

LOCATION: SWANSEA

LLEOLIAD: ABERTAWE

LEVEL 5

LEFEL 5

EXAMINATION PERIOD

MAY 2025

CYFNOD ARHOLIADAU

MAI 2025

SOLUTION

MODULE CODE: GOEE5001

ELECTRICAL PRINCIPLES THEORY

TIME: 2 HOURS

ANSWER ALL QUESTIONS

SHOW ALL CALCULATIONS AND WORKINGS CLEARLY

Equipment or materials required – Calculator, Formula Sheet

Question One (30 marks)

- a. A series RLC circuit has the following total impedance in the Laplace domain:

$$Z(s) = 2s + \frac{1000}{s} + 6$$

Determine the values of the resistor R, inductor L and the capacitor C in the circuit.

[5 marks]

Solution

The general form of the impedance for a series RLC circuit in the Laplace domain is:

$$Z(s) = R + sL + \frac{1}{sC}$$

The expression $2s + \frac{1000}{s} + 6$ can be separated into three distinct terms:

1. The term $2s$ is the sL term (inductive term).
2. The term $\frac{1000}{s}$ corresponds to the capacitive term $\frac{1}{sC}$.
3. The constant term 6 corresponds to the resistor R.

$$\therefore L = 2H$$

$$\frac{1}{C} = 1000$$

$$\therefore C = \frac{1}{1000} F$$

$$R = 6 \Omega$$

Thus, the component values in the series RLC circuit are: $R = 6, L = 2H, C = \frac{1}{1000} F$

- b. Compute the Laplace transform of the given functions:

[10 marks]

i. $3e^{3t} \cos(2t) + t^2 + 0.75$

ii. $\sin(4t) e^t + 5e^{-3t}$

iii. $e^{2t}(3 - t^2) + \frac{3\pi}{2}$

iv. $t \cosh(2t) + e^{-2t}$

v. $e^2 e^{-4t} t^3 - 3e^t \sin(e^2 t)$

Solution

i. $L\{3e^{3t} \cos(2t) + t^2 + 0.75\} = 3 \cdot \frac{s-3}{(s-3)^2+4} + \frac{2}{s^3} + \frac{0.75}{s}$

ii. $L\{\sin(4t) e^t + 5e^{-3t}\} = \frac{4}{(s-1)^2+(4)^2} + \frac{5}{s+3}$

iii. $L\left\{e^{2t}(3 - t^2) + \frac{3\pi}{2}\right\} = L\left\{3e^{2t} - t^2 e^{2t} + \frac{3\pi}{2}\right\}$
 $= \frac{3}{s-2} - \frac{2}{(s-2)^3} + \frac{3\pi}{2s}$

iv. $L\{t \cosh(2t) + e^{-2t}\} = \frac{2s}{(s^2+4)^2} + \frac{1}{s+2}$

v. $L\{e^2 e^{-4t} t^3 - 3e^t \sin(e^2 t)\} = e^2 \frac{3!}{(s+4)^4} - 3 \cdot \frac{e^2}{(s-1)^2+(e^2)^2}$
 $= \frac{6e^2}{(s+4)^4} - \frac{3e^2}{(s-1)^2+e^4}$

[10 marks]

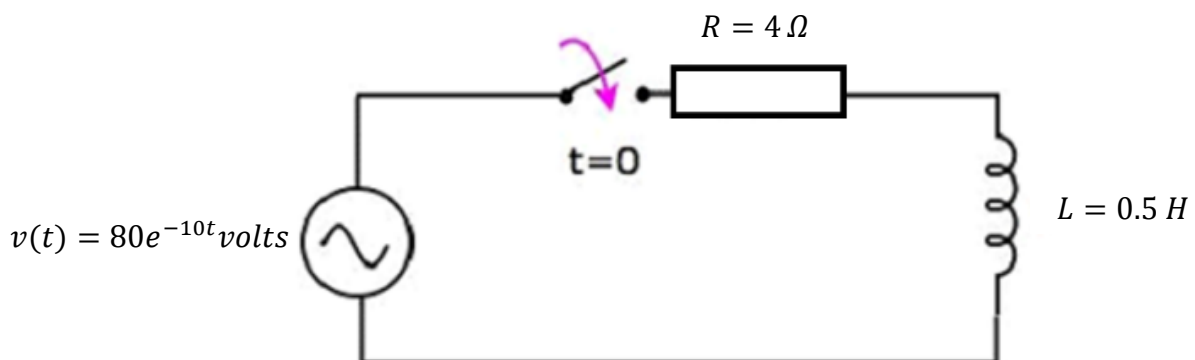
- c. A series RL circuit, as shown in Figure 1, is connected to an exponential voltage source $v(t) = 80e^{-10t}$ volts for $t \geq 0$. The circuit contains a resistance $R = 4 \Omega$ and an inductance $L = 0.5H$.

