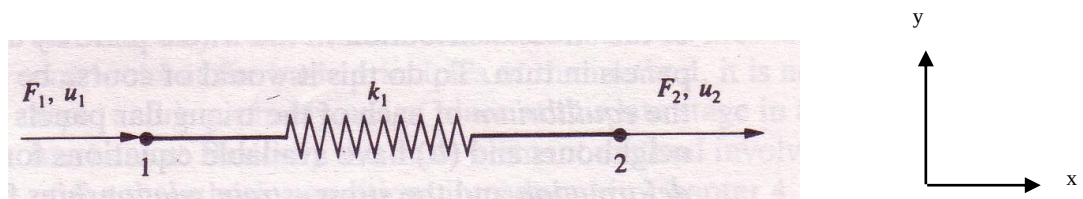


Simple 1D Examples – Analysis of spring elements

To show what sort of calculations the computer does we will consider a very basic problem as follows;



When a tie or strut is part of a structure, its ends will be able to move due to displacement of the structure and deformation of the member. It may thus be modelled as above. So this model has one element (the spring) and two nodes (one at either end). If the strut has a length L and an area A , then the stiffness is given by;

$$k_1 = \frac{AE}{L}$$

Where A is the area of the rod, E is the young's modulus and L is the length. This single element can then be represented in matrix form in the following way;

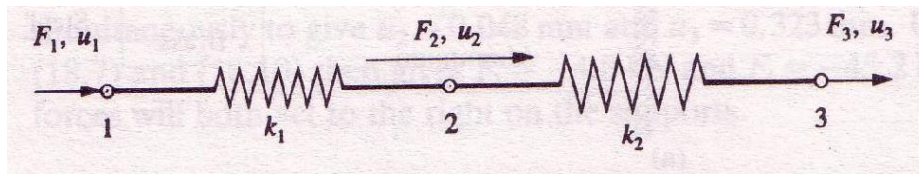
$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

This can be written in shorthand as

$$\{F\} = [K^e] \{u\}$$

Where $[K^e]$ is referred to as the **stiffness matrix** and notice, as is always the case, that the stiffness matrix is symmetrical.

If we were to extend our example to consider two elements and three nodes we would see the following;



In the same way we could represent the two springs with the following matrices;

$$\text{Spring 1} \quad \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\text{Spring 2} \quad \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

If these are expanded so that they are in an equivalent form they could be written as;

$$\text{Spring 1} \quad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\text{Spring 2} \quad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

As these springs are in series we can add these matrices to give

$$\text{Both Springs} \quad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Once again this could be written in shorthand as

$$\{F\} = [K^e] \{u\}$$

Where $[K^e]$ is our stiffness matrix of our complete structure and is again symmetrical.

If the diagonal terms in the element stiffness matrix $[K^e]$ are referred to as direct stiffness and the off-diagonal terms in $[K^e]$ are indirect stiffness the rules for the formation of the matrix can be summarised as follows;

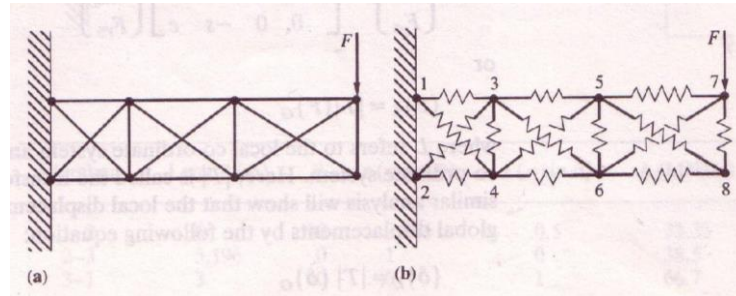
- The term in location ii consists of the sum of the direct stiffnesses of all the elements meeting at node i .
- The term in location ij consists of the sum of the indirect stiffness relating to nodes i and j of all the elements joining node i to node j .

This approach to a solution is the basis of the finite element method.

