

DIFFERENTIATION PRACTICE

- 1 a** $y = \ln x^2 = 2 \ln x$
(using properties of logs)

$$\therefore \frac{dy}{dx} = 2 \times \frac{1}{x} = \frac{2}{x}$$

Alternative method:

$$\text{When } y = \ln f(x), \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

(by the chain rule)

$$\therefore y = \ln x^2 \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2} = \frac{2}{x}$$

- b** $y = x^2 \sin 3x$

Using the product rule,

$$\begin{aligned} \frac{dy}{dx} &= x^2(3 \cos 3x) + (\sin 3x) \times 2x \\ &= 3x^2 \cos 3x + 2x \sin 3x \end{aligned}$$

- 2 a** $2y = x - \sin x \cos x$

$$\therefore y = \frac{x}{2} - \frac{1}{2} \sin x \cos x$$

Using the product rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} - \frac{1}{2}(\sin x(-\sin x) + \cos x \cos x) \\ &= \frac{1}{2} + \frac{1}{2} \sin^2 x - \frac{1}{2} \cos^2 x \\ &= \frac{1}{2}(1 - \cos^2 x) + \frac{1}{2} \sin^2 x \\ &= \frac{1}{2} \sin^2 x + \frac{1}{2} \sin^2 x \\ &= \sin^2 x \end{aligned}$$

- b** $y = \frac{x}{2} - \frac{1}{2} \sin x \cos x$

$$\frac{dy}{dx} = \sin^2 x$$

$$\frac{d^2y}{dx^2} = 2 \sin x \cos x = \sin 2x$$

$$\text{At points of inflection } \frac{d^2y}{dx^2} = 0$$

$$\text{i.e. } \sin 2x = 0$$

$$2x = \pi, 2\pi \text{ or } 3\pi$$

$$x = \frac{\pi}{2}, \pi \text{ or } \frac{3\pi}{2}$$

$$\text{When } x = \frac{\pi}{2}, y = \frac{\pi}{4}$$

$$\text{At } x = \frac{\pi}{3}, \frac{d^2y}{dx^2} > 0; \text{ at } x = \frac{3\pi}{4}, \frac{d^2y}{dx^2} < 0$$

$$\text{So } \frac{d^2y}{dx^2} \text{ changes sign either side of } x = \frac{\pi}{2}$$

$$\text{When } x = \pi, y = \frac{\pi}{2}$$

$$\text{At } x = \frac{3\pi}{4}, \frac{d^2y}{dx^2} < 0; \text{ at } x = \frac{5\pi}{4}, \frac{d^2y}{dx^2} > 0$$

$$\text{So } \frac{d^2y}{dx^2} \text{ changes sign either side of } x = \pi$$

$$\text{When } x = \frac{3\pi}{2}, y = \frac{3\pi}{4}$$

$$\text{At } x = \frac{5\pi}{4}, \frac{d^2y}{dx^2} > 0; \text{ at } x = \frac{7\pi}{4}, \frac{d^2y}{dx^2} < 0$$

$$\text{So } \frac{d^2y}{dx^2} \text{ changes sign either side of } x = \frac{3\pi}{2}$$

Hence the points of inflection are

$$\left(\frac{\pi}{2}, \frac{\pi}{4}\right), \left(\pi, \frac{\pi}{2}\right) \text{ and } \left(\frac{3\pi}{2}, \frac{3\pi}{4}\right)$$

3 a $y = \frac{\sin x}{x}$

Using the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{x \cos x - \sin x \times 1}{x^2} \\ &= \frac{x \cos x - \sin x}{x^2}\end{aligned}$$

b $y = \ln \frac{1}{x^2 + 9} = \ln 1 - \ln(x^2 + 9)$
 $= -\ln(x^2 + 9)$

(by the laws of logarithms)

Using the chain rule:

$$\frac{dy}{dx} = -\frac{1}{x^2 + 9} \times 2x = -\frac{2x}{x^2 + 9}$$

4 a $f(x) = \frac{x}{x^2 + 2}$
 $f'(x) = \frac{(x^2 + 2) \times 1 - x \times 2x}{(x^2 + 2)^2} = \frac{2 - x^2}{(x^2 + 2)^2}$
 The function is increasing when $f'(x) \geq 0$

i.e. $\frac{2 - x^2}{(x^2 + 2)^2} \geq 0$

$$x^2 \leq 2$$

$$-\sqrt{2} \leq x \leq \sqrt{2}$$

Hence $f(x)$ is increasing on the interval $[-k, k]$ where $k = \sqrt{2}$.