The switch is closed at $t = 0$. The circuit is described by the following differential equation:

$$L \frac{di}{dt} + Ri = v(t)$$

Using the Laplace transform method, determine the current $i(t)$ for $t \geq 0$, assuming $i(0) = 0$.

[15 marks]



Solution

$$L \frac{di}{dt} + Ri = v$$

$$0.5 \frac{di}{dt} + 4i = 80e^{-10t} \quad [1]$$

$$0.5 \mathcal{L}\left[\frac{di}{dt}\right] + 4 \mathcal{L}[i] = 80 \mathcal{L}[e^{-10t}]$$

$$0.5 sI(s) - i(0) + 4I(s) = \frac{80}{s+10} \quad [3]$$

$$0.5 sI(s) + 4I(s) = \frac{80}{s+10}$$

$$I(s)\{0.5 s + 4\} = \frac{80}{s+10} \quad [2]$$

$$I(s) = \frac{80}{(0.5s+4)(s+10)} \quad [2]$$

$$I(s) = \frac{160}{(s+8)(s+10)}$$

$$\frac{160}{(s+8)(s+10)} = \frac{A1}{(s+8)} + \frac{A2}{(s+10)} \quad [2]$$

$$160 = A1(s+10) + A2(s+8)$$

$$s = -10$$

$$160 = A1(-10+10) + A2(-2) \dots \dots \dots A2 = -80 \quad [1]$$

$$s = -8$$

$$160 = A1(-8+10) + A2(-8+8) \dots \dots \dots A1 = 80 \quad [1]$$

$$I(s) = \frac{160}{(s+8)(s+10)} = \frac{80}{(s+8)} + \frac{-80}{(s+10)} \quad [1]$$

$$I(s) = \frac{80}{(s+8)} - \frac{80}{(s+10)}$$

$$i(t) = 80e^{-8t} - 80e^{-10t} \quad [2]$$

Question Two (30 marks)

a. Determine the fundamental frequency of the following signals:

i. $x(t) = 2 + 3 \cos(0.2t) + \cos(0.25t + \frac{\pi}{2}) + 4 \cos(0.3t - \pi)$

[3 marks]

ii. $x(t) = 1 + 10 \cos\left(2\pi(60)t + \frac{\pi}{8}\right) + 2 \cos(2\pi(300)t + \frac{\pi}{4})$

[2 marks]

Solutions

i. $x(t) = 2 + 3 \cos(0.2t) + \cos(0.25t + \frac{\pi}{2}) + 4 \cos(0.3t - \pi)$

Frequencies are:

$$\omega_1 = 0.2 \text{ rad/sec}$$

$$\omega_2 = 0.25 \text{ rad/sec}$$

$$\omega_3 = 0.3 \text{ rad/sec}$$

The greatest common divisor is 0.05 rad/sec

$$\therefore \text{fundamental frequency } \omega_0 = 0.05 \text{ rad/sec}$$

ii.

$$x(t) = 1 + 10 \cos\left(2\pi(60)t + \frac{\pi}{8}\right) + 2 \cos(2\pi(300)t + \frac{\pi}{4})$$

$$f_1 = 60 \text{ Hz}$$

$$f_2 = 300 \text{ Hz}$$

$$\therefore \text{fundamental frequency } f_0 = 60 \text{ Hz}$$

b. Figure 2 below shows the waveform graph of a periodic square wave $f(x)$, defined by the following piecewise function:

$$f(x) = \begin{cases} 1 & \text{if } -\pi \leq x < 0 \\ 0 & \text{if } 0 \leq x < \pi \end{cases}$$

And the function is periodic with $f(x + 2\pi) = f(x)$.

Write down the real Fourier series expression for $f(x)$, explicitly including the first five non-zero terms.

