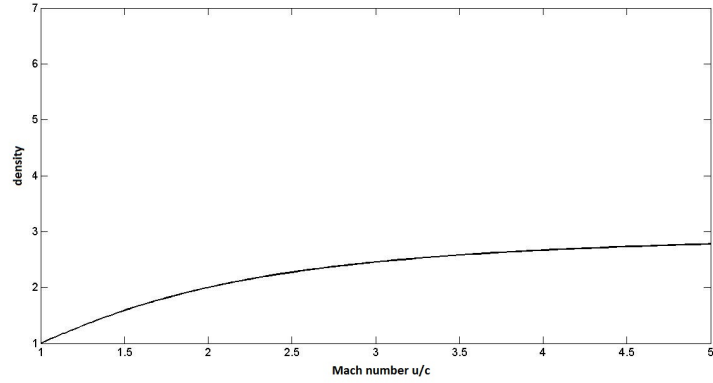


Figure 2.6: variation of density ratio as a function of Mach number

relation between pressure difference ratio can be determined by:

$$\frac{p_2 - p_1}{p_1} = \frac{2\gamma(M^2 - 1)}{\gamma + 1} \quad (2.131)$$

this relation can also be plotted in figure 2.7.

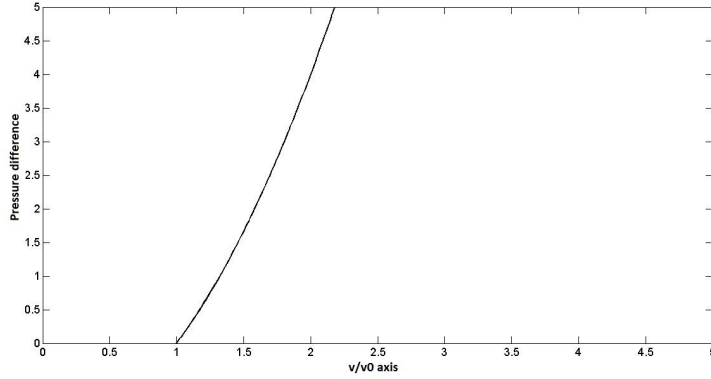


Figure 2.7: variation of pressure difference with respect to Mach number

Reactive Euler

For the case of reactive Euler equation the sequence of study is the same with a difference in the energy which is now $e + yq$. Thus the relation will remain the same except for the equation of enthalpy which will be:

$$\begin{aligned} \left[\rho v \left(H + \frac{1}{2} v^2 \right) \right] &= 0 \\ \left[\rho v \left(h + yq + \frac{1}{2} v^2 \right) \right] &= 0 \end{aligned} \quad (2.132)$$

$$h_2 + y_2 q_2 + \frac{1}{2} v_2^2 = h_1 + y_1 q_1 + \frac{1}{2} v_1^2 \quad (2.133)$$

$$h_2 + \frac{1}{2} v_2^2 = h_1 + q_1 + \frac{1}{2} v_1^2 \quad (2.134)$$

The relation with the speed of sound remains the same and thus :

$$c^2 = \frac{p\gamma}{\rho}$$

the Hugoniot curve equation changes a bit by substituting instead of h a value of $h + qy$:

$$H_2 + \frac{1}{2} v_2^2 = H_0 + \frac{1}{2} v_1^2 \quad (2.135)$$

$$h_2 + y_2 q + \frac{1}{2} v_2^2 = h_1 + y_1 q + \frac{1}{2} v_1^2 \quad (2.136)$$

$$h_2 + \frac{1}{2} v_2^2 = h_1 + q + \frac{1}{2} v_1^2 \quad (2.137)$$

it is known that :

$$h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \quad (2.138)$$

then we get:

$$\frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2} - \frac{\gamma}{\gamma-1} \frac{p_1}{\rho_1} = q + \frac{1}{2} (v_1^2 - v_2^2) \quad (2.139)$$

$$\frac{\gamma}{\gamma-1} \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) = q + \frac{1}{2} \left(v_1^2 - \frac{p_1 - p_2 + \rho_1 v_1^2}{\rho_2} \right) \quad (2.140)$$

$$\frac{\gamma}{\gamma-1} \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) = q + \frac{1}{2} \left(v_1^2 \left(1 - \frac{\rho_1}{\rho_2} \right) - \frac{p_1}{\rho_2} + \frac{p_2}{\rho_2} \right) \quad (2.141)$$

$$(2.142)$$

from previous relation it is known that :

$$U^2 = \frac{p_2 - p_1}{\rho_1 \left(1 - \frac{\rho_1}{\rho_2} \right)}. \quad (2.143)$$

Knowing that $v_1 = U$ then the relation becomes:

$$\frac{1}{\gamma-1} \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} \right) = q + \frac{1}{2} \left(\frac{p_2 - p_1}{\rho_1 \left(1 - \frac{\rho_1}{\rho_2} \right)} \left(1 - \frac{\rho_1}{\rho_2} \right) - \frac{p_1}{\rho_2} + \frac{p_2}{\rho_2} \right) \quad (2.144)$$

$$\frac{1}{\gamma-1} p_1 v_1 (xy - 1) = q + \frac{1}{2} \left(\frac{p_2 - p_1}{\rho_1} - p_1 v_2 + p_2 v_2 \right), \quad (2.145)$$

$$\frac{1}{\gamma-1} p_1 v_1 (xy - 1) = q + \frac{1}{2} (p_2 v_1 - p_1 v_1 - p_1 v_2 + p_2 v_2), \quad (2.146)$$

$$\frac{1}{\gamma-1} p_1 v_1 (xy - 1) = q + \frac{1}{2} (p_1 (v_2 + v_1) - p_2 (v_2 + v_1)), \quad (2.147)$$

$$\frac{1}{\gamma-1} p_1 v_1 (xy - 1) = q + \frac{1}{2} ((v_2 + v_1)(p_1 - p_2)), \quad (2.148)$$

$$\frac{1}{\gamma-1} p_1 v_1 (xy - 1) = q + \frac{1}{2} v_1 p_1 (1 - x)(1 + y), \quad (2.149)$$

$$\frac{1}{\gamma-1} (xy - 1) = \frac{q}{p_1 v_1} + \frac{1}{2} (1 - x)(1 + y), \quad (2.150)$$

this is the new hugoniot curve equation for a reactive case in its implicit form , lets assume a new variable $\tilde{q} = \frac{q}{p_1 v_1}$. then the equation becomes:

$$\frac{1}{\gamma-1} (xy - 1) = \tilde{q} + \frac{1}{2} (1 - x)(1 + y), \quad (2.151)$$

Let's plot the reactive-Euler hugoniot curve for a $\gamma = 1.4$, heat release value of $\tilde{q} = 1$, this conditions and their corresponding Hugoniot curve are shown in figure 2.8.

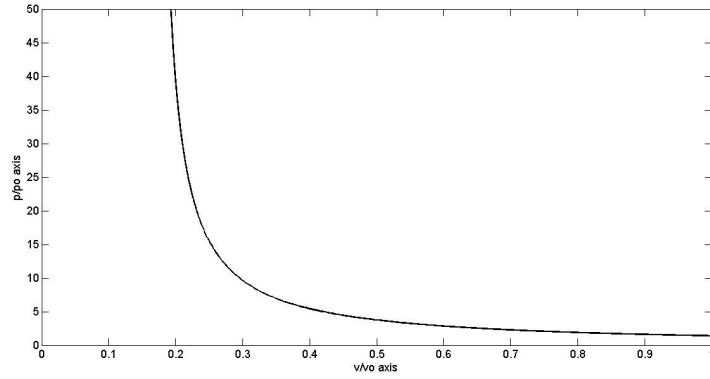
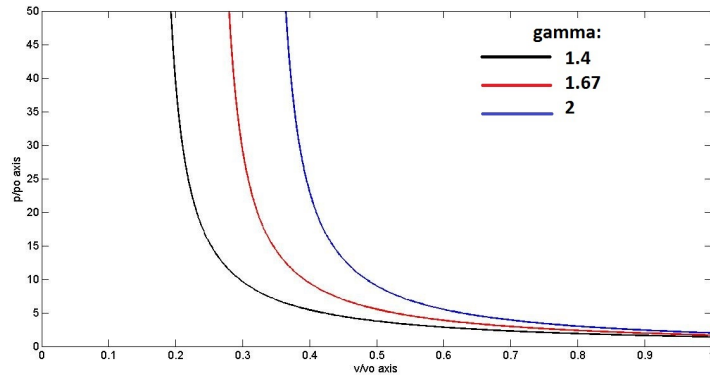
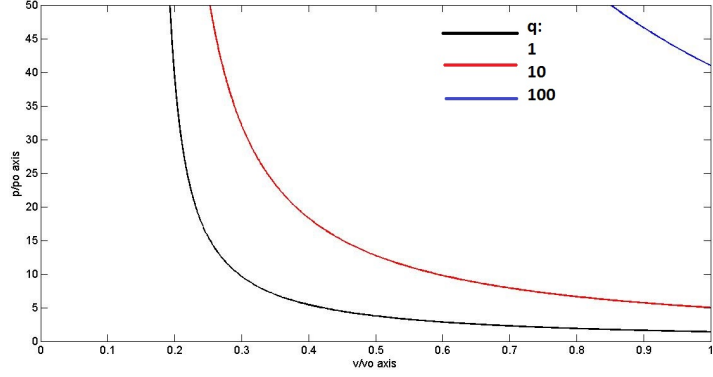


Figure 2.8: The hugoniot curve for the reactive Euler

This reactive Hugoniot curve has a new induced variable which is q thus we can draw different Hugoniot curves according to values of $g\gamma$ as we did for the non-reactive case for 1.4, 1.67, and 2 which is plotted in figure 2.9, and we can also plot for different values of q as 1, 2 and 3 as in figure 2.10.

Figure 2.9: Hugoniot curve for a reactive Euler for different γ

Figure 2.10: Hugoniot curve for a reactive Euler for different q

Other relations for the velocity, pressure, density and enthalpy ratio can also be written as a function of Mach number:

$$\frac{u_2}{c} = \frac{2(M^2 - 1)}{(\gamma + 1)M} \quad (2.152)$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M^2}{(\gamma - 1)M^2 + 2} \quad (2.153)$$

$$\frac{p_2 - p_1}{p_1} = \frac{2\gamma(M^2 - 1)}{\gamma + 1} \quad (2.154)$$

those relations can also be plotted as shown in the figures 2.11, 2.12, 2.13.