b $f''(x) = \frac{-2x(x^2 + 2)^2 - 4x(x^2 + 2)(2 - x^2)}{(x^2 + 2)^4}$
 $= \frac{2x(x^2 + 2)(-(x^2 + 2) - 2(2 - x^2))}{(x^2 + 2)^4}$
 $= \frac{2x(x^2 + 2)(x^2 - 6)}{(x^2 + 2)^4}$

$f''(x)$ changes sign when the numerator $2x(x^2 + 2)(x^2 - 6)$ is zero

i.e. at $x = 0$ and $x = \pm\sqrt{6}$

where $y = 0$ and $y = \frac{\pm\sqrt{6}}{6 + 2}$

Points of inflection are

$(0, 0)$ and $\left(\pm\sqrt{6}, \pm\frac{\sqrt{6}}{8}\right)$

5 a $f(x) = 12\ln x + x^{\frac{3}{2}}, \quad x > 0$

$$f'(x) = \frac{12}{x} + \frac{3}{2}x^{\frac{1}{2}} = \frac{12}{x} + \frac{3}{2}\sqrt{x}$$

$f(x)$ is an increasing function when $f'(x) \geq 0$

As $x > 0$, $\frac{12}{x} + \frac{3}{2}\sqrt{x}$ is always positive.

$\therefore f(x)$ is increasing for all $x > 0$.

b $f''(x) = -\frac{12}{x^2} + \frac{3}{4}x^{-\frac{1}{2}} = -\frac{12}{x^2} + \frac{3}{4\sqrt{x}}$

At a point of inflection $f''(x) = 0$

$$-\frac{12}{x^2} + \frac{3}{4\sqrt{x}} = 0$$

$$\frac{12}{x^2} = \frac{3}{4\sqrt{x}}$$

$$x^2 = 16\sqrt{x}$$

$$x^{\frac{3}{2}} = 16$$

$$x = \sqrt[3]{256}$$

$$f\left(\sqrt[3]{256}\right) = 12\ln(256)^{\frac{1}{3}} + 256^{\frac{1}{2}}$$

$$= 4\ln 256 + 16$$

$$= 4\ln 2^8 + 16 = 32\ln 2 + 16$$

Coordinates of the point of inflection are $\left(\sqrt[3]{256}, 32\ln 2 + 16\right)$

6 $y = \cos^2 x + \sin x$

$$\begin{aligned}\frac{dy}{dx} &= -2 \cos x \sin x + \cos x \\ &= \cos x(1 - 2 \sin x)\end{aligned}$$

At stationary points $\frac{dy}{dx} = 0$

$$\cos x(1 - 2 \sin x) = 0$$

$$\cos x = 0 \text{ or } \sin x = \frac{1}{2}$$

Solutions in the interval $(0, 2\pi]$ are

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6} \text{ and } \frac{3\pi}{2}$$

$$x = \frac{\pi}{6} \Rightarrow y = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$$x = \frac{\pi}{2} \Rightarrow y = 1$$

$$x = \frac{5\pi}{6} \Rightarrow y = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$$x = \frac{3\pi}{2} \Rightarrow y = -1$$

So the stationary points are

$$\left(\frac{\pi}{6}, \frac{5}{4}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{5}{4}\right) \text{ and } \left(\frac{3\pi}{2}, -1\right)$$

7 $y = x\sqrt{\sin x} = x(\sin x)^{\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= x \times \frac{1}{2}(\sin x)^{-\frac{1}{2}} \cos x + (\sin x)^{\frac{1}{2}} \times 1 \\ &= \frac{1}{2}(\sin x)^{-\frac{1}{2}}(x \cos x + 2 \sin x)\end{aligned}$$

At the maximum point $\frac{dy}{dx} = 0$

$$\frac{1}{2}(\sin x)^{-\frac{1}{2}}(x \cos x + 2 \sin x) = 0$$

$$\therefore x \cos x + 2 \sin x = 0$$

$$(\text{as } (\sin x)^{-\frac{1}{2}} = \frac{1}{\sqrt{\sin x}} \neq 0)$$

Dividing through by $\cos x$ gives

$$x + 2 \tan x = 0$$

So the x -coordinate of the maximum point satisfies $2 \tan x + x = 0$.