[25 marks]

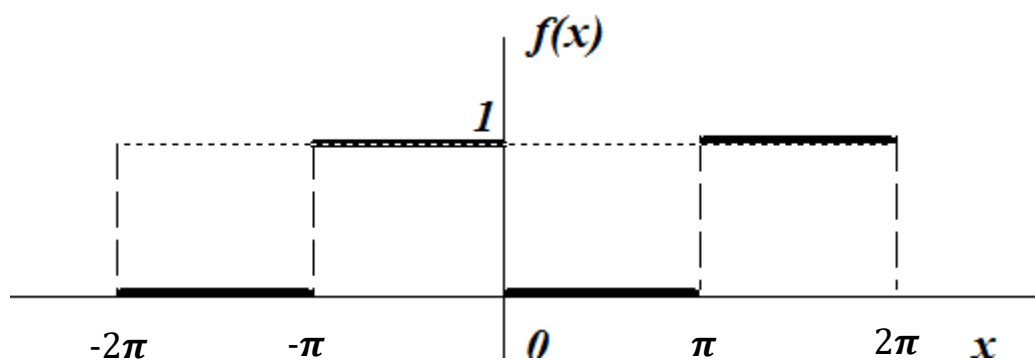


Figure 2

Solution

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$a_o = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_o = \frac{1}{\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{\pi} \int_0^{\pi} f(x) dx$$

$$a_o = \frac{1}{\pi} \int_{-\pi}^0 1 dx + \frac{1}{\pi} \int_0^{\pi} 0 dx$$

$$a_o = \frac{1}{\pi} [x]_{-\pi}^0 + 0$$

$$a_o = \frac{1}{\pi} (0 - (-\pi))$$

$$a_o = \frac{1}{\pi} (0 - (-\pi))$$

$$a_o = \frac{1}{\pi} (\pi)$$

$$\therefore a_o = 1$$

[7 marks]

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^0 1 \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} 0 \cos nx \, dx \\
&= \frac{1}{\pi} \left(\left[\frac{\sin nx}{n} \right]_{-\pi}^0 \right) \\
&= \frac{1}{n\pi} ([\sin nx]_{-\pi}^0) \\
&= \frac{1}{n\pi} [\sin 0 - \sin(-n\pi)] \\
&= \frac{1}{n\pi} [0 + \sin(n\pi)] \\
&= \frac{1}{n\pi} [0 + 0] \\
\mathbf{a_n} &= \mathbf{0}
\end{aligned}$$

[7 marks]

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Integration will be performed in two parts from 0 to π and from π to 2π

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^0 f(x) \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\
&= \frac{1}{\pi} \int_{-\pi}^0 1 \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} 0 \sin nx \, dx \\
&= \frac{1}{\pi} \left[\left(-\frac{\cos nx}{n} \right) \right]_{-\pi}^0 \\
&= \frac{-1}{n\pi} [(\cos nx)]_{-\pi}^0 \\
&= -\frac{1}{n\pi} (1 - \cos(-n\pi)) \\
\mathbf{b_n} &= \mathbf{\frac{-1}{n\pi} (1 - (-1)^n)}
\end{aligned}$$

$$\mathbf{b_n = \frac{-1}{n\pi} (1 - (-1)) = \frac{-2}{n\pi} \quad for \, n \, odd}$$

$$\mathbf{b_n = \frac{-1}{n\pi} (1 - (1)) = 0 \quad for \, n \, even}$$

[7 marks]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$a_o = 1 \quad a_n = 0 \quad b_n = -\frac{1}{n\pi} (1 - (-1)^n)$$

The Fourier series up to the first five non-zero terms is:

$$f(x) = \left(\frac{1}{2}\right) - \frac{2}{\pi} \left(\sin(x) + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) + \frac{1}{7} \sin(7x) + \frac{1}{9} \sin(9x) \dots \right)$$

[4 marks]

Question Three (20 marks)

For the circuit shown in Figure 3, determine:

- a. The Thevenin equivalent circuit with respect to terminals A and B.

[10 marks]

- b. The load impedance Z_L that maximizes the power drawn.

[3 marks]

- c. The maximum average power.

[7 marks]