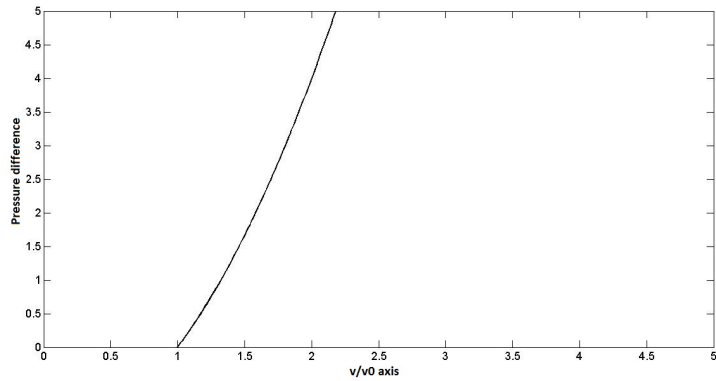


Figure 2.11: variation of pressure difference with respect to Mach number for reactive Euler

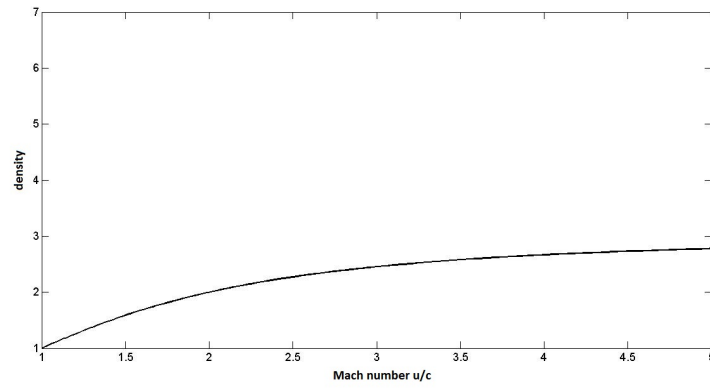


Figure 2.12: variation of density ratio as a function of Mach number for reactive Euler

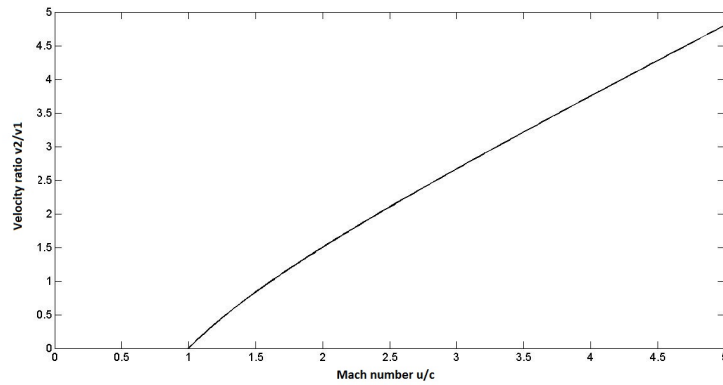


Figure 2.13: Variation of velocity ratio with respect to Mach number for reactive Euler

2.4.2 Shallow Water Wave

Let's derive these relation to both reactive and non-Reactive shallow water wave.

Non-Reactive Shallow Water Wave

thus the relation between F_r which is the Froude number which is $f_r = U/c$:

$$U^2 \left(-\frac{1}{x} \right) + \frac{1}{2} h_0 g x \left(1 + \frac{1}{x} \right) = 0 \quad (2.155)$$

$$U^2 = \frac{1}{2} h_0 g x^2 \left(1 + \frac{1}{x} \right) \quad (2.156)$$

$$\frac{U^2}{h_0 g} = \frac{1}{2} x^2 \left(1 + \frac{1}{x} \right) \quad (2.157)$$

$$\frac{U^2}{h_0 g} = \frac{1}{2} x (x + 1) \quad (2.158)$$

$$F_r^2 = \frac{1}{2} x (x + 1) \quad (2.159)$$

now it is easy to solve out this quadratic equation to get a relation between F_r and h/h_0 the relation is an explicit form of the Hugoniot curve that should be drawn :

$$F_r^2 = \frac{1}{2} x (x + 1)$$

$$\frac{1}{2} x (x + 1) - F_r^2 = 0 \quad (2.160)$$

$$x^2 + x - 2F_r^2 = 0 \quad (2.161)$$

$$x_{1,2} = \frac{-1 \pm \sqrt{1 + 8F_r^2}}{1(2)} \quad (2.162)$$

$$x_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{1 + 8F_r^2}}{2} \quad (2.163)$$

this is the explicit form of the equation lets draw for the + to be sure it is positive increasing curve shown in figure fig-hugo-sww .

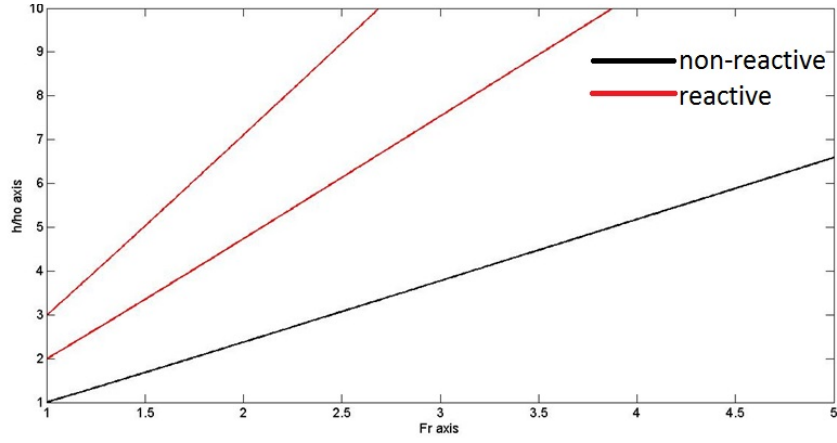


Figure 2.14: the variation of height ratio as a function of Froude number

Figure 2.14 seems like a straight line passing by the point (1,1), but let's try to zoom that line to take a very small region between a Froude number [1,1.04] just to see the real direction.

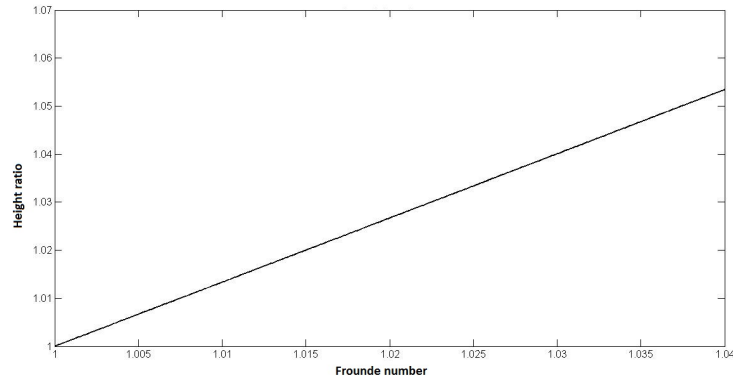


Figure 2.15: Zooming of height to Froude number ratio

The fact that the height ratio to Froude number remarkably seems like a straight line, and is almost like that is due to the 8 factor that is multiplied by the Froude number that gives a very big ratio to F_r in comparison with the 1 number meaning that the x ratio really when approximated becomes:

$$x = -\frac{1}{2} + \frac{\sqrt{8F_r^2}}{2} \quad (2.164)$$

$$x = -\frac{1}{2} + F_r\sqrt{2} \quad (2.165)$$

This is the actual reason why it actually seems like a straight line although if looking for a Froude number less than 1 the curve structure will be perfectly shown. We are also able to find a relation between the u/c_0 and F_r to be :

$$\frac{u}{c_0} = F_r \left[1 - \frac{1}{x} \right], \quad (2.166)$$

it can also be written only in terms of F_r as:

$$\frac{u}{c_0} = F_r \left[1 - \left(-\frac{1}{2} + \frac{\sqrt{1 + 8F_r^2}}{2} \right) \right], \quad (2.167)$$

and its plot is shown in figure 2.16.

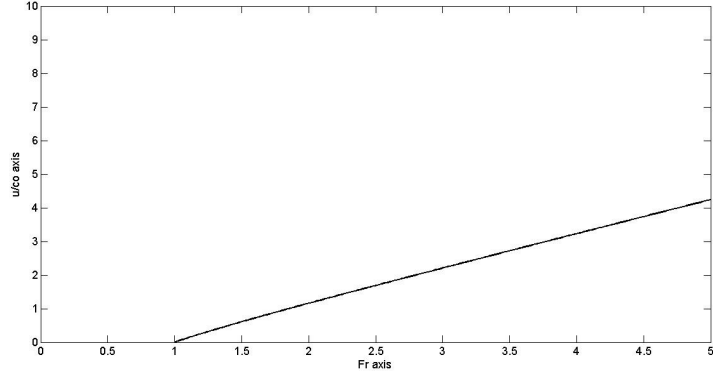


Figure 2.16: the variation of velocity ratio as a function of F_r

Reactive Shallow Water Wave

Finding the same curve for the reactive shallow water wave equations is now easy to be found we only need to replace h with H and it is known from prior studies that $H = h + yq$:

$$x = \frac{H}{H_0} \quad (2.168)$$

$$x = \frac{h + yq}{h_0 + y_0q} \quad (2.169)$$

$$x = \frac{h}{h_0 + q} \quad (2.170)$$

$$x = \frac{h/q}{h_0/q + 1} \quad (2.171)$$

$$(2.172)$$

this is the only difference that is recognizable between both equations, note that the equation of F_r remains the same lets find now the relation between the new ratio of $\frac{h/q}{h_0/q+1}$ and F_r :

$$x = -1 + \sqrt{1 + 8F_r^2} \quad (2.173)$$

$$\frac{h/q}{h_0/q+1} = -\frac{1}{2} + \frac{\sqrt{1 + 8F_r^2}}{2} \quad (2.174)$$

This relation is similar to the know reactive one with a difference in the y axis the ratio .

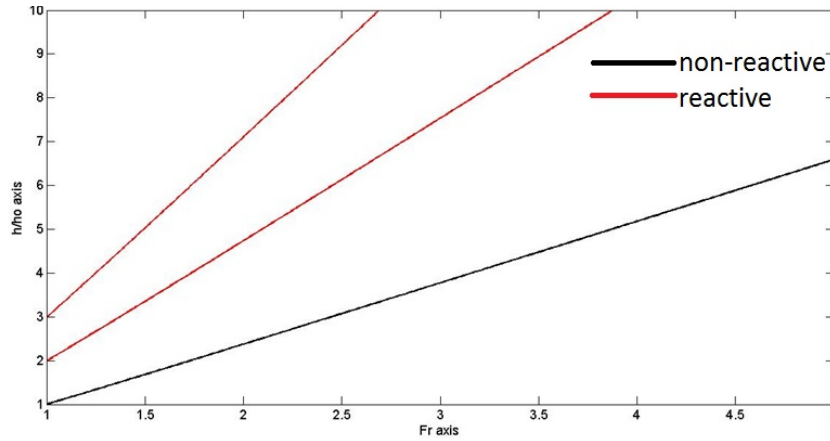


Figure 2.17: the variation of reactive height ratio as a function of Froude number

2.5 Comparison of Results

figure 2.3, shows a sample for the Hugoniot curve for a non-reactive Euler equation for a particular value of $\gamma = 1.4$, it is known that the specific volume ratio should start from one, and can not start from a value less than that for that lets zoom the figure a bit as shown in figure 2.18:

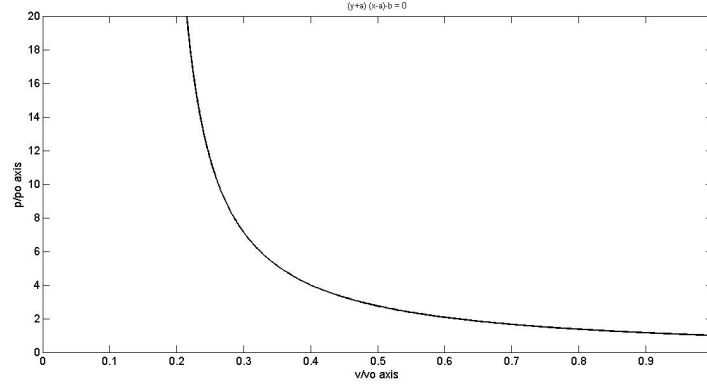


Figure 2.18: variation of pressure ratio as a function of specific volume ratio for non-reactive Euler

for an initial value of $v/v_0 = 1$ it is shown that the pressure ratio $p/p_0 = 1$ meaning that the initial condition of p_0 and v_0 is satisfied, then as the specific volume ratio increases which is proportional to saying that the specific volume increases or that the density decreases the pressure ratio which decreases in a slight manner at the beginning and then decreases slightly, the pressure ratio never reaches zero but rather tends to have an asymptotic direction parallel to the specific volume ratio axis.

