

General : Opto waves

$$\begin{array}{l} \lambda \approx 0.5 \mu\text{m} \\ f = 10^{14} - 10^{15} \text{ Hz} \end{array} \parallel \begin{array}{l} \text{photon energy} \\ \approx 2 \text{ eV} \end{array}$$

Radio

$$\begin{array}{l} \lambda \approx 3 \text{ m} \\ f \approx 150 \text{ MHz} \end{array} \parallel 10^{-6} \text{ eV}$$

Waves in Three Dimensions

So far, you have considered mainly waves in one dimension (e.g. on strings or transmission lines) governed by equations of motion of the form

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

This is an idealised wave equation, because it is scalar, loss-less and non-dispersive (the phase velocity, c , is not a function of frequency).

Clearly waves may also propagate in two dimensions (on sheets or surfaces) and most generally in three dimensions (electro-magnetic waves, for example).

We might guess that a three dimensional wave equation (loss-less, dispersion-less) would be of the form

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

or, more generally

$$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$$

where ∇^2 can be expressed in a number of coordinate systems (e.g. cylindrical, spherical). Two particular, and very simple, solutions of the three dimensional equation may be the plane wave, where surfaces of constant phase are parallel planes, i.e.

$$f = A \exp j(\omega t - [k_x x + k_y y + k_z z])$$

or

$$f = A \exp j(\omega t - (k_x x + k_y y + k_z z))$$

or the spherical wave, emanating from a point source, in which surfaces of constant phase are spheres, i.e.

$$f = A \exp j(\omega t - k(x^2 + y^2 + z^2)^{1/2}).$$

Some of the problems we shall consider later involve plane waves meeting an interface between two uniform, semi-infinite but different media - from this we will derive laws of reflection, refraction, and a description of polarization changes at reflection. Also we shall study waves propagating in partially restricted spaces, i.e. waveguides, such as hollow metal pipes and glass fibres.

In all these problems, boundary conditions at discontinuities are the key to analysis and understanding.

You have already met boundary conditions in connection with standing waves, for example. For a string, length l , fixed at its ends, we must have

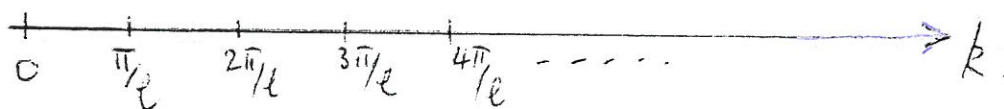
$$y = \sum A_n \sin k_n x \sin(\omega_n t + \phi_n),$$

with

$$k_n = \frac{n\pi}{l},$$

i.e. the wavenumbers can have only certain values, which are separated by a fixed amount π/l , i.e.

(Allowed k)



An equivalent analysis for, say, an electron in a one-dimensional potential well also gives

$$k_n = \frac{n\pi}{l}$$

but since

$$p = \frac{h}{\lambda} = \left(\frac{h}{2\pi}\right) k$$

and

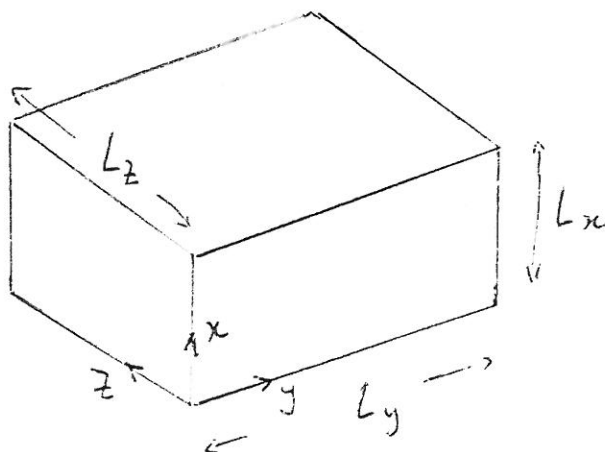
$$E = p^2/2m \quad (\text{kinetic energy})$$

we can say that the electron energy is quantised with values

$$E_n = \frac{h^2}{8me^2} n^2$$

This is a somewhat surprising result because it suggests that the interval between energy states increases with n , whereas we might expect the opposite, going over eventually to a classical continuum.

We can have a simple introduction to boundary conditions, as applied to three dimensional wave systems, by considering waves in a rectangular box, where we suppose that the wave amplitude must be zero at all points on the surface of the box.



Wave equation (for, say, E-M waves)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = -\frac{\omega^2}{c^2} f$$

For rectangular coordinates we may expect the solution to be separable, i.e.

$$\text{put } f = X(x) Y(y) Z(z)$$

hence

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\frac{\omega^2}{c^2}$$

Putting $\frac{d^2 X}{dx^2} = -k_x^2$ etc,

gives $\frac{d^2 X}{dx^2} + k_x^2 X = 0$

$X = X_0 \sin k_x x$ ($X=0$ at $x=0$)

and $k_x = \frac{n_x \pi}{L_x}$ ($X=0$ at $x=L_x$)