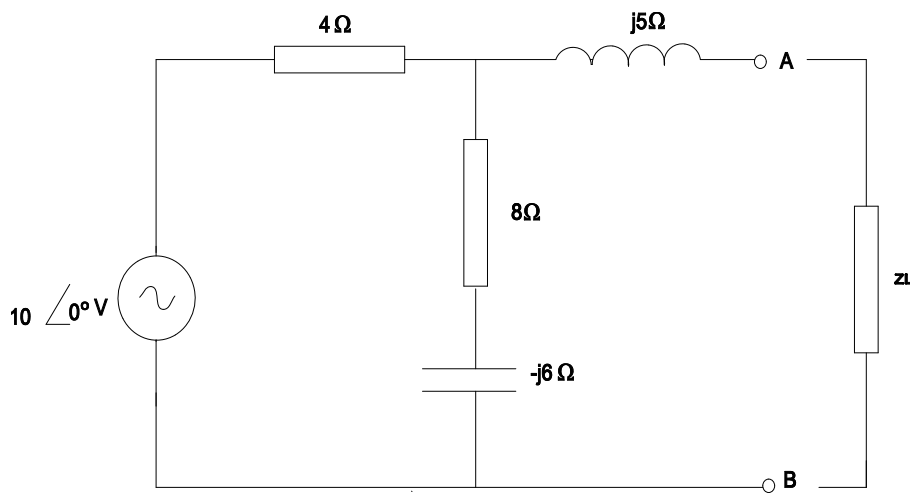


Figure 3

Solution

- a. Thevenin equivalent circuit with respect to terminals AB.

$$V_{Th} = \left(\frac{8 - j6}{(8 - j6) + 4} \right) 10 \angle 0^\circ$$

$$V_{Th} = \left(\frac{8 - j6}{12 - j6} \right) 10 \angle 0^\circ$$

$$V_{Th} = \left(\frac{10 \angle -36.9^\circ}{13.42 \angle -26.6^\circ} \right) 10 \angle 0^\circ$$

$$V_{Th} = (0.74 \angle -10.3^\circ) 10 \angle 0^\circ$$

$$V_{Th} = 7.4 \angle -10.3^\circ \text{ V} \quad [5]$$

$$Z_{th} = \frac{4(8-j6)}{4+8-j6} + j5$$

$$Z_{th} = \frac{32-j24}{12-j6} + j5$$

$$Z_{th} = \frac{40 \angle -36.9^\circ}{13.4 \angle -26.6^\circ} + j5$$

$$Z_{th} = (3 \angle -10.3^\circ) + j5$$

$$Z_{th} = 2.95 - j0.54 + j5$$

$$Z_{th} = 2.95 + j4.46 \text{ } \Omega \text{ or } Z_{th} = 5.35 \angle 56.5^\circ \text{ } \Omega \quad [5]$$

b. The load impedance Z_L that maximize the power drawn.

Z_L for maximum power

$$Z_L = Z_{th}^* = 2.95 - j4.46 \text{ } \Omega \text{ or } 5.34 \angle -56.6^\circ \quad [3]$$

c. The maximum average power.

$$I = \frac{7.4 \angle -10.3^\circ}{2.95 + j4.46 + 2.95 - j4.46}$$

$$I = \frac{7.4 \angle -10.3^\circ}{5.9} = \frac{7.4 \angle -10.3^\circ}{5.9 \angle 0^\circ} = 1.25 \angle -10.3^\circ \text{ A}$$

$$I = 1.25 \angle -10.3^\circ \text{ A} \quad [4]$$

$$P_{avg \max} = I^2 Z_L$$

$$P_{avg \max} = \frac{1}{2} (1.25)^2 (2.95)$$

$$P_{avg \max} = 2.304 \text{ W} \quad [3]$$

Question Four (20 marks)

- a. In the UK, three-phase systems are often configured as either star (Y) or delta (Δ). When dealing with motor loads, it is common for the motor to start in star configuration and then be switched to delta configuration after reaching a specific speed to provide higher torque.

Consider the star-connected load in Figure 4 below with the following impedance:

$$Z_A = 10 + j6 \, \Omega$$

$$Z_B = 8 + j4 \, \Omega$$

$$Z_C = 12 + j8 \, \Omega$$

Provide the equivalent delta network impedance in both Polar and Cartesian (Rectangular) forms.

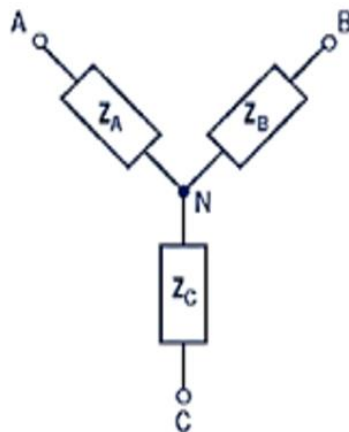
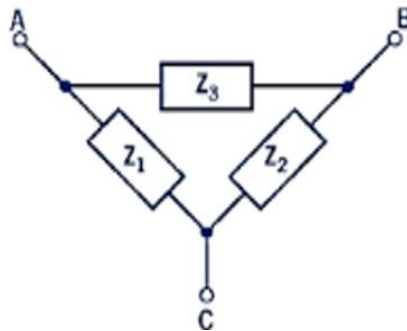


Figure 4

[10 marks]