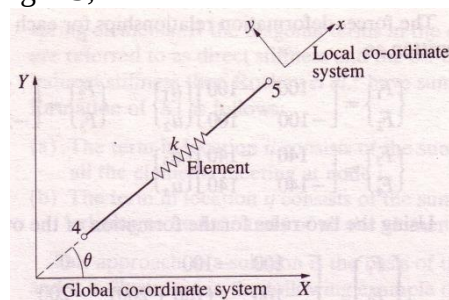
Simple 2D Examples – Analysis of Pin jointed frameworks

So far we have considered co-linear springs. In order to consider a more complex situation such as shown in (a), below, we need to express the forces and deformations for each element (calculated for it's own local co-ordinate system x,y) to the deformation of the whole framework relative to the global co-ordinate system X,Y.

Eg



If we consider spring 4-5;



So far we have seen for a co-linear spring in its LOCAL co-ordinate system that ;

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

In a pin jointed structure the local y forces for each element will be zero, hence we can expand this to;

$$\begin{Bmatrix} F_{x4} \\ F_{y4} \\ F_{x5} \\ F_{y5} \end{Bmatrix} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{x4} \\ u_{y4} \\ u_{x5} \\ u_{y5} \end{Bmatrix}$$

To relate the local forces to the global forces the following relationships are true;

$$F_{x4} = F_{X4} \cos\theta + F_{Y4} \sin\theta$$

$$F_{y4} = -F_{X4} \sin\theta + F_{Y4} \cos\theta$$

$$F_{x5} = F_{X5} \cos\theta + F_{Y5} \sin\theta$$

$$F_{y5} = -F_{X5} \sin\theta + F_{Y5} \cos\theta$$

This can be represented in matrix form as follows, using $c = \cos\theta$ and $s = \sin\theta$

$$\begin{Bmatrix} F_{x4} \\ F_{y4} \\ F_{x5} \\ F_{y5} \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & -c \end{bmatrix} \begin{Bmatrix} F_{X4} \\ F_{Y4} \\ F_{X5} \\ F_{Y5} \end{Bmatrix}$$

Once again this could be written in shorthand as

$$\{F\}_L = [T] \{F\}_G$$

Where $\{F\}_L$ are the forces represented in the local co-ordinate system, $\{F\}_G$ are the forces represented in the global co-ordinate system and $[T]$ is the **transformation** matrix.

It can be shown similarly that the local displacements are related to the global displacements as follows;

$$\{u\}_L = [T] \{u\}_G$$

Since $\{F\}_L = [T] \{F\}_G$ and $\{F\}_L = [K^e]_L \{u\}_L$

Then our global forces multiplied by the transformation matrix may be equated to our local stiffness matrix, multiplied by our local displacements

$$[K^e]_L \{u\}_L = [T] \{F\}_G$$

Replacing our local displacements with global displacements;

$$\therefore [K^e]_L [T] \{u\}_G = [T] \{F\}_G$$

$$\therefore \{F\}_G = [T]^{-1} [K^e]_L [T] \{u\}_G$$

Or

$$\{F\}_G = [K^e]_G \{u\}_G$$

Where $[K^e]_G$ is the stiffness matrix for the element in global co-ordinates, i.e.

$$[K^e]_G = [T]^{-1} [K^e]_L [T]$$

Or since that $[T]$ is orthogonal;

$$[K^e]_G = [T]^T [K^e]_L [T]$$

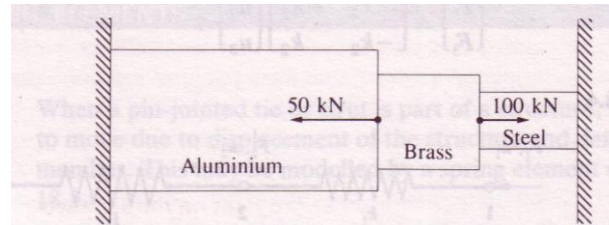
It can thus be shown that;

$$[K^e]_G = k \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Where $k = \frac{AE}{L}$

Examples

- 1) Three dissimilar materials are friction welded together and placed between rigid end supports as shown below. If forces of 50kN and 100kN are applied as indicated calculate the movement of the interfaces between the material and the force exerted on the end supports.