8 a $f(x) = e^{0.5x} - x^2$

$$f'(x) = 0.5e^{0.5x} - 2x$$

b $f'(6) = -1.957... < 0$

$$f'(7) = 2.557... > 0$$

As the sign changes between $x = 6$ and $x = 7$ and $f'(x)$ is continuous, $f'(x) = 0$ has a root p between 6 and 7.

Therefore $y = f(x)$ has a stationary point at $x = p$ where $6 < p < 7$.

9 a $f(x) = e^{2x} \sin 2x$

$$\begin{aligned}f'(x) &= e^{2x}(2 \cos 2x) + \sin 2x(2e^{2x}) \\ &= 2e^{2x}(\cos 2x + \sin 2x)\end{aligned}$$

At turning points $f'(x) = 0$

$$2e^{2x}(\cos 2x + \sin 2x) = 0$$

$$\cos 2x + \sin 2x = 0$$

$$\sin 2x = -\cos 2x$$

Divide both sides by $\cos 2x$:

$$\tan 2x = -1$$

$$2x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$$

$$\therefore x = \frac{3\pi}{8} \text{ or } \frac{7\pi}{8} \text{ (in the interval } 0 < x < \pi)$$

$$\text{When } x = \frac{3\pi}{8}, y = \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}$$

$$\text{When } x = \frac{7\pi}{8}, y = -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}$$

So the coordinates of the turning points

$$\text{are } \left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right) \text{ and } \left(\frac{7\pi}{8}, -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right).$$

9 b $f'(x) = 2e^{2x}(\cos 2x + \sin 2x)$

$$\begin{aligned} f''(x) &= 2e^{2x}(-2\sin 2x + 2\cos 2x) \\ &\quad + 4e^{2x}(\cos 2x + \sin 2x) \\ &= e^{2x}(-4\sin 2x + 4\cos 2x \\ &\quad + 4\cos 2x + 4\sin 2x) \\ &= 8e^{2x} \cos 2x \end{aligned}$$

c $f''\left(\frac{3\pi}{8}\right) = 8e^{\frac{3\pi}{4}} \cos \frac{3\pi}{4}$

$$= 8e^{\frac{3\pi}{4}} \left(-\frac{\sqrt{2}}{2}\right) = -4\sqrt{2} e^{\frac{3\pi}{4}} < 0$$

$\therefore \left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}} e^{\frac{3\pi}{4}}\right)$ is a maximum.

$$\begin{aligned} f''\left(\frac{7\pi}{8}\right) &= 8e^{\frac{7\pi}{4}} \cos \frac{7\pi}{4} \\ &= 8e^{\frac{7\pi}{4}} \left(\frac{\sqrt{2}}{2}\right) = 4\sqrt{2} e^{\frac{7\pi}{4}} > 0 \end{aligned}$$

$\therefore \left(\frac{7\pi}{8}, -\frac{1}{\sqrt{2}} e^{\frac{7\pi}{4}}\right)$ is a minimum.

d At points of inflection $f''(x) = 0$

$$8e^{2x} \cos 2x = 0$$

$$\cos 2x = 0$$

$$2x = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore x = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\text{When } x = \frac{\pi}{4}, y = e^{\frac{\pi}{2}} \sin \frac{\pi}{2} = e^{\frac{\pi}{2}}$$

$$\text{When } x = \frac{3\pi}{4}, y = e^{\frac{3\pi}{2}} \sin \frac{3\pi}{2} = -e^{\frac{3\pi}{2}}$$

Points of inflection are

$$\left(\frac{\pi}{4}, e^{\frac{\pi}{2}}\right) \text{ and } \left(\frac{3\pi}{4}, -e^{\frac{3\pi}{2}}\right).$$