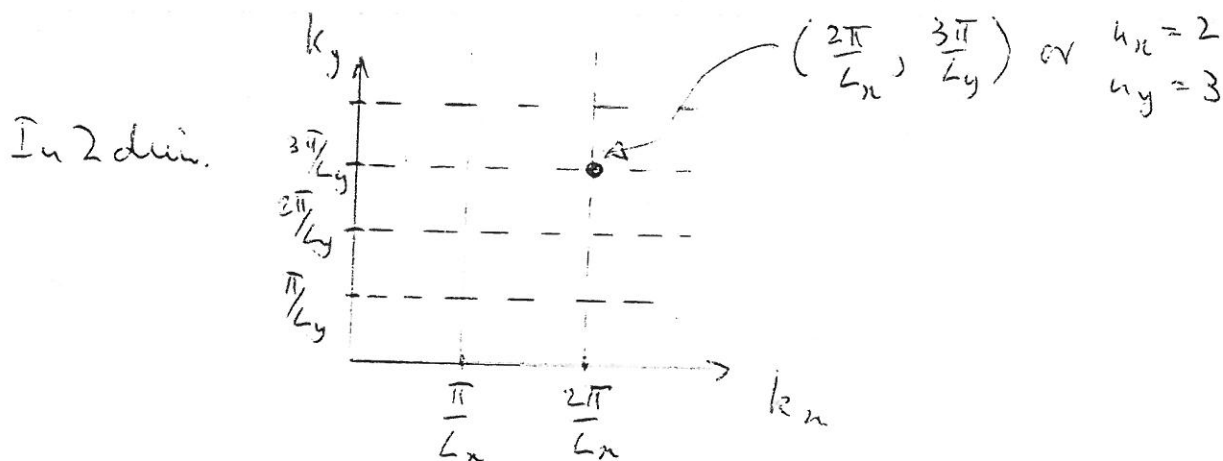
Similarly $k_y = \frac{n_y \pi}{L_y}$

and $k_z = \frac{n_z \pi}{L_z}$

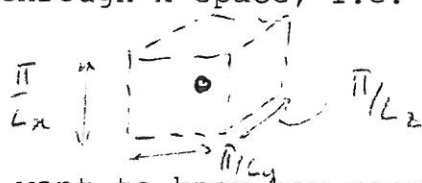
Also $k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$ ($= k^2$)

We can interpret k as a wavenumber, and k_x, k_y, k_z as the components of k in the x, y, z directions.

The modes are now three dimensional shapes and we need three mode numbers n_x, n_y, n_z to describe each mode. We can represent each of these modes as points in k -space



In three dimensions, each solution surrounded by empty volume of $\pi^3 / L_x L_y L_z$ - they are distributed regularly and uniformly through k-space, i.e.



If we want to know how many modes exist with $k \leq k_0$, then we want the number of solutions in the octant, radius k_0 .

$$\text{Volume of octant} = \frac{1}{8} \cdot \frac{4}{3} \pi k_0^3$$

$$\therefore \text{No. of solutions} = \frac{\frac{1}{8} \cdot \frac{4}{3} \pi k_0^3}{\pi^3 / L_x L_y L_z}$$

Alternatively, the number of solutions, dn , with k value between k and $k+dk$ is

$$dn = \frac{\frac{1}{8} \cdot \frac{4}{3} \pi [(k+dk)^3 - k^3]}{\pi^3 / L_x L_y L_z} = \frac{k^2 dk}{2\pi^2} \cdot L_x L_y L_z$$

because dk small, neglect dk^2, dk^3

This is essentially a simple density of states function, which appears in very many contexts. Let us consider two forms.

(i) Electro-magnetic waves

We can readily determine the distribution of electro-magnetic energy as a function of frequency in what amounts to a black body cavity.

First, we can transform from k to frequency f :

$$f\lambda = c; \quad f = c/\lambda = \frac{c}{2\pi} k$$

and

$$df = \frac{c}{2\pi} dk$$

Hence

$$dn = \frac{4\pi f^2}{c^3} df \quad (\text{per unit volume})$$

Each state (if filled) would constitute an energy of $E = hf$. Therefore if there are dn states in df , this would constitute an energy dE of

$$dE = \frac{4\pi h f^3}{c^3} df$$

But each state is not necessarily filled. The probability of a state of energy E being filled is given by the Bose-Einstein probability function,

$$P(E) = \frac{1}{\left(e^{hf/kT} - 1\right)}$$

Further, each state can be filled independently twice over, in effect, because we can have two possible polarisations at right angles, hence the overall energy density result is

$$dE = \frac{8\pi h f^3}{c^3} \frac{df}{\left(e^{hf/kT} - 1\right)} \quad (\text{Planck's Black Body Radiation Law})$$

(ii) Electron waves

We still have $dn = \frac{k^2}{2\pi^2} dk$, (per unit volume) and from this we can work out the number of electrons in an energy interval dE

$$p = h/\lambda = \left(\frac{h}{2\pi}\right) k$$

and

$$p^2/2m = E$$

$$\therefore \left(\frac{h}{2\pi}\right)^2 \frac{1}{2m} k^2 = E$$

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and

$$k = \left(\frac{2\pi}{h}\right) (2m)^{1/2} E^{1/2}$$

$$dk = \left(\frac{2\pi}{h}\right) (2m)^{1/2} \frac{1}{2} E^{-1/2} dE$$

$$\therefore du = \frac{1}{2\pi^2} \left(\frac{2\pi}{h}\right)^2 2m E \left(\frac{2\pi}{h}\right) (2m)^{1/2} \frac{1}{2} E^{-1/2} dE$$

$$du = \frac{1}{2} \left\{ \frac{8\pi\sqrt{2}}{h^3} E^{1/2} \right\} dE$$

Note that now, in three dimensions, the number of electron states in dE increases with E as we would expect. To complete the story we should multiply this density of states function by the Fermi-Dirac probability function

$$P(E) = \frac{1}{1 + \exp \frac{E - E_F}{kT}}$$

and noting that we can have two electrons in each state with opposite spin, gives

$$dN = \frac{8\pi\sqrt{2}}{h^3} E^{1/2} \frac{dE}{1 + \exp \frac{E - E_F}{kT}}$$

the energy distribution of electrons in a three dimensional box.