A graph comparing different values of γ was plotted and shown in figure 2.4, the purpose of this figure is to show the variation of the Hugoniot curve for different values of γ . This figure indicates that in the region of specific ratio greater than one which is the detonation region, as γ increases the pressure ratio for a specific value of v/v_0 increases knowing that all the curves intersect at the well known point (1, 1) which is a necessary point to pass by due to the fact that the reaction is non-reactive and that $q = 0$.

The same analysis was done for the reactive Euler equation for certain values assuming in figure 2.8 such as the $\gamma = 1.4$ and that $q = 1$, the Hugoniot curve remarkably also started from the initial point (1,1) presenting the initial conditions and then the pressure ratio starts decreasing as the specific ratio increases, reaching an asymptotic direction parallel to the specific ratio axis.

For a Hugoniot reactive equation there are two factors that affect the processes: there is the value of γ which is shown in figure 2.9 where the slope of the Hugoniot curve decreases as the value of γ increases, the figure shows the plot for three different values of $\gamma = 1.4, 1.67, \text{ and } 2$.

The other factor that affects the Hugoniot curve is the value of q , figure 2.10 shows the plot for different values of q for 1, 10 and 100. It is recognized that changing the value of q shifts the curve upward changing the initial condition point, as q increases the initial pressure ratio at $v/v_0 = 1$ increases meaning that the initial pressure is increasing as q increases, in other words the pressure no longer starts from the initial pressure p_0 but rather starts from a higher

value. We can also compare between the reactive and non-reactive Hugoniot curve for a same value of γ . Let's take a case where $\gamma = 1.4$ and for the reactive euler a value of $q = 20$, and plot both reactive and non-reactive Hugoniot in one figure as shown in figure 2.19.

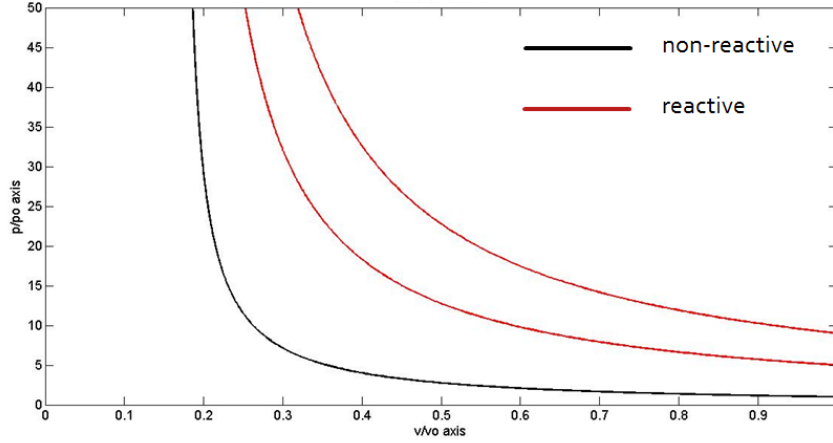


Figure 2.19: Reactive and non-reactive hugoniot curve

Figure 2.19 shows two curves on for the reactive and one of the non-reactive Hugoniot curve for a constant value of $\gamma = 1.4$, it is shown that the reactive Hugoniot curve is shifted at the beginning of the curve at the initial conditions where pressure starts from a value of p_0 for the non-reactive curve, while the reactive curve starts from a value higher than p_0 this is due to the presence of the constant q value.

For a shallow water wave system the only figure that can be observed and established is that of the height ratio versus the Froude number which seems like a straight line passing by the point (1,1), but which is actually a hyperbolic curve with a very small tilt that appears before the Froude number of 1.

Our target from this study is to find an analogy between Euler and Shallow water wave systems for that it is important to find a shape that is relevant and similar to the height ratio versus Froude number in the Shallow water model. For this study the most apparent similarity is to compare between the velocity versus Mach number shown in figure 2.13 and between the height and Froude number shown in figure 2.14. Both figures start from an initial value of (1,1) and both continue as a straight line, with a small slope compared to others. Comparing the reactive and non-reactive shallow water height to Froude number ratio, it can be found that it is the same relation and same plot. But the difference is actually in the height ratio where for the non-reactive case it is $\frac{h}{h_0}$ while in the case of reactive it is $\frac{h/q}{h_0/q+1}$.

Chapter 3

Constant Volume Explosion

In this chapter a constant volume solution for both models is to be formulated.

3.1 Euler Solution

The differential reactive Euler equation found from equations 2.26 , 2.35 and 2.36 :

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial \rho}{\partial x_j} = 0 \quad (3.1)$$

$$\rho \left(\frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial p}{\partial x_i} = 0 \quad (3.2)$$

$$\rho \frac{\partial e}{\partial t} + \rho u_j \frac{\partial e}{\partial x_j} + u_j \frac{\partial p}{\partial x_j} = -qw \quad (3.3)$$

The constant volume solution is based on the assumption that the volume is constant and thus no change in density and velocity happens in this domain this lead us to a simple differential equation :

$$\rho \frac{\partial e}{\partial t} = -qw \quad (3.4)$$

solving this equation:

$$\frac{\partial e}{\partial t} = -\frac{qw}{\rho} \quad (3.5)$$

using equation 2.33 to substitute instead of $\frac{w}{\rho}$ and assuming that y does not vary with respect to x since constant volume:

$$\frac{\partial e}{\partial t} = -q \frac{\partial y}{\partial t} \quad (3.6)$$

then:

$$\frac{\partial e}{\partial y} = -q \quad (3.7)$$

$$\partial e = -q \partial y \quad (3.8)$$

$$e(y) = \int_1^y -q \partial y \quad (3.9)$$

$$e - e_0 = q(1 - y) + c \quad (3.10)$$

where c is a constant number with respect to y . This is a dimensional equation we can also write this equation in its non-dimensional form by dividing all over by q we get:

$$\frac{e}{q} = 1 + \frac{c}{q} - y \quad (3.11)$$

this is an equation of a straight line with a negative slope. An attempt for drawing it will require an information about the initial conditions and the value of q . for an initial condition of no energy at the beginning of the reaction where $y = 1$ we get:

$$0 = 1 + \frac{c}{q} - 1 \quad (3.12)$$

then we conclude that $c = 0$ the equation of the straight line then becomes:

$$\frac{e - e_0}{q} = 1 - y \quad (3.13)$$

and it can be presented in the following figure:

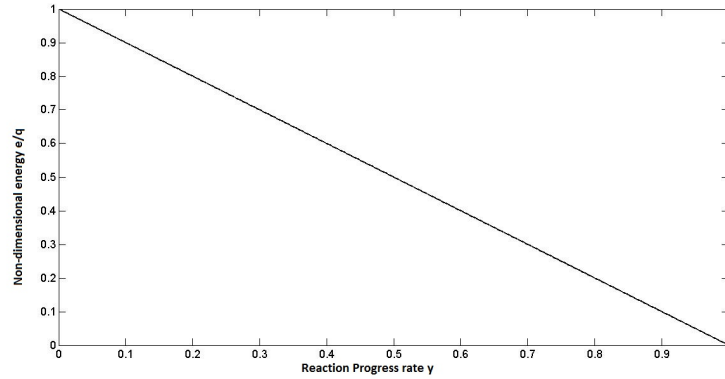


Figure 3.1: the variation of e/q as a function of y

From this figure we can recognize that the variation of e/q as a function of y is a straight line that starts from the value 0 when $y = 1$ and ends by a value of 1 when y reaches 0.

There are two other equations can be written to be able to solve the Euler equation which are the perfect gas equation and the ideal gas equations presented in equation 3.14 and 3.15 respectively:

$$e = c_v T \quad (3.14)$$

$$p = \rho R T \quad (3.15)$$

lets use equation 3.14 to derive the equation:

$$\frac{de}{dt} = c_v \frac{dT}{dt} \quad (3.16)$$

lets eliminate the energy from this equation using equation 3.5:

$$-qr + e_0 = c_v \frac{dT}{dt} \quad (3.17)$$

then as a final form we can write :

$$\frac{dT}{dt} = \frac{-qr + e_0}{c_v} \quad (3.18)$$

where r is the Arrhenius form of [12] and can be written as:

$$r = -Ay \exp^{\frac{-E_a}{RT}} . \quad (3.19)$$

where: A is the pre-exponential factor that controls the time scale, T is the temperature (in kelvin), T_0 is the initial temperature, E_a is the activation energy, and R is the Universal gas constant ($R = 8.314472 \frac{J}{K.mol}$). For reactants to transform into products, they must first acquire a minimum amount, activation energy. At an absolute temperature T, the fraction of molecules that have a kinetic energy greater than E_a can be calculated from statistical mechanics, in this study we will only calculate the zero-order solution. Another form can be written that relates temperature to y:

$$e = (1 - y)q + e_0 \quad (3.20)$$

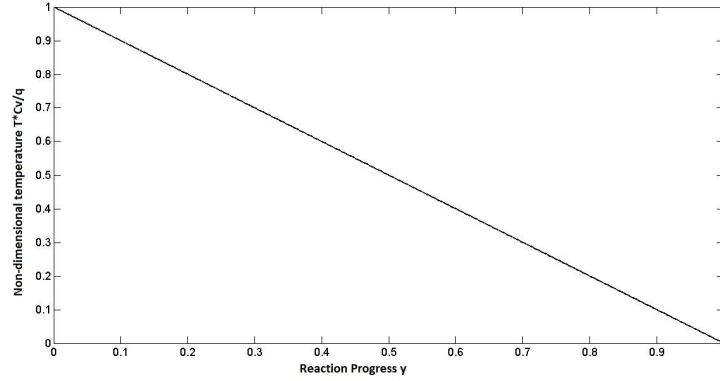
$$c_v T = (1 - y)q + e_0 \quad (3.21)$$

$$T = (1 - y) \frac{q}{c_v} + \frac{e_0}{c_v} \quad (3.22)$$

to write it in an non-dimensional form :

$$\frac{Tc_v}{q} = (1 - y) + \frac{e_0}{q} \quad (3.23)$$

plotting this function in its non-dimensional form we give:

Figure 3.2: variation of Tc_v/q as a function of y

This figure shows the variation of Tc_v/q as a function of y . Temperature varies linearly as a function of y in a decreasing manner starting from a value of 1 when y was 0 reaching a value of 0 when y equals to 1, but taking into consideration the variation of temperature with respect to time will be increasing since y , the reaction progress starts at time 0 from a value of 1 and reaches at time infinity a value of 0. This non-dimensional analysis can give us an overview of the behavior of temperature as a function of y , with a factor of c_v/q which is a constant quantity. We can also do the same derivation but with respect to time:

$$\frac{\partial y}{\partial t} = \frac{w}{\rho} \quad (3.24)$$

$$\partial t = \frac{\rho}{w} \partial y \quad (3.25)$$

assuming r is defined by:

$$r = \frac{w}{\rho} \quad (3.26)$$

then the equation becomes:

$$\partial t = \frac{1}{r} \partial y \quad (3.27)$$

lets assume that r is a function of pressure, temperature and y

$$r = f(R, T, y) \quad (3.28)$$

substituting the Arrhenius form in equation 3.19 we get:

$$\frac{\partial t}{\partial y} = \frac{y}{-Ay \exp \frac{-E_a}{RT}} \quad (3.29)$$