10 $y = 2e^x + 3x^2 + 2$

$$\frac{dy}{dx} = 2e^x + 6x$$

When $x = 0$, $y = 4$ and $\frac{dy}{dx} = 2$

Equation of normal at $(0, 4)$ is

$$y - 4 = -\frac{1}{2}(x - 0)$$

$$2y - 8 = -x$$

$$\text{or } x + 2y - 8 = 0$$

11 a $f(x) = 3 \ln x + \frac{1}{x}$

$$f'(x) = \frac{3}{x} - \frac{1}{x^2}$$

At a stationary point $\frac{dy}{dx} = 0$

$$\frac{3}{x} - \frac{1}{x^2} = 0$$

$$3x - 1 = 0$$

$$x = \frac{1}{3}$$

So the x -coordinate of the stationary point P is $\frac{1}{3}$

b At the point Q , $x = 1$ so $y = f(1) = 1$

The gradient of the curve at point Q is

$$f'(1) = 3 - 1 = 2$$

So the gradient of the normal to the curve

at Q is $-\frac{1}{2}$

Equation of the normal at Q is

$$y - 1 = -\frac{1}{2}(x - 1)$$

i.e. $y = -\frac{1}{2}x + \frac{3}{2}$

12 a Let $f(x) = e^{2x} \cos x$

$$\begin{aligned}\text{Then } f'(x) &= e^{2x}(-\sin x) + \cos x(2e^{2x}) \\ &= e^{2x}(2\cos x - \sin x)\end{aligned}$$

Turning points occur when $f'(x) = 0$

$$\begin{aligned}e^{2x}(2\cos x - \sin x) &= 0 \\ \sin x &= 2\cos x\end{aligned}$$

Dividing both sides by $\cos x$ gives

$$\tan x = 2$$

b When $x = 0, y = f(0) = e^0 \cos 0 = 1$

The gradient of the curve at $(0, 1)$ is
 $f'(0) = e^0(2 - 0) = 2$

This is also the gradient of the tangent at $(0, 1)$.

So the equation of the tangent at $(0, 1)$ is

$$y - 1 = 2(x - 0)$$

$$y = 2x + 1$$

13 a $x = y^2 \ln y$

Using the product rule:

$$\frac{dx}{dy} = y^2 \left(\frac{1}{y} \right) + \ln y \times 2y = y + 2y \ln y$$

b $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{y + 2y \ln y}$

When $y = e$,

$$\frac{dy}{dx} = \frac{1}{e + 2e \ln e} = \frac{1}{3e}$$

14 a $f(x) = (x^3 - 2x)e^{-x}$

$$\begin{aligned}f'(x) &= (x^3 - 2x)(-e^{-x}) + (3x^2 - 2)e^{-x} \\ &= e^{-x}(-x^3 + 3x^2 + 2x - 2)\end{aligned}$$

b When $x = 0, f'(x) = -2$

Gradient of normal is $\frac{1}{2}$

$\therefore$ equation of normal to the curve at the origin is

$$y = \frac{1}{2}x$$

This line will intersect the curve again when

$$\frac{1}{2}x = (x^3 - 2x)e^{-x}$$

$$1 = 2(x^2 - 2)e^{-x}$$

$$e^x = 2x^2 - 4$$

$$2x^2 = e^x + 4$$

15 a $f(x) = x(1+x) \ln x = (x+x^2) \ln x$

$$\begin{aligned}f'(x) &= (x+x^2) \times \frac{1}{x} + \ln x \times (1+2x) \\ &= 1+x + (1+2x) \ln x\end{aligned}$$

b At minimum point $A, f'(x) = 0$

$$1+x + (1+2x) \ln x = 0$$

$$(1+2x) \ln x = -(1+x)$$

$$\ln x = -\frac{1+x}{1+2x}$$

So x -coordinate of A is the solution to the equation $x = e^{-\frac{1+x}{1+2x}}$

16 a $x = 4t - 3, y = \frac{8}{t^2} = 8t^{-2}$

$$\frac{dx}{dt} = 4, \frac{dy}{dt} = -16t^{-3}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-16t^{-3}}{4} = \frac{-4}{t^3}$$

16 b When $t = 2$, the curve has gradient

$$\frac{dy}{dx} = \frac{-4}{2^3} = -\frac{1}{2}$$

$\therefore$ the normal has gradient 2.

Also, when $t = 2$, $x = 5$ and $y = 2$,
so the point A has coordinates $(5, 2)$.

$\therefore$ the equation of the normal at A is

$$y - 2 = 2(x - 5)$$

$$\text{i.e. } y = 2x - 8$$

17 $x = 2t$, $y = t^2$

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t$$

At the point P where $t = 3$, the gradient of
the curve is $\frac{dy}{dx} = 3$

$\therefore$ gradient of the normal is $-\frac{1}{3}$

Also, when $t = 3$, the coordinates are $(6, 9)$.