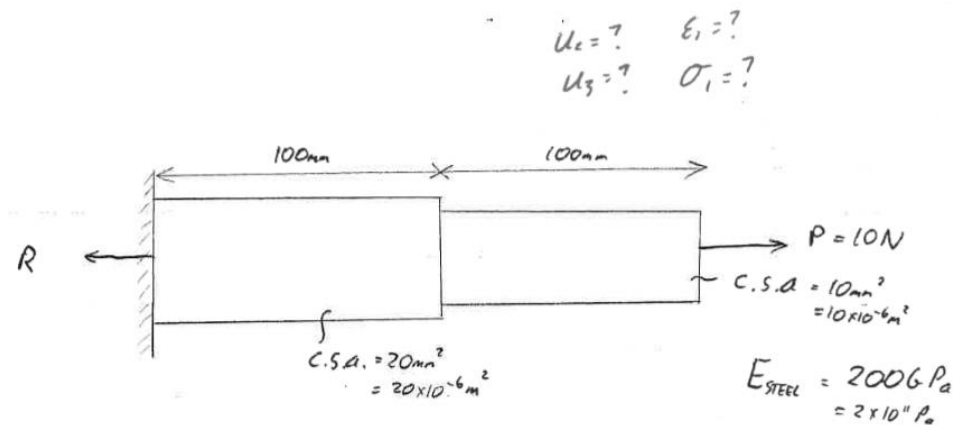
For aluminium	For brass	For steel
Area = 400 mm ²	Area = 200 mm ²	Area = 70 mm ²
Length = 280 mm	Length = 100 mm	Length = 100 mm
E = 70 GN/m ²	E = 100 GN/m ²	E = 200 GN/m ²

[0.323mm, 0.049mm, -4.9kN, -45.22kN]

- 2) Repeat the previous question but make the 50kN load 100kN and make the 100kN load 50kN. What are the new displacements and the reaction forces?

[-0.387mm, -0.081mm, 38.7kN, 11.34kN]

3)

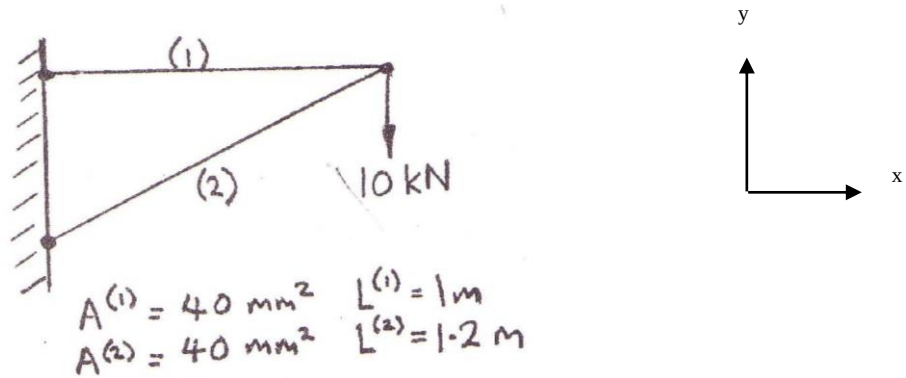


In the above example find the displacement at the end and the interface of the two sections and the stress and the strain in the 200mm² section.

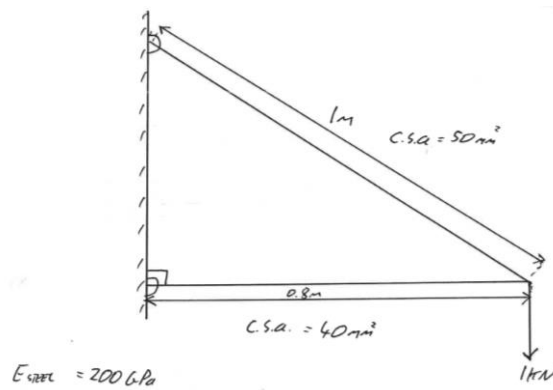
[2.5×10^{-7} m, 7.5×10^{-7} m, 2.5×10^{-6} , 0.5MPa]

- 4) Use the finite element method to find the global displacements at the loading point below and the axial forces in the struts. Assume both members are steel of Young's Modulus 200GPa.

$$[U_{2X}=1.88\text{mm}, U_{2Y}=-7.73\text{mm}, F_1=15.04\text{kN}, F_2=-18.07\text{kN}]$$



5)



In the above example, find the stress and the strain in the member that is in tension.

$$[167 \times 10^{-6}, 33.4 \text{ MPa}]$$