The relation between temperature and y can be determined from equation 3.23, then the above equation becomes:

$$\frac{\partial t}{\partial y} = \frac{1}{-Ay \exp \frac{-E_a}{RT}} \quad (3.30)$$

$$\frac{\partial t}{\partial y} = \frac{1}{-Ay \exp \frac{-E_a}{R(1-y)\frac{q}{c_v} + R\frac{e_0}{c_v}}} \quad (3.31)$$

$$\frac{\partial t}{\partial y} = \frac{1}{-yA \exp \frac{-E_a c_v}{R(1-y)q + Re_0}} \quad (3.32)$$

$$\frac{\partial t}{\partial y} = \frac{\exp \frac{E_a c_v}{R(e_0 + (1-y)q)}}{-Ay} \quad (3.33)$$

This is the dimensional form of the relation between time and y . Now lets write it in the non-dimensional form:

$$\frac{\partial t}{\partial y} = \frac{\exp \frac{E_a c_v}{R(e_0 + (1-y)q)}}{-Ay} \quad (3.34)$$

Lets write the above equations in the non-dimensional form, lets assume some non-dimensional parameters at the beginning:

$$T^* = \frac{T}{T_0} \quad (3.35)$$

$$r^* = \frac{r}{A} \quad (3.36)$$

$$\theta_a = \frac{E_a}{RT_0} \quad (3.37)$$

$$q^* = \frac{q}{c_v T_0} \quad (3.38)$$

$$\tau = tA \quad (3.39)$$

lets start with equation 3.30 :

$$\frac{\partial t}{\partial y} = \frac{\exp \frac{E_a c_v}{R(e_0 + (1-y)q)}}{-Ay} \quad (3.40)$$

$$\frac{\partial t}{\partial \tau} \frac{\partial \tau}{\partial y} = \frac{\exp \frac{E_a c_v}{RT}}{-Ay} \quad (3.41)$$

$$A \frac{\partial \tau}{\partial y} = \frac{\exp \frac{E_a c_v}{R \frac{T}{T_0} T}}{-Ay} \quad (3.42)$$

$$\frac{\partial \tau}{\partial y} = \frac{\exp \frac{\theta_a}{T^*}}{-y} \quad (3.43)$$

$$\frac{\partial \tau}{\partial y} = \frac{\exp \frac{\theta_a}{T^*}}{-y} \quad (3.44)$$

now lets write the temperature equation in a non-dimensional form:

$$\frac{\partial T}{\partial t} = \frac{-qr}{c_v} \quad (3.45)$$

$$\frac{\partial T}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{-qr}{c_v} \quad (3.46)$$

$$\frac{\partial T}{\partial \tau} A = \frac{-qrT_0}{c_v T_0} \quad (3.47)$$

$$\frac{\partial T}{\partial \tau} = \frac{q^* T_0}{A} \quad (3.48)$$

$$\frac{\partial T}{\partial \tau} = q^* r^* \quad (3.49)$$

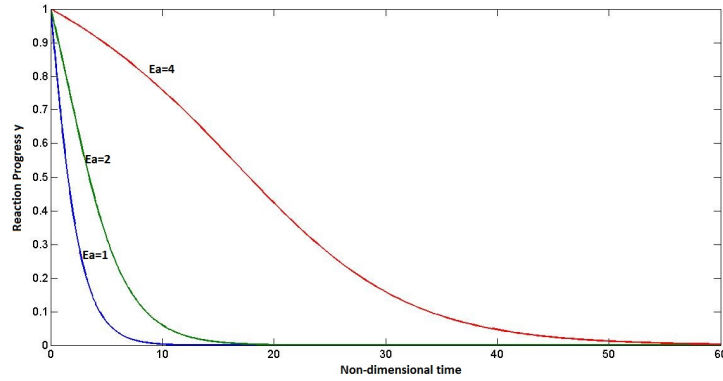


Figure 3.3: variation of reaction progress y as a function of non-dimensional time T/τ for different E_a

knowing that A and y are unit less factors and that the exponential power ends up to be dimensionless, then this form can be called a non-dimensional form. Lets plot this integral for different values of E_a setting all other factors to be equal to one. lets take for example an initial value of E_a to be 1 and vary by the magnitude of 2.

Figure 3.3 shows the variation of y which is the reaction progress with respect to time where y starts with a value of 1 at the initial state and reaches a value of zero at the end of the reaction, this figure shows three different cases where E_a varies from 1 to 4. It can be shown that as E_a increases the time of the reaction to end increases for example when $E_a = 1$ it took around 12 sec for the process to end which when $E_a = 2$ it took about 20 sec to end and the time rapidly increases to 75 sec when $E_a = 4$. In addition as the activation energy is increasing a slight change at the beginning of the reaction is starting to be shown, it's clear at the beginning of the curve where E_a is 4 the significant decrease in the slop is near 20 second. This change is shown more in figure 3.4.

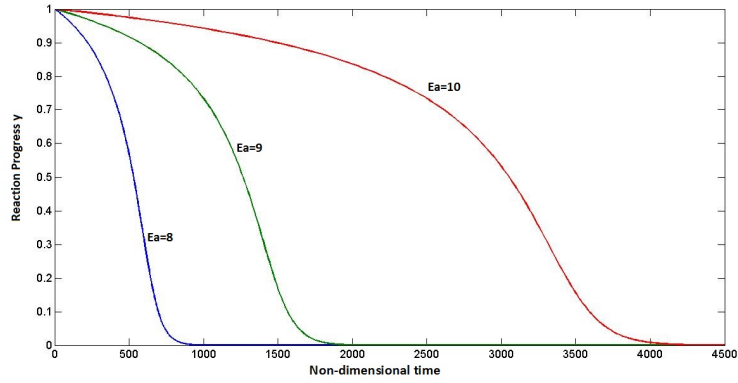


Figure 3.4: variation of reaction progress y as a function of non-dimensional time T/τ for different E_a

In figure 3.4 it is shown more clearly that rapid change in the slope. This figure shows also the variation of y as a function of time, for different values of activation energy in this case we took three values 8, 9, and 10. The rapid change is well shown in the curve of activation energy 10, this change in slope is apparent at the 3000 sec for $E_a = 10$ and at 1000 sec for $E_a = 9$ and at 500 sec for $E_a = 8$. Another graph can be conducted for a high activation energy of 100.

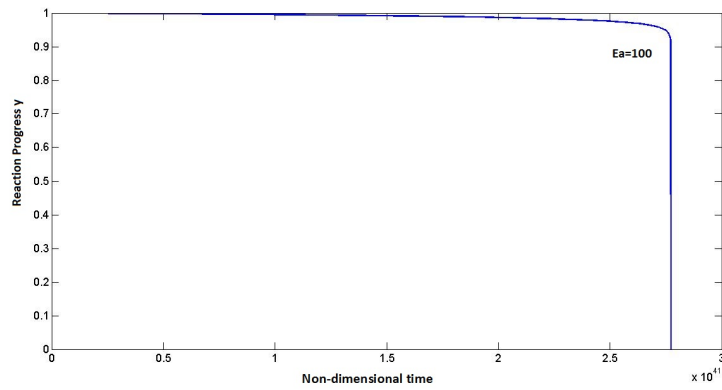


Figure 3.5: variation of reaction progress y as a function of non-dimensional time T/τ for different E_a

3.2 Shallow Water Wave Equation

The Differential reactive shallow water wave equation is given by equations 2.47 and 2.48 :

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = -gq \frac{\partial y}{\partial x}. \quad (3.50)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = -qw. \quad (3.51)$$

for a constant volume solution, the velocity doesn't change with respect to the time or x dimension thus the equation 2.47 becomes:

$$0 = 0 \quad (3.52)$$

we are now left with equation 2.48 which becomes:

$$\frac{\partial h}{\partial t} = -qw. \quad (3.53)$$

from the reaction rate equation given in equation 2.54 .

$$\frac{\partial y}{\partial t} + \frac{\partial (yu)}{\partial x} = w. \quad (3.54)$$

and applying the constant volume condition we get:

$$\frac{\partial y}{\partial t} = w. \quad (3.55)$$

substituting in equation 3.53 :

$$\frac{\partial h}{\partial t} = -q \left[\frac{\partial y}{\partial t} \right]. \quad (3.56)$$

$$\partial h = -q \partial y \quad (3.57)$$

$$h(y) = \int_1^y -q \partial y. \quad (3.58)$$

$$h(y) + h_0 = q(1 - y) + c \quad (3.59)$$

where c is a constant number with respect to y. This is a dimensional equation we can also write it in the non-dimensional form by dividing all other by q:

$$\frac{h + h_0}{q} = 1 + \frac{c}{q} - y. \quad (3.60)$$

this is a straight line with a negative slope where c is a constant number that can be determined from the initial condition where initially the height was zero at $y = 1$

$$0 = 1 + \frac{c}{q} - 1. \quad (3.61)$$

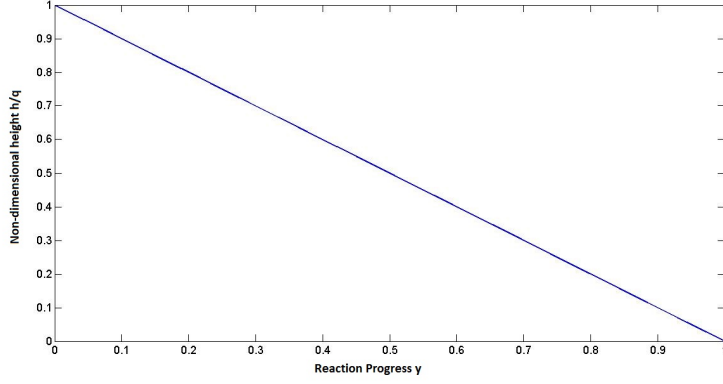


Figure 3.6: The variation of h/q as a function of the reaction progress y

then for this initial condition it is calculated that $c = 0$. This straight line is presented in figure 3.6, then the equation of the straight line becomes:

$$\frac{h - h_0}{q} = 1 - y, \quad (3.62)$$

$$y = -\frac{h - h_0}{q} + 1. \quad (3.63)$$

Plotting this line we get a decreasing straight line starting from an $y = 1$ where $h/q = 0$ reaching $y = 0$ where $h/q = 1$, the slope of the straight line is known to be -1. We can also do the same derivation but with respect to time, starting again from equation constant volume reactive equation:

$$\frac{\partial y}{\partial t} = w, \quad (3.64)$$

$$\frac{\partial t}{\partial y} = \frac{1}{w}. \quad (3.65)$$

Assuming a new variable which is r where r is defined by:

$$r = w, \quad (3.66)$$

then the equation becomes:

$$\frac{\partial t}{\partial y} = \frac{1}{r}, \quad (3.67)$$

lets assume that r is a function of height and y

$$r = f(h, y) \quad (3.68)$$

solving such integration requires choosing a form of r , lets choose a typical form of r :

$$r = -Ay \exp \frac{-h_a}{h}. \quad (3.69)$$

a relation between h and y was conducted before, lets substitute it:

$$h - h_0 = q(1 - y) \quad (3.70)$$

$$r = -Ay \exp^{\frac{-h_a}{q(1-y)+h_0}} \quad (3.71)$$

Lets do the same non-dimensional analysis for equation 3.67:

$$\frac{\partial t}{\partial y} = \frac{1}{r} \quad (3.72)$$

$$\frac{\partial t}{\partial y} = \frac{1}{-Ay \exp^{\frac{-h_a}{h}}} \quad (3.73)$$

$$\frac{\partial t}{\partial \tau} \frac{\partial \tau}{\partial y} = \frac{\exp^{\frac{h_a}{h}}}{-Ay} \quad (3.74)$$

$$\frac{\partial \tau}{\partial y} \frac{1}{A} = \frac{\exp^{\theta_{sa}}}{-Ay} \quad (3.75)$$

$$\frac{\partial \tau}{\partial y} = \frac{\exp^{\theta_{sa}}}{-y} \quad (3.76)$$

where $\theta_{sa} = \frac{h_a}{h}$, this is the non-dimensional form of the relation between time and reaction progress y for the shallow water wave equations. lets plot the variation of y as a function of time from equation 3.76 for different values of h_a assuming that q , A , and h_0 are equal to 1.