$\therefore$ the equation of the normal at P is

$$y - 9 = -\frac{1}{3}(x - 6)$$

$$\text{i.e. } 3y + x = 33$$

18 $x = t^3$, $y = t^2$

$$\frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2} = \frac{2}{3t}$$

At the point $(1, 1)$ the value of t is 1.

$\therefore$ the gradient of the curve is $\frac{2}{3}$, which is
also the gradient of the tangent.

$\therefore$ the equation of the tangent is

$$y - 1 = \frac{2}{3}(x - 1)$$

$$\text{i.e. } y = \frac{2}{3}x + \frac{1}{3}$$

19 a $x = 2 \cos t + \sin 2t$, $y = \cos t - 2 \sin 2t$

$$\frac{dx}{dt} = -2 \sin t + 2 \cos 2t$$

$$\frac{dy}{dt} = -\sin t - 4 \cos 2t$$

$$\text{b } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t - 4 \cos 2t}{-2 \sin t + 2 \cos 2t}$$

$$\text{When } t = \frac{\pi}{4}, \quad \frac{dy}{dx} = \frac{-\frac{1}{\sqrt{2}} - 0}{-\frac{2}{\sqrt{2}} + 0} = \frac{1}{2}$$

- 19 c** The gradient of the normal at the point P where $t = \frac{\pi}{4}$ is -2 .

The coordinates of P are found by substituting $t = \frac{\pi}{4}$ into the parametric equations:

$$x = \frac{2}{\sqrt{2}} + 1, \quad y = \frac{1}{\sqrt{2}} - 2$$

$\therefore$ the equation of the normal at P is

$$y - \left(\frac{1}{\sqrt{2}} - 2 \right) = -2 \left(x - \left(\frac{2}{\sqrt{2}} + 1 \right) \right)$$

$$y - \frac{1}{\sqrt{2}} + 2 = -2x + 2\sqrt{2} + 2$$

$$\text{i.e. } y + 2x = \frac{5\sqrt{2}}{2}$$

- 20 a** $x = 2t + 3, \quad y = t^3 - 4t$

At point A , where $t = -1$,
 $x = 1$ and $y = 3$.

$\therefore$ the coordinates of A are $(1, 3)$.

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 3t^2 - 4$$

$$\therefore \frac{dy}{dx} = \frac{3t^2 - 4}{2}$$

At the point A , $\frac{dy}{dx} = -\frac{1}{2}$

$\therefore$ gradient of the tangent at A is $-\frac{1}{2}$

Equation of the tangent at A is

$$y - 3 = -\frac{1}{2}(x - 1)$$

$$2y - 6 = -x + 1$$

$$\text{i.e. } 2y + x = 7$$

- b** The tangent line l meets the curve C at points A and B .

Substitute $x = 2t + 3$ and $y = t^3 - 4t$ into the equation of l :

$$2(t^3 - 4t) + (2t + 3) = 7$$

$$2t^3 - 6t = 4$$

$$t^3 - 3t - 2 = 0$$

At point A , $t = -1$, so $t = -1$ is a root of this equation, and hence $(t + 1)$ is a factor of the left-hand side expression.

$$\begin{aligned} t^3 - 3t - 2 &= (t + 1)(t^2 - t - 2) \\ &= (t + 1)(t + 1)(t - 2) \\ &= 2(t + 1)^2(t - 2) \end{aligned}$$

So line l meets the curve C at $t = -1$ (repeated root because the line is tangent to the curve there) and at $t = 2$.

Therefore, at point B , $t = 2$.

- 21** The rate of change of V is $\frac{dV}{dt}$

$$\therefore \frac{dV}{dt} \propto V$$

$$\text{i.e. } \frac{dV}{dt} = -kV$$

where k is a positive constant.
(The negative sign is needed as the value of the car is *decreasing*.)

- 22** The rate of change of mass is $\frac{dM}{dt}$

$$\therefore \frac{dM}{dt} \propto M$$

$$\text{i.e. } \frac{dM}{dt} = -kM$$

where k is a positive constant.
(The negative sign represents *loss* of mass.)