Solution



$$Z_1 = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_B}$$

$$Z_1 = \frac{(10 + j6)(8 + j4) + (8 + j4)(12 + j8) + (12 + j8)(10 + j6)}{(8 + j4)}$$

$$Z_1 = \frac{(80 + j40 + j48 - 24) + (96 + j64 + j48 - 32) + (120 + j72 + j80 - 48)}{(8 + j4)}$$

$$Z_1 = \frac{(56 + j88) + (64 + j112) + (72 + j152)}{(8 + j4)}$$

$$Z_1 = \frac{192 + j352}{8 + j4}$$

$$Z_1 = \frac{400.96 \angle 61.39^\circ}{8.944 \angle 26.57^\circ}$$

$$Z_1 = 44.83 \angle 34.82^\circ$$

$$Z_1 = 36.80 + j25.60 \, \Omega$$

[5]

$$Z_2 = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_A}$$

$$Z_2 = \frac{(10 + j6)(8 + j4) + (8 + j4)(12 + j8) + (12 + j8)(10 + j6)}{(10 + j6)}$$

$$Z_2 = \frac{(80 + j40 + j48 - 24) + (96 + j64 + j48 - 32) + (120 + j72 + j80 - 48)}{(10 + j6)}$$

$$Z_2 = \frac{(56 + j88) + (64 + j112) + (72 + j152)}{(10 + j6)}$$

$$Z_2 = \frac{192 + j352}{(10 + j6)}$$

$$Z_2 = \frac{400.96 \angle 61.39^\circ}{11.66 \angle 30.96^\circ}$$

$$Z_2 = 34.39 \angle 1.98^\circ$$

$$Z_2 = 34.37 + j1.19 \, \Omega$$

[3]

$$Z_3 = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_c}$$

$$Z_3 = \frac{(10 + j6)(8 + j4) + (8 + j4)(12 + j8) + (12 + j8)(10 + j6)}{(12 + j8)}$$

$$Z_3 = \frac{(80 + j40 + j48 - 24) + (96 + j64 + j48 - 32) + (120 + j72 + j80 - 48)}{(12 + j8)}$$

$$Z_3 = \frac{(56 + j88) + (64 + j112) + (72 + j152)}{(12 + j8)}$$

$$Z_3 = \frac{192 + j352}{(12 + j8)}$$

$$Z_3 = \frac{400.96 \angle 61.39^\circ}{14.42 \angle 33.69^\circ}$$

$$Z_3 = 38.48 \angle 1.82^\circ$$

$$Z_3 = 38.46 + j1.22 \, \Omega$$

[2]

b. A periodic voltage source is represented by the waveform:

$$v(t) = 300 \cos(2\pi 120t) + 100 \cos(4\pi 120t) \, V,$$

and is applied across a circuit. The resulting current through the circuit is given by the Fourier series:

$$i(t) = 15 \cos(2\pi 120t - 30^\circ) + 5 \cos(6\pi 120t + 60^\circ) \, A$$

Determine:

i. The rms values of both the voltage and the current.

[4 marks]

ii. The real (average) power delivered to the circuit and the power factor of the circuit.

[6 marks]

Solution

i.

$$v(t) = 300 \cos(2\pi 120t) + 100 \cos(4\pi 120t) \text{ V}$$

$$i(t) = 15 \cos(2\pi 120t - 30^\circ) + 5 \cos(6\pi 120t + 60^\circ) \text{ A}$$

The rms value of the voltage.

$$V_{rms} = \sqrt{V_0^2 + \sum_{n=1}^{\infty} \left(\frac{V_n^2}{2}\right)} = \sqrt{0 + \frac{300^2}{2} + \frac{100^2}{2}} = \mathbf{223.607 \text{ V}} \quad [2]$$

The rms value of the current,

$$I_{rms} = \sqrt{I_0^2 + \sum_{n=1}^{\infty} \left(\frac{I_n^2}{2}\right)} = \sqrt{0 + \frac{15^2 + 5^2}{2}} = \mathbf{11.180 \text{ A}} \quad [2]$$

ii.

The power absorbed by the load.

$$P_{av} = V_0 I_0 + \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \cos \phi_n - \theta_n$$
$$P_{av} = 0 + \frac{(300)(15)}{2} \cos(30^\circ) + 0 = \mathbf{1948.557 \text{ W}} \quad [3]$$

The power factor of the load.

$$S = V_{rms} \times I_{rms} = 223.607 \times 11.180 = 2499.926 \text{ VA}$$

$$pf = \frac{P}{S} = \frac{1948.557}{2499.926}$$

$$\mathbf{pf = 0.779.} \quad [3]$$

End