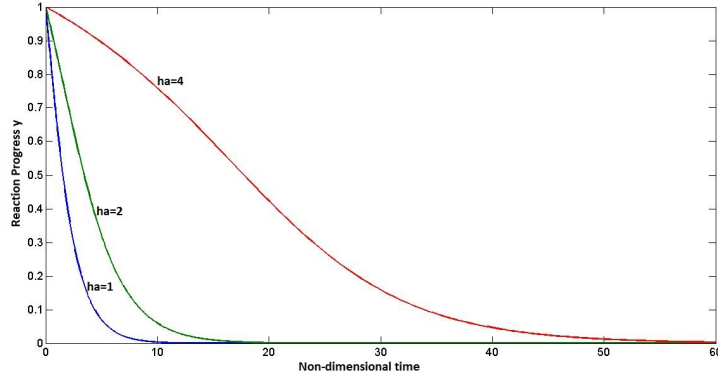


Figure 3.7: variation of reaction progress y as a function of non-dimensional time t/τ for different activation height

Figure 3.7 shows the variation of y as a function of time for different values of activation height, the figure shows three different values of activation height 1, 2, and 4, it is shown that the shallow water wave equations has the same resonance as the Euler equation shown in figure 3.3. We can also do the same

analysis done for the Euler Equation here, let's draw for $h_a = 8, 9, 10$, and for an value of $h_a = 100$.

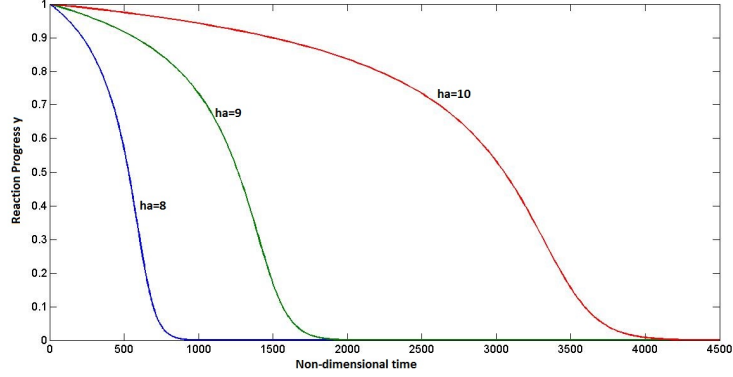


Figure 3.8: variation of reaction progress y as a function of non-dimensional time t/τ for different activation height

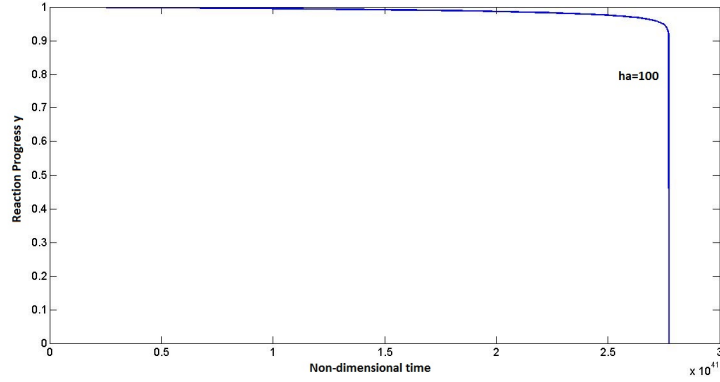


Figure 3.9: variation of reaction progress y as a function of non-dimensional time t/τ for different activation height

3.3 Comparison of Results

For an Euler system the relation between energy and reaction progress y is linear as indicated in figure 3.1. In addition the relation between temperature and reaction progress is also linear due to the fact that energy and temperature are proportional, relation between temperature and reaction progress is shown in figure 3.2. Thus both figures can have the same analysis, the energy or

temperature increase as the reaction takes place, knowing that the reaction starts from a reaction progress value of 1 and ends with a reaction progress of 0.

Another important relation to look at is the relation between reaction progress with respect to time which varies with the variation of the E_a , this was plotted to the non-reactive case in three different figure for seven different activation energy values in figures 3.3, 3.4 and 3.5. In the figure 3.3 the activation energy is 1, 2, and 4. The reaction progress y increases as the activation energy increases, taking into consideration that all curves must start from the same point which is (0,1) which is the condition for the reaction. Also, it is quit shown that the time to reach a steady state or in other words to reach the end of the reaction where $y = 0$ increases in a large way as the activation energy increases when the activation energy is doubled the time to reach a reaction progress of 0 is tripled. Figure 3.4 shows a tilt at the beginning of the reaction meaning that for bigger activation energies the reaction progress doesn't decrease rapidly but instead decreases in three stages the first with a low rate, then the second rapidly, and the third is slow to reach the steady state. Now we can compare the duration of each of these states as the activation energy increases, stage one duration increases as the activation energy increase and it's direction tends toward begin horizontal. For the second stage the duration decreases as the activation energy increases this indicates that the direction of the curve in this stage tends toward being vertical and thus the reaction happens in less time. Figure 3.5 shows a huge activation energy to show the limits of the curve and how the process tends to happen at huge rates. the first stage becomes almost horizontal with a small slope at the end of the first stage, the second stage tends to become almost vertical meaning that the reaction becomes rapid and happens in no time but it also means that the reaction takes a lot of time to become steady due to the fact that stage one takes a lot of time, when the activation energy was multiplied by 100 the time to reach a steady state increased by 10^{37} times.

Doing the same study for the Shallow water wave equation we get only two relations. First relation is the relation between height and reaction progress y , which is linear as shown in figure 3.6. The height increases as the reaction proceeds, the drown graph represents an inverse relation since it is plotted from $y = 0$ to $y = 1$ while in reality the reaction proceeds from a reaction progress 1 to 0.

The second relation that can be drawn is the relation between reaction progress and time, this relation which is also plotted in three different figures as shown in figures 3.7, 3.8, and 3.9. These three figures are identical to these of Euler shown in figures 3.3, 3.4 and 3.5. In other words the variation of reaction progress as a function of time behaves the same for both system Euler and Shallow Water, for the case of Euler it is a function of activation energy while in the case of Shallow Water it is a function of activation height.

Chapter 4

Detonation wave

The leading element is a shock, which instantaneously heats the material and triggers the reaction. Immediately behind the shock is the relatively short reaction zone in which the reaction proceeds to completion. The reaction zone is steady in the coordinate frame attached to the shock. Following the steady reaction zone is a time-dependent rarefaction wave that reduces the pressure down to the low value ordinarily obtaining at the rear boundary [9]. The constant volume solution is based on the assumption that reaction happens homogeneously in the whole volume this means mathematically that there is no space dependence. The constant volume reaction has the same Rayleigh line as an infinitely fast detonation. Using this assumption certain equations in the equation will become zero and thus we are left with energy or height equation as a function of certain variables. More detailed work will come in the next section.

4.1 CJ solution

Referring to figure 4.1 we can observe the two states of the shock.

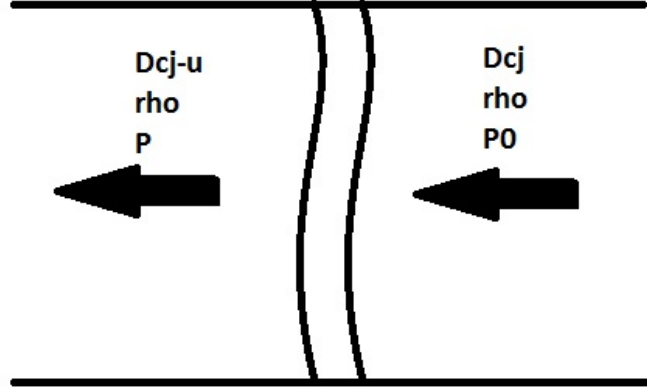


Figure 4.1: The two states of a shock

using the shock conditions:

$$\rho_0 D_{cj} = \rho(D_{cj} - u), \quad (4.1)$$

$$P - P_0 = \rho_0 D_{cj} u, \quad (4.2)$$

$$E - E_0 = \frac{1}{2}(p - p_0)(v_0 - v). \quad (4.3)$$

Let's start by solving equation 4.1:

$$\rho_0 D_{cj} = \rho(D_{cj} - u), \quad (4.4)$$

$$\frac{\rho_0}{\rho} D_{cj} = D_{cj} - u, \quad (4.5)$$

$$x D_{cj} - D_{cj} = -u, \quad (4.6)$$

$$D_{cj}(1 - x) = u. \quad (4.7)$$

Now with equation 4.2 and using equation 4.7:

$$P - P_0 = \rho_0 D_{cj} u, \quad (4.8)$$

$$P - P_0 = \rho_0 D_{cj} D_{cj}(1 - x), \quad (4.9)$$

$$\frac{P - P_0}{p_0} = \frac{\gamma \rho_0}{\gamma p_0} D_{cj}^2 (1 - x), \quad (4.10)$$

$$y - 1 = \gamma M_{cj}^2 (1 - x). \quad (4.11)$$

In the wave frame we know that:

$$D_{cj} - u = c, \quad (4.12)$$

then from equation 4.1 we can eliminate u to get:

$$\begin{aligned}\rho_0 D_{cj} &= \rho(D_{cj} - u), \\ \rho_0 D_{cj} &= \rho c,\end{aligned}\tag{4.13}$$

$$\rho_0 D_{cj} = \rho c_0 \frac{c}{c_0},\tag{4.14}$$

$$\frac{\rho_0}{\rho} \frac{D_{cj}}{c_0} = \frac{c}{c_0},\tag{4.15}$$

$$x M_{cj} = \left(\frac{p}{p_0} \frac{v}{v_0} \right)^{\frac{1}{2}},\tag{4.16}$$

$$x M_{cj} = (y x)^{\frac{1}{2}},\tag{4.17}$$

$$x M_{cj}^2 = y.\tag{4.18}$$

Solving equations 4.11 and 4.18 as a system of equations we have:

$$y - 1 = \gamma M_{cj}^2 (1 - x),\tag{4.19}$$

$$y - 1 = \gamma M_{cj}^2 \left(1 - \frac{y}{M_{cj}^2}\right),\tag{4.20}$$

$$y - 1 = \gamma M_{cj}^2 - \gamma y,\tag{4.21}$$

$$y - 1 + \gamma y = \gamma M_{cj}^2,\tag{4.22}$$

$$y(\gamma + 1) = \gamma M_{cj}^2 + 1,\tag{4.23}$$

$$y = \frac{\gamma M_{cj}^2 + 1}{\gamma + 1}.\tag{4.24}$$

Let's limit the Mach value to infinity and see what is the value of y :

$$y = \frac{\rho_0 D_{cj}^2}{(\gamma + 1) p_0},\tag{4.25}$$

in other words:

$$p = \frac{\rho_0 D_{cj}^2}{\gamma + 1},\tag{4.26}$$

now let's find x of the CJ by substituting in equation 4.18:

$$x = \frac{y^2}{M_{cj}^2},\tag{4.27}$$

$$x = \frac{1}{M_{cj}^2} \frac{\gamma M_{cj}^2 + 1}{\gamma + 1},\tag{4.28}$$

$$x = \frac{\gamma + \frac{1}{M_{cj}^2}}{\gamma + 1}.\tag{4.29}$$

Now let's also try to see the behavior of x_{cj} as Mach tends to infinity:

$$x = \frac{\gamma}{\gamma + 1}.\tag{4.30}$$

Now lets solve equation 4.3:

$$E - E_0 = \frac{1}{2}(p + p_0)(v_0 - v), \quad (4.31)$$

$$E = \frac{pv}{\gamma - 1} + yq, \quad (4.32)$$

$$E - E_0 = \frac{pv}{\gamma - 1} - \frac{p_0 v_0}{\gamma - 1} - q, \quad (4.33)$$

$$\frac{pv}{\gamma - 1} - \frac{p_0 v_0}{\gamma - 1} - q = \frac{1}{2}(p + p_0)(v_0 - v), \quad (4.34)$$

$$\frac{xy}{\gamma - 1} - \frac{1}{\gamma - 1} - \frac{q}{p_0 v_0} = \frac{1}{2}(y + 1)(1 - x), \quad (4.35)$$

$$\frac{xy - 1}{\gamma - 1} - \tilde{q} =$$

$$\frac{1}{2}[y + 1 - xy - x], \quad (4.36)$$

$$\frac{xy - 1}{\gamma - 1} - \tilde{q} + \frac{xy}{2} - \frac{1}{2}[y - x + 1] = 0, \quad (4.37)$$

$$xy \left[\frac{1}{\gamma - 1} + \frac{1}{2} \right] - \tilde{q} - \frac{1}{\gamma - 1} - \frac{1}{2} - \frac{1}{2}(y - x) = 0, \quad (4.38)$$

$$xy \frac{\gamma + 1}{2(\gamma - 1)} - \tilde{q} - \frac{\gamma + 1}{2(\gamma - 1)} - \frac{1}{2}(y - x) = 0, \quad (4.39)$$

$$xy(\gamma + 1) - 2(\gamma - 1)\tilde{q} - (\gamma + 1) - (\gamma - 1)(y - x) = 0. \quad (4.40)$$

It is also known that:

$$xy = M_{cj}^2 \left[\frac{\gamma + \frac{1}{M_{cj}^2}}{\gamma + 1} \right]^2, \quad (4.41)$$

$$y - x = \frac{\gamma M_{cj}^2 + 1}{\gamma + 1} - \frac{1}{M_{cj}^2} \frac{\gamma M_{cj}^2 + 1}{\gamma + 1}, \quad (4.42)$$

$$y - x = \frac{1}{\gamma + 1} \left[\gamma M_{cj}^2 + 1 - \gamma - \frac{1}{M_{cj}^2} \right], \quad (4.43)$$

$$y - x = \frac{1}{\gamma + 1} \left[\gamma(M_{cj}^2 - 1) + 1 - \frac{1}{M_{cj}^2} \right], \quad (4.44)$$

$$y - x = \frac{1}{\gamma + 1} \left[\gamma(M_{cj}^2 - 1) + \frac{1}{M_{cj}^2} (M_{cj}^2 - 1) \right], \quad (4.45)$$

$$y - x = \frac{1}{\gamma + 1} \left[\left(\gamma + \frac{1}{M_{cj}^2} \right) (M_{cj}^2 - 1) \right]. \quad (4.46)$$

Moreover:

$$M_{cj}^2 \left[\frac{\gamma + \frac{1}{M_{cj}^2}}{\gamma + 1} \right]^2 (\gamma + 1) - 2(\gamma - 1)\tilde{q} - (\gamma + 1) \\ - (\gamma - 1) \left(\frac{\gamma M_{cj}^2 + 1 - \gamma - \frac{1}{M_{cj}^2}}{\gamma + 1} \right) = 0, \quad (4.47)$$

$$M_{cj}^2 \left(\gamma + \frac{1}{M_{cj}^2} \right)^2 + [-2(\gamma - 1)\tilde{q} - (\gamma + 1)] \\ (\gamma + 1) - (\gamma - 1) \left(\gamma M_{cj}^2 + 1 - \gamma - \frac{1}{M_{cj}^2} \right) = 0, \quad (4.48)$$

$$\gamma^2 M_{cj}^4 + 2\gamma M_{cj}^2 + 1 - M_{cj}^2 \left[(\gamma + 1)^2 + 2(\gamma^2 - 1)\tilde{q} \right] - \\ M_{cj}^2 (\gamma - 1) \left(\gamma M_{cj}^2 + 1 - \gamma - \frac{1}{M_{cj}^2} \right) = 0, \quad (4.49)$$

$$(\gamma^2 - (\gamma - 1)\gamma) M_{cj}^4 - \\ M_{cj}^2 (\gamma + 1)^2 - 2(\gamma^2 - 1)\tilde{q} M_{cj}^2 + 1 + (\gamma - 1) = 0, \quad (4.50)$$

$$\gamma M_{cj}^4 + [2\gamma + 2\tilde{q}(\gamma^2 - 1)] M_{cj}^2 + \gamma = 0. \quad (4.51)$$

Solving this equation will lead to two values of M_{cj}^2 as follows:

$$M_{cj}^2_{1,2} = \frac{2\gamma + 2\tilde{q}(\gamma^2 - 1) \pm \sqrt{(2\gamma + 2\tilde{q}(\gamma^2 - 1))^2 - 4\gamma^2}}{2\gamma}, \quad (4.52)$$

$$= 1 + \frac{\tilde{q}(\gamma^2 - 1)}{\gamma} \pm \sqrt{\frac{\tilde{q}(\gamma^2 - 1)}{\gamma} \left[\frac{\tilde{q}(\gamma^2 - 1)}{\gamma} + 2 \right]}, \quad (4.53)$$

we should take the biggest root for that:

$$M_{cj}^2 = 1 + \frac{\tilde{q}(\gamma^2 - 1)}{\gamma} + \sqrt{\frac{\tilde{q}(\gamma^2 - 1)}{\gamma} \left[\frac{\tilde{q}(\gamma^2 - 1)}{\gamma} + 2 \right]}, \quad (4.54)$$

to make sure that what we have derived till now is right lets see the limits when p_0 tends to 0, meaning that $\tilde{q}$ will tend to infinity:

$$\begin{aligned} M_{cj}^2 &= 1 + \frac{\tilde{q}(\gamma^2 - 1)}{\gamma} + \sqrt{\frac{\tilde{q}(\gamma^2 - 1)}{\gamma} \left[\frac{\tilde{q}(\gamma^2 - 1)}{\gamma} + 2 \right]}, \\ M_{cj}^2 &= \frac{\tilde{q}(\gamma^2 - 1)}{\gamma} + \frac{\tilde{q}(\gamma^2 - 1)}{\gamma}, \end{aligned} \quad (4.55)$$

$$\frac{D_{cj}^2}{\gamma p_0 v_0} = 2 \frac{\tilde{q}(\gamma^2 - 1)}{\gamma}, \quad (4.56)$$

$$D_{cj}^2 = 2q(\gamma^2 - 1), \quad (4.57)$$

$$(4.58)$$

which is the same solution as the one mentioned in Fickett and Davis book [2]. then the Mach number at the CJ condition is:

$$M_{cj} = \sqrt{1 + \frac{\tilde{q}(\gamma^2 - 1)}{\gamma} + \sqrt{\frac{\tilde{q}(\gamma^2 - 1)}{\gamma} \left[\frac{\tilde{q}(\gamma^2 - 1)}{\gamma} + 2 \right]}}. \quad (4.59)$$

For example if $\tilde{q} = 1$ then M_{cj} is 1.7444, figure 4.2 shows the variation of M_{cj} with respect to $\tilde{q}$ which is proportional to the variation of M_{cj} with respect to q .

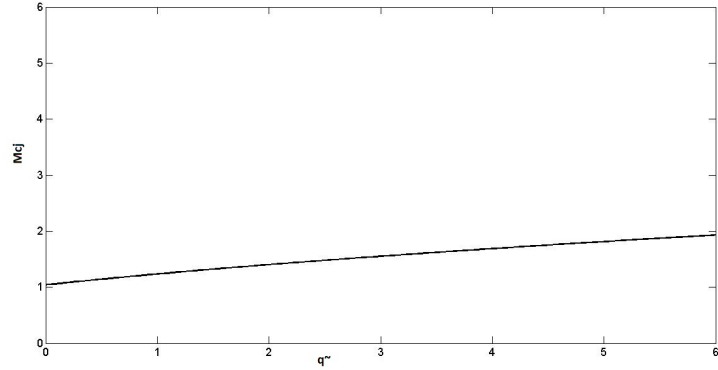


Figure 4.2: the variation of CJ Mach number as a function of $\tilde{q}$

from figure 4.2 it is shown that M_{cj} increases as $\tilde{q}$ increases from an initial value of 2.8983 when $\tilde{q} = 0$ and increasing. Now lets establish the VN conditions

for the same condition where $\tilde{q} = 1$ and $M_{cj} = 1.7444$:

$$\frac{p_{VN}}{p_0} = 3.3832 \quad (4.60)$$

$$\frac{u_{VN}}{c_0} = 2.2699, \quad (4.61)$$

$$\frac{\rho_{VN}}{\rho} = 1.4034. \quad (4.62)$$

after we have the VN condition now we can integrate to find other terms.

4.1.1 Reactive Shallow Water

The reactive shallow water wave equation should be solved in the same manner as the reactive Euler by finding a common point between the curve and the line: the condition equations are known as:

$$(h_0 + q)D = h(D - u), \quad (4.63)$$

$$= hc,$$

$$= h\sqrt{gh},$$

$$= g^{\frac{1}{2}}h^{\frac{3}{2}},$$

$$h = \left[\frac{(h_0 + q)D}{g^{\frac{1}{2}}} \right]^{\frac{2}{3}}, \quad (4.64)$$

$$(4.65)$$

$$(h_0 + q)D^2 + \frac{1}{2}g(h_0 + q)^2 = h(D - u)^2 + \frac{1}{2}gh^2, \quad (4.66)$$

$$(h_0 + q)D^2 + \frac{1}{2}g(h_0 + q)^2 = h \left(\frac{(h_0 + q)D}{h} \right)^2 + \frac{1}{2}gh^2, \quad (4.67)$$

$$(h_0 + q)D^2 + \frac{1}{2}g(h_0 + q)^2 = \frac{(h_0 + q)^2 D^2}{h} + \frac{1}{2}gh^2, \quad (4.68)$$

$$(h_0 + q)D^2 + \frac{1}{2}g(h_0 + q)^2 = g^{\frac{1}{3}}(h_0 + q)^{\frac{4}{3}}D^{\frac{4}{3}} + \frac{1}{2}g^{\frac{1}{3}}(h_0 + q)^{\frac{4}{3}}D^{\frac{4}{3}}, \quad (4.69)$$

$$(h_0 + q)D^2 + \frac{1}{2}g(h_0 + q)^2 = \frac{3}{2}g^{\frac{1}{3}}(h_0 + q)^{\frac{4}{3}}D^{\frac{4}{3}}, \quad (4.70)$$

$$(4.71)$$

$$h_0(1 + \tilde{q})D^2 + \frac{1}{2}gh_0^2(1 + \tilde{q})^2 = \frac{3}{2}g^{\frac{1}{3}}h_0^{\frac{4}{3}}(1 + \tilde{q})^{\frac{4}{3}}D^{\frac{4}{3}}, \quad (4.72)$$

$$(1 + \tilde{q})F_r^2 + \frac{1}{2}(1 + \tilde{q})^2 = \frac{3}{2}g^{\frac{1}{3}}(1 + \tilde{q})^{\frac{4}{3}}F_r^{\frac{4}{3}}, \quad (4.73)$$

$$F_r^6 - \frac{15}{8}(1 + \tilde{q})F_r^4 + \frac{3}{4}(1 + \tilde{q})^2F_r^2 + \frac{1}{8}(1 + \tilde{q})^3 = 0. \quad (4.74)$$

solving the following equation but transferring it into a cubic equation where $w = F_r^2$ will lead to:

$$w^3 - \frac{15}{8}(1 + \tilde{q})w^2 + \frac{3}{4}(1 + \tilde{q})^2w + \frac{1}{8}(1 + \tilde{q})^3 = 0, \quad (4.75)$$

finding the factorized form of the equation we have:

$$(w - (1 + \tilde{q}))^2 \left(w + \frac{1}{8}(1 + \tilde{q}) \right) = 0, \quad (4.76)$$

finding the roots we have:

$$\begin{aligned} w &= -\frac{1}{8}(1 + \tilde{q}), \\ &= 1 + q, \\ &= -(1 + q), \end{aligned} \quad (4.77)$$

we need the positive root, then F_r becomes:

$$F_r = \sqrt{1 + q}. \quad (4.78)$$

The plot of F_r as a function of $\tilde{q}$ is shown in figure 4.3.