23 The rate of change of pondweed is $\frac{dP}{dt}$

The growth rate is proportional to P :

$$\text{growth rate} \propto P$$

$$\text{i.e. growth rate} = kP$$

where k is a positive constant.

But pondweed is also being removed at a constant rate Q .

$$\therefore \frac{dP}{dt} = \text{growth rate} - \text{removal rate}$$

$$\frac{dP}{dt} = kP - Q$$

24 The rate of increase of the radius is $\frac{dr}{dt}$

$$\therefore \frac{dr}{dt} \propto \frac{1}{r}, \text{ as the rate is inversely}$$

proportional to the radius.

$$\text{Hence } \frac{dr}{dt} = \frac{k}{r}$$

where k is the constant of proportion.

25 The rate of change of temperature is $\frac{d\theta}{dt}$

$$\therefore \frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\text{i.e. } \frac{d\theta}{dt} = -k(\theta - \theta_0),$$

where k is a positive constant.

(The negative sign indicates that the temperature is decreasing, i.e. *loss* of temperature.)

26 a $x = 4 \cos 2t$, $y = 3 \sin t$

The point $A \left(2, \frac{3}{2} \right)$ is on the curve, so

$$4 \cos 2t = 2 \quad \text{and} \quad 3 \sin t = \frac{3}{2}$$

$$\cos 2t = \frac{1}{2} \quad \text{and} \quad \sin t = \frac{1}{2}$$

The only value of t in the interval

$$-\frac{\pi}{2} < t < \frac{\pi}{2} \text{ that satisfies both equations}$$

is $\frac{\pi}{6}$. Therefore $t = \frac{\pi}{6}$ at the point A .

$$\text{b } \frac{dx}{dt} = -8 \sin 2t, \quad \frac{dy}{dt} = 3 \cos t$$

$$\therefore \frac{dy}{dx} = \frac{3 \cos t}{-8 \sin 2t}$$

$$= -\frac{3 \cos t}{16 \sin t \cos t}$$

(using a double angle formula)

$$= -\frac{3}{16 \sin t}$$

$$= -\frac{3}{16} \operatorname{cosec} t$$

$$\text{c At point } A, \text{ where } t = \frac{\pi}{6}, \quad \frac{dy}{dx} = -\frac{3}{8}$$

$$\therefore \text{gradient of the normal at } A \text{ is } \frac{8}{3}$$

Equation of the normal is

$$y - \frac{3}{2} = \frac{8}{3}(x - 2)$$

Multiply through by 6 and rearrange to give:

$$6y - 9 = 16x - 32$$

$$6y - 16x + 23 = 0$$

- 26 d** To find where the normal cuts the curve, substitute $x = 4 \cos 2t$ and $y = 3 \sin t$ into the equation of the normal:

$$6(3 \sin t) - 16(4 \cos 2t) + 23 = 0$$

$$18 \sin t - 64 \cos 2t + 23 = 0$$

$$18 \sin t - 64(1 - 2 \sin^2 t) + 23 = 0$$

(using a double angle formula)

$$128 \sin^2 t + 18 \sin t - 41 = 0$$

But $\sin t = \frac{1}{2}$ is one solution of this equation, as point A lies on the line and on the curve. So

$$128 \sin^2 t + 18 \sin t - 41$$

$$= (2 \sin t - 1)(64 \sin t + 41)$$

$$\therefore (2 \sin t - 1)(64 \sin t + 41) = 0$$

Therefore, at point B , $\sin t = -\frac{41}{64}$

$\therefore$ the y -coordinate of point B is

$$3 \times \left(-\frac{41}{64} \right) = -\frac{123}{64}$$

- 27 a** $x = a \sin^2 t$, $y = a \cos t$

$$\frac{dx}{dt} = 2a \sin t \cos t, \quad \frac{dy}{dt} = -a \sin t$$

$$\therefore \frac{dy}{dx} = \frac{-a \sin t}{2a \sin t \cos t} = \frac{-1}{2 \cos t} = -\frac{1}{2} \sec t$$

- b** As $P \left(\frac{3}{4}a, \frac{1}{2}a \right)$ lies on the curve,

$$a \sin^2 t = \frac{3}{4}a \quad \text{and} \quad a \cos t = \frac{1}{2}a$$

$$\sin t = \pm \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos t = \frac{1}{2}$$

The only value of t in the interval

$0 \leq t \leq \frac{\pi}{2}$ that satisfies both equations

is $\frac{\pi}{3}$. Therefore $t = \frac{\pi}{3}$ at the point P .

Gradient of the curve at point P is

$$-\frac{1}{2} \sec \frac{\pi}{3} = -1.$$

$\therefore$ equation of the tangent at P is

$$y - \frac{1}{2}a = -1 \left(x - \frac{3}{4}a \right)$$

$$y - \frac{1}{2}a = -x + \frac{3}{4}a$$

Multiply equation by 4 and rearrange to give

$$4y + 4x = 5a$$

- c** Equation of the tangent at C is

$$4y + 4x = 5a$$

$$\text{At } A, x = 0 \Rightarrow y = \frac{5a}{4}$$

$$\text{At } B, y = 0 \Rightarrow x = \frac{5a}{4}$$

$$\text{Area of } AOB = \frac{1}{2} \left(\frac{5a}{4} \right)^2 = \frac{25}{32}a^2,$$

which is of the form ka^2 with $k = \frac{25}{32}$

$$28 \quad x = (t+1)^2, \quad y = \frac{1}{2}t^3 + 3$$

$$\frac{dx}{dt} = 2(t+1), \quad \frac{dy}{dt} = \frac{3}{2}t^2$$

$$\therefore \frac{dy}{dx} = \frac{\frac{3}{2}t^2}{2(t+1)} = \frac{3t^2}{4(t+1)}$$

$$\text{When } t = 2, \quad \frac{dy}{dx} = \frac{3 \times 4}{4 \times 3} = 1$$

$\therefore$ gradient of the normal at the point P , where $t = 2$, is -1 .

The coordinates of P are $(9, 7)$.

$\therefore$ equation of the normal is

$$y - 7 = -1(x - 9)$$

$$y - 7 = -x + 9$$

$$\text{i.e. } y + x = 16$$

$$29 \quad 5x^2 + 5y^2 - 6xy = 13$$

Differentiate with respect to x :

$$10x + 10y \frac{dy}{dx} - 6 \left(x \frac{dy}{dx} + y \right) = 0$$

$$(10y - 6x) \frac{dy}{dx} = 6y - 10x$$

$$\therefore \frac{dy}{dx} = \frac{6y - 10x}{10y - 6x}$$

At the point $(1, 2)$

$$\frac{dy}{dx} = \frac{12 - 10}{20 - 6} = \frac{2}{14} = \frac{1}{7}$$

So the gradient of the curve at $(1, 2)$ is $\frac{1}{7}$

$$30 \quad e^{2x} + e^{2y} = xy$$

Differentiate with respect to x :

$$2e^{2x} + 2e^{2y} \frac{dy}{dx} = x \frac{dy}{dx} + y \times 1$$

$$2e^{2y} \frac{dy}{dx} - x \frac{dy}{dx} = y - 2e^{2x}$$

$$(2e^{2y} - x) \frac{dy}{dx} = y - 2e^{2x}$$

$$\therefore \frac{dy}{dx} = \frac{y - 2e^{2x}}{2e^{2y} - x}$$

$$31 \quad y^3 + 3xy^2 - x^3 = 3$$

Differentiate with respect to x :

$$3y^2 \frac{dy}{dx} + \left(3x \times 2y \frac{dy}{dx} + y^2 \times 3 \right) - 3x^2 = 0$$

$$(3y^2 + 6xy) \frac{dy}{dx} = 3x^2 - 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{3(x^2 - y^2)}{3y(y + 2x)} = \frac{x^2 - y^2}{y(y + 2x)}$$

Turning points occur when $\frac{dy}{dx} = 0$

$$\frac{x^2 - y^2}{y(y + 2x)} = 0$$

$$x^2 = y^2$$

$$x = \pm y$$

$$\text{When } x = y, \quad y^3 + 3y^3 - y^3 = 3$$

$$\text{so } 3y^3 = 3$$

$$y = 1 \quad \text{and hence } x = 1$$

$$\text{When } x = -y, \quad y^3 - 3y^3 + y^3 = 3$$

$$\text{so } -y^3 = 3$$

$$y = \sqrt[3]{-3} \quad \text{and hence } x = -\sqrt[3]{-3}$$

$\therefore$ the coordinates of the turning points are $(1, 1)$ and $(-\sqrt[3]{-3}, \sqrt[3]{-3})$.