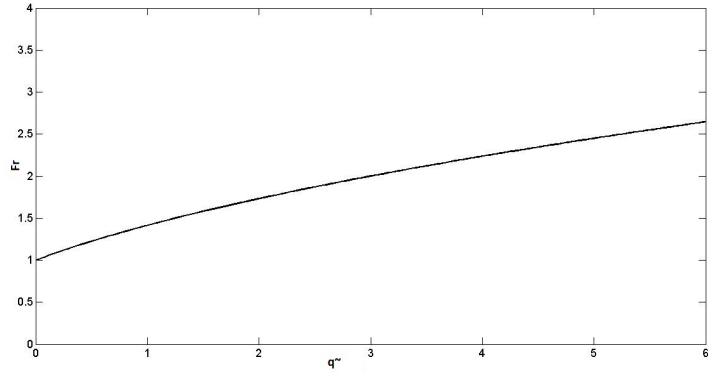


Figure 4.3: The variation of F_r as a function of $\tilde{q}$

4.2 ZND Solution

Let's derive the ZND solution for both models.

4.2.1 Reactive Euler Equation

In this section we will solve the ZND model for both reactive Euler and reactive Shallow Water Wave models. We shall first analyze the ZND structure for

a perfect gas with constant γ . The chemical reaction will be described by a single-step Arrhenius rate law.

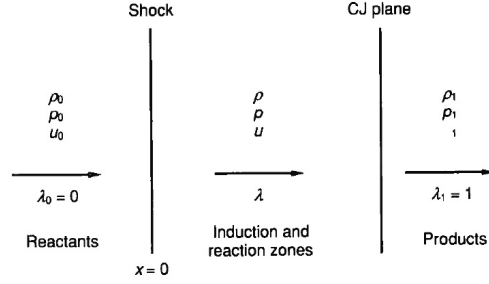


Figure 4.4: Flow across a detonation with respect to a coordinate fixed to the detonation wave [13]

From figure 4.4 we can derive the conservation equation for a steady one-dimensional flow [13] :

$$\frac{d}{dx}(\rho u) = 0 \quad (4.79)$$

$$\frac{d}{dx}(p + \rho u^2) = 0 \quad (4.80)$$

$$\frac{d}{dx} \left(h + \frac{u^2}{2} \right) = 0 \quad (4.81)$$

where

$$h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} + yq. \quad (4.82)$$

Lets start by solving the equation 4.79:

$$\rho u = \rho_{VN} u_{VN}, \quad (4.83)$$

$$\rho = \frac{\rho_{VN} u_{VN}}{u}, \quad (4.84)$$

we can also non-dimensional the equation:

$$\frac{\rho}{\rho_0} = \frac{\rho_{VN} u_{VN}}{u \rho_0}, \quad (4.85)$$

$$x = x_{VN} \frac{u}{u_{VN}}. \quad (4.86)$$

where:

$$x = \frac{\rho_0}{\rho}, \quad (4.87)$$

$$x_{VN} = \frac{\rho_0}{\rho_{VN}}, \quad (4.88)$$

$$u_{VN} = \sqrt{\gamma} u_{VN}. \quad (4.89)$$

Similar solution can be found for equation 4.80:

$$p = p_{VN} + \rho_{VN} u_{VN} (u_{VN} - u), \quad (4.90)$$

and in the non-dimensional form it can be expressed as:

$$y = y_{VN} + \frac{1}{x_{VN}} u \tilde{v}_N (u \tilde{v}_N - u) \quad (4.91)$$

where:

$$y = \frac{p}{p_0}, \quad (4.92)$$

$$y_{VN} = \frac{p_{VN}}{p_0}. \quad (4.93)$$

from equation 4.81 a similar relation for the reaction progress rate y can be written, we assume that the reaction progress is named λ because y is already used:

$$\lambda = \frac{1}{\tilde{q}} \frac{\gamma}{\gamma - 1} x_{VN} y_{VN} + \tilde{q} + \frac{1}{2} u \tilde{v}_N^2 - \frac{\gamma}{\gamma - 1} yx - \frac{1}{2} u^2. \quad (4.94)$$

It is known that q is the chemical energy per unit mass and $0 \leq y \leq 1$ let's differentiate equation 4.81 :

$$\frac{d}{dx} \left(h + \frac{u^2}{2} \right) = 0 \quad (4.95)$$

$$\frac{d}{dx} h + \frac{d}{dx} \left(\frac{u^2}{2} \right) = 0 \quad (4.96)$$

$$\frac{d}{dx} h + \frac{1}{2} u \frac{du}{dx} = 0 \quad (4.97)$$

substituting the value of h from equation 4.82 :

$$\frac{d}{dx} h + \frac{1}{2} u \frac{du}{dx} = 0 \quad (4.98)$$

$$\frac{d}{dx} \left(\frac{\gamma}{\gamma - 1} \frac{p}{\rho} + yq \right) + \frac{1}{2} u \frac{du}{dx} = 0 \quad (4.99)$$

$$\frac{d}{dx} \left(\frac{\gamma}{\gamma - 1} \frac{p}{\rho} \right) + \frac{d}{dx} (yq) + u \frac{du}{dx} = 0 \quad (4.100)$$

$$\frac{\gamma}{\gamma - 1} \left(\frac{1}{\rho} \frac{dp}{dx} - \frac{p}{\rho^2} \frac{d\rho}{dx} \right) + q \frac{dy}{dx} + u \frac{du}{dx} = 0 \quad (4.101)$$

$$(4.102)$$

now differentiating equations 4.79 and 4.80 we get:

$$\frac{d\rho}{dx} = -\frac{\rho}{u} \frac{du}{dx} \quad (4.103)$$

$$\frac{dp}{dx} = -\rho u \frac{du}{dx} \quad (4.104)$$

using those equation to reduce equation 4.101 we get:

$$\frac{\gamma}{\gamma-1} \left(\frac{1}{\rho} \frac{dp}{dx} - \frac{p}{\rho^2} \frac{d\rho}{dx} \right) + q \frac{dy}{dx} + u \frac{du}{dx} = 0 \quad (4.105)$$

$$\frac{\gamma}{\gamma-1} \left[\frac{1}{\rho} \left(-\rho u \frac{du}{dx} \right) + \frac{p}{\rho^2} \frac{\rho}{u} \frac{du}{dx} \right] + q \frac{dy}{dx} + u \frac{du}{dx} = 0 \quad (4.106)$$

$$\left(-\frac{\gamma}{\gamma-1} + 1 \right) u \frac{du}{dx} + \frac{\gamma}{\gamma-1} \frac{p}{\rho u} \frac{du}{dx} + q \frac{dy}{dx} = 0 \quad (4.107)$$

$$\frac{-1}{\gamma-1} u \frac{du}{dx} + \frac{\gamma}{\gamma-1} \frac{p}{\rho u} \frac{du}{dx} + q \frac{dy}{dx} = 0 \quad (4.108)$$

$$\left(\frac{u}{1-\gamma} + \frac{c^2}{(\gamma-1)u} \right) \frac{du}{dx} = -q \frac{dy}{dx} \quad (4.109)$$

$$\left(-\frac{\gamma}{\gamma-1} \right) \left(u - \frac{c^2}{u} \right) \frac{du}{dx} = -q \frac{dy}{dx} \quad (4.110)$$

$$\frac{(1-\gamma)u}{c^2 - u^2} \frac{dy}{dx} q = \frac{du}{dx} \quad (4.111)$$

$$\frac{(1-\gamma)u}{c^2(1-M^2)} \frac{dy}{dx} q = \frac{du}{dx} \quad (4.112)$$

where $c^2 = \gamma p / \rho$ and $M = c / u$ A simple rate law that is widely used in theoretical and numerical detonation is the single-step Arrhenius expression:

$$u \frac{dy}{dx} = -A y \exp \frac{-E_a}{RT} \quad (4.113)$$

The two constants k and E_a have to be specified. The detonation velocity can first be obtained from the CJ theory for given values of y and q . The von Neumann state behind the leading shock front can then be found from the Rankine-Hugoniot normal shock relations.

4.2.2 Shallow Water Wave Equation

lets start first with deriving the deferential of u and h with respect to x for a steady state: the reactive shallow water wave equation are defined as:

$$\frac{\partial H}{\partial t} + \frac{\partial(Hu)}{\partial x} = 0 \quad (4.114)$$

$$\frac{\partial(Hu)}{\partial t} + \frac{\partial(Hu^2 + \frac{1}{2}gH^2)}{\partial x} = 0 \quad (4.115)$$

for a steady state all derivative with respect to time will be zero:

$$\frac{\partial(Hu)}{\partial x} = 0 \quad (4.116)$$

$$\frac{\partial(Hu^2 + \frac{1}{2}gH^2)}{\partial x} = 0 \quad (4.117)$$

it is also known that $H = h + yq$ then:

$$\frac{\partial(hu + yqu)}{\partial x} = 0 \quad (4.118)$$

$$\frac{\partial(hu^2 + yqu^2 + \frac{1}{2}g(h + yq)^2)}{\partial x} = 0 \quad (4.119)$$

solving the first equation we get:

$$\frac{\partial(Hu)}{\partial x} = 0 \quad (4.120)$$

$$H \frac{\partial u}{\partial x} + u \left(\frac{\partial h}{\partial x} + q \frac{\partial y}{\partial x} \right) = 0 \quad (4.121)$$

Now solving the second equation we get:

$$\frac{\partial(Hu^2 + \frac{1}{2}gH^2)}{\partial x} = 0 \quad (4.122)$$

$$2uH \frac{\partial u}{\partial x} + \frac{1}{2}gH \left(\frac{\partial h}{\partial x} + q \frac{\partial y}{\partial x} \right) = 0 \quad (4.123)$$

$$2uH \frac{\partial u}{\partial x} + \frac{1}{2}gH \left(\frac{H}{u} \frac{\partial u}{\partial x} \right) = 0 \quad (4.124)$$

$$\frac{\partial u}{\partial x} \left(\frac{1}{2}g \frac{H^2}{u} + 2uH \right) = 0 \quad (4.125)$$

This means in other words that $\frac{\partial u}{\partial x}$ is zero because the bracket value isn't always equal to zero, thus the velocity u is constant value with respect to the position x . Now we can write an expression of the height h :

$$\frac{\partial Hu}{\partial x} = 0, \quad (4.126)$$

$$u \frac{\partial H}{\partial x} + H \frac{\partial u}{\partial x} = 0, \quad (4.127)$$

$$u_{VN} \frac{\partial H}{\partial x} = 0, \quad (4.128)$$

$$\frac{\partial H}{\partial x} = 0, \quad (4.129)$$

this also indicates that the height does not vary with respect to position and that H has a constant value, let's see how does this affect the quantity h :

$$\frac{\partial H}{\partial x} = 0, \quad (4.130)$$

$$\frac{\partial h}{\partial x} + q \frac{\partial y}{\partial x} = 0, \quad (4.131)$$

$$\frac{\partial h}{\partial x} = -q \frac{\partial y}{\partial x},$$

but in the wave frame:

$$u_l = D - u, \quad (4.132)$$

deriving along space we get:

$$\frac{du_l}{dx} = \frac{du}{dx} \quad (4.133)$$

which lead to the conclusion that u is constant. from the reaction rate equation of shallow water wave model we can find a relation between Arrhenius rate and $\frac{\partial y}{\partial x}$.

$$\frac{\partial y}{\partial t} + \frac{\partial y u}{\partial x} = r, \quad (4.134)$$

$$u \frac{\partial y}{\partial x} + y \frac{\partial u}{\partial x} = r, \quad (4.135)$$

$$u_{VN} \frac{\partial y}{\partial x} = r, \quad (4.136)$$

$$\frac{\partial y}{\partial x} = \frac{r}{u_{VN}}, \quad (4.137)$$

in the wave frame it can be written as:

$$\frac{\partial y}{\partial x} = \frac{r}{D - u_{VN}}, \quad (4.138)$$

$$\frac{\partial y}{\partial x} = \frac{r}{\frac{c_0(D - u_{VN})}{c_0}}, \quad (4.139)$$

$$\frac{\partial y}{\partial x} = \frac{r}{c_0(F_r c_j - u_{VN})}, \quad (4.140)$$

substituting instead of r its equation:

$$\frac{\partial y}{\partial x} = -y \exp \frac{-h_a}{h} \frac{1}{c_0(F_r c_j - u_{VN})}, \quad (4.141)$$

$$\frac{\partial x}{\partial y} = -\frac{1}{y} \exp \theta_a c_0 (F_r c_j - u_{VN}), \quad (4.142)$$

now the relation between the height and the Arrhenius rate can be written as:

$$\frac{\partial h}{\partial x} = -q \frac{r}{u_{VN}} \quad (4.143)$$

let's recall the Arrhenius rate for the shallow water wave equation:

$$r = -A y \exp \frac{-h_a}{h}$$

then the relation of height becomes:

$$\frac{\partial h}{\partial x} = q A y \frac{\exp \frac{-h_a}{h}}{D - u_{VN}}, \quad (4.144)$$

which can also be written as:

$$\frac{\partial h}{\partial x} = qAy \frac{\exp \frac{-h_a}{h}}{\frac{c_0(D-u_{VN})}{c_0}}, \quad (4.145)$$

$$\frac{\partial h}{\partial x} = qAy \frac{\exp \frac{-h_a}{h}}{c_0(F_r - u_{VN})}, \quad (4.146)$$

$$(4.147)$$

The variation of the reaction progress as a function of x is shown in figure 4.5 for different activation energies.

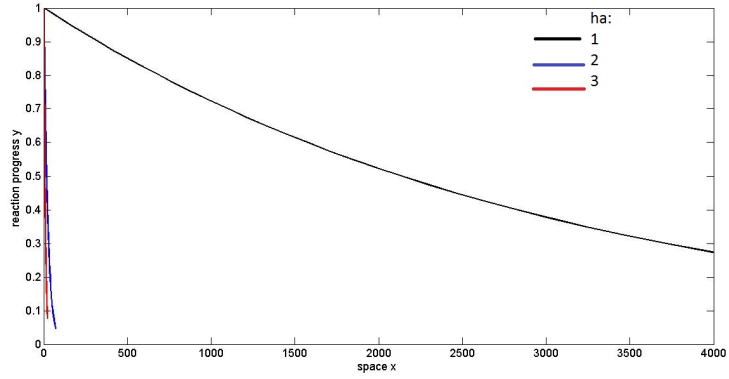


Figure 4.5: variation of y as a function of x for different h_a values

we can also plot the for different heat release:

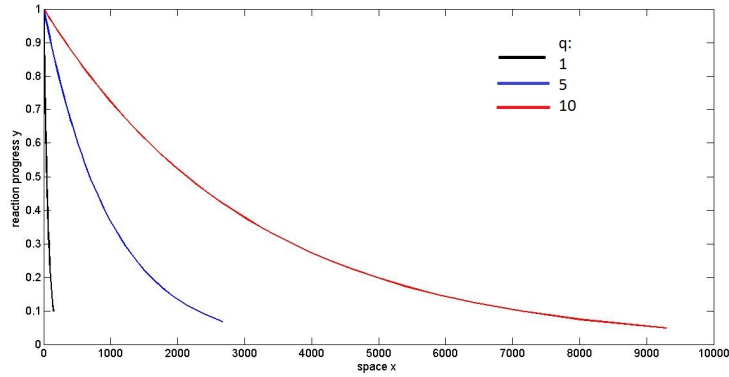


Figure 4.6: variation of y as a function of x for different q values

Similar graph can be established for the variation of the height h as a function of space x which is shown in figure 4.7

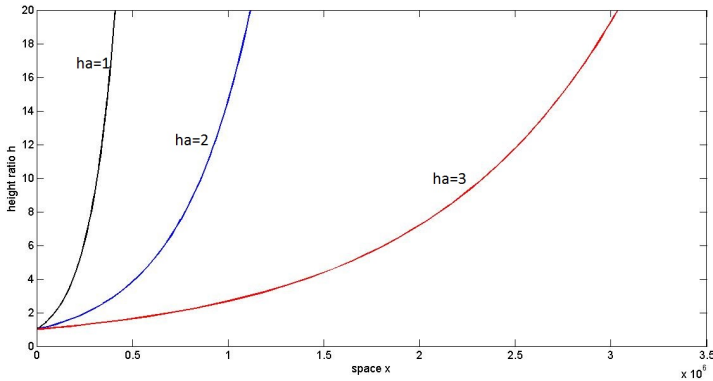


Figure 4.7: variation of height h as a function of space x for different h_a

similar figure can be conducted for different heat release values:

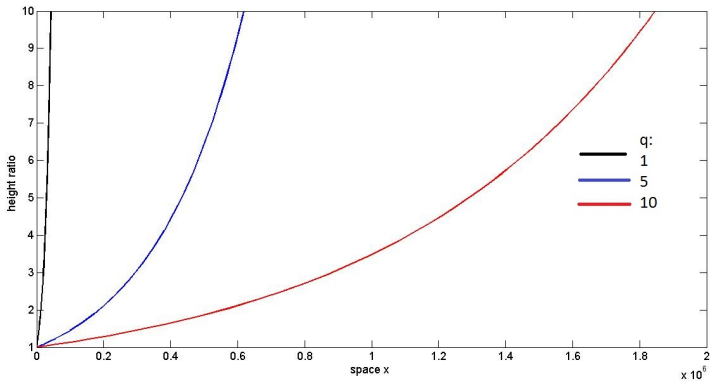


Figure 4.8: variation of height h as a function of space x for different q

Appendix A

Derivation of Rayleigh Line

The basic conservation equations of mass, momentum and energy for one-dimensional steady flow across a combustion wave with respect to a coordinate system fixed to the wave is given by [13]:

$$\rho_0 u_0 = \rho_1 u_1 \quad (\text{A.1})$$

$$p_0 + \rho_0 u_0^2 = p_1 + \rho_1 u_1^2 \quad (\text{A.2})$$

$$h_0 + \frac{u_0^2}{2} = h_1 + \frac{u_1^2}{2} \quad (\text{A.3})$$

The thermodynamic path along which the initial state proceeds to the final state across the combustion wave can be readily obtain from equations A.1 , A.2 :

$$\frac{p_1 - p_0}{v_0 - v_1} = \rho_0^2 u_0^2 = \rho_1^2 u_1^2 = \dot{m}^2 \quad (\text{A.4})$$

also it is known that $v = 1/\rho$ and that $\dot{m} = \rho u$ so it can be derived that:

$$\dot{m} = \sqrt{\frac{p_1 - p_0}{v_0 - v_1}} \quad (\text{A.5})$$

if we define $x = v_1/v_0 = \rho_0/\rho_1$ and $y = p_1/p_0$ then we can write:

$$\dot{m} = \sqrt{\frac{(y - 1) p_0}{(1 - x) v_0}} \quad (\text{A.6})$$

the speed of sound in the upstream flow and the Mach number of the combustion wave are given by:

$$c_0 = \sqrt{\gamma_0 p_0 v_0} \quad (\text{A.7})$$

$$M_0 = \frac{u_0}{c_0} \quad (\text{A.8})$$

then we can write the equation of the mass flow rate as:

$$\gamma_0 M_0^2 = \frac{y-1}{1-x} \quad (\text{A.9})$$

$$y = (1 + \gamma_0 M_0^2) - (\gamma_0 M_0^2) x \quad (\text{A.10})$$

this equation represents a straight line with a slop of $-(\gamma_0 M_0^2)$. It defines the thermodynamic path across the combustion wave and is refereed to as the Rayleigh line. Thus the slope of the Rayleigh line can be written as:

$$\left(\frac{dy}{dx} \right)_R = \frac{y-1}{1-x} \quad (\text{A.11})$$

Appendix B

CJ Calculation

The CJ solutions can be obtained from the conservation laws alone without the need to consider the detailed propagation mechanism of the combustion wave. First we should equate the slopes of the Rayleigh line and the Hugoniot curve. Lets assume CJ (a,b) then [13]:

$$b = \frac{-a}{\gamma_1 - (\gamma_1 + 1)a} \quad (\text{B.1})$$

substituting in the Rayleigh line we get :

$$\left(\frac{dy}{dx}\right)_R = \frac{b-1}{1-a} = \frac{\gamma_1}{\gamma_1 - (\gamma_1 + 1)a} \quad (\text{B.2})$$

multiplying with a in the top and bottom the slope can be written as:

$$\left(\frac{dy}{dx}\right)_R = \frac{\gamma_1 a}{[\gamma_1 - (\gamma_1 + 1)a]a} = \frac{\gamma_1 b}{a} = \left(\frac{dy}{dx}\right)_H \quad (\text{B.3})$$

for an isentrope relation $xy^{\gamma_1} = \text{constant}$, then the isentrope slope becomes:

$$(dydx)_s = -\frac{\gamma_1 y}{x} \quad (\text{B.4})$$

where s refers to isentrope. All the slopes must be equal:

$$\left(\frac{dy}{dx}\right)_R = \left(\frac{dy}{dx}\right)_H = (dydx)_s \quad (\text{B.5})$$

the slope is related to the sound speed by:

$$c_1^2 = \left(\frac{dp}{d\rho}\right)_s = -v_1^2 \left(\frac{dp}{dv}\right)_s = -p_0 v_0 x^2 \left(\frac{dy}{dx}\right)_s \quad (\text{B.6})$$

then the Rayleigh line slope can be written as:

$$\left(\frac{dy}{dx}\right)_R = -\frac{u_1^2}{p_0 v_0 x_1^2} \quad (\text{B.7})$$

finally it can be written that:

$$\left(\frac{u_1}{c_1}\right)^2 = M_1^2 = 1 \quad (\text{B.8})$$

Therefore, the flow Mach number downstream of a CJ detonation or deflagration M_1 is equal to unity.

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