32 a $(1+x)(2+y) = x^2 + y^2$

Differentiate with respect to x :

$$(1+x)\left(\frac{dy}{dx}\right) + (2+y)(1) = 2x + 2y\frac{dy}{dx}$$

$$(1+x-2y)\frac{dy}{dx} = 2x - y - 2$$

$$\therefore \frac{dy}{dx} = \frac{2x - y - 2}{1 + x - 2y}$$

b When the curve meets the y -axis, $x = 0$.

Substitute $x = 0$ into the equation of the curve:

$$2 + y = y^2$$

$$\text{i.e. } y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$

$$y = 2 \text{ or } y = -1$$

$$\text{At } (0, 2), \frac{dy}{dx} = \frac{0-2-2}{1+0-4} = \frac{4}{3}$$

$$\text{At } (0, -1), \frac{dy}{dx} = \frac{0+1-2}{1+0+2} = -\frac{1}{3}$$

c A tangent that is parallel to the y -axis has infinite gradient.

$$\text{For } \frac{dy}{dx} = \frac{2x - y - 2}{1 + x - 2y} \text{ to be infinite,}$$

$$\text{the denominator } 1 + x - 2y = 0,$$

$$\text{i.e. } x = 2y - 1$$

Substitute $x = 2y - 1$ into the equation of the curve:

$$(1 + 2y - 1)(2 + y) = (2y - 1)^2 + y^2$$

$$2y^2 + 4y = 4y^2 - 4y + 1 + y^2$$

$$3y^2 - 8y + 1 = 0$$

$$y = \frac{8 \pm \sqrt{64 - 12}}{6} = \frac{4 \pm \sqrt{13}}{3}$$

$$\text{When } y = \frac{4 + \sqrt{13}}{3}, x = \frac{5 + 2\sqrt{13}}{3}$$

$$\text{When } y = \frac{4 - \sqrt{13}}{3}, x = \frac{5 - 2\sqrt{13}}{3}$$

$\therefore$ there are two points at which the tangents are parallel to the y -axis.

$$\text{They are } \left(\frac{5 + 2\sqrt{13}}{3}, \frac{4 + \sqrt{13}}{3} \right) \text{ and}$$

$$\left(\frac{5 - 2\sqrt{13}}{3}, \frac{4 - \sqrt{13}}{3} \right).$$

33 $7x^2 + 48xy - 7y^2 + 75 = 0$

Implicit differentiation with respect to x gives

$$14x + 48\left(x \frac{dy}{dx} + y\right) - 14y \frac{dy}{dx} = 0$$

$$(48x - 14y) \frac{dy}{dx} = -14x - 48y$$

$$\therefore \frac{dy}{dx} = \frac{-14x - 48y}{48x - 14y} = \frac{7x + 24y}{7y - 24x}$$

When $\frac{dy}{dx} = \frac{2}{11}$,

$$\frac{7x + 24y}{7y - 24x} = \frac{2}{11}$$

$$14y - 48x = 77x + 264y$$

$$125x + 250y = 0$$

$$\therefore x + 2y = 0$$

So the coordinates of the points at which the gradient is $\frac{2}{11}$ satisfy $x + 2y = 0$,

which means that the points lie on the line $x + 2y = 0$.

34 $y = x^x$

Take natural logs of both sides:

$$\ln y = \ln x^x$$

$$\ln y = x \ln x \quad (\text{using properties of logarithms})$$

Differentiate with respect to x :

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= x \times \frac{1}{x} + \ln x \times 1 \\ &= 1 + \ln x \end{aligned}$$

$$\therefore \frac{dy}{dx} = y(1 + \ln x)$$

But $y = x^x$

$$\therefore \frac{dy}{dx} = x^x(1 + \ln x)$$

35 a $a^x = e^{kx}$

Take natural logs of both sides:

$$\ln a^x = \ln e^{kx}$$

$$x \ln a = kx$$

As this is true for all values of x , $k = \ln a$.

b Taking $a = 2$,

$$y = 2^x = e^{kx} \quad \text{where } k = \ln 2$$

$$\frac{dy}{dx} = ke^{kx} = (\ln 2)e^{(\ln 2)x} = 2^x \ln 2$$

c At the point $(2, 4)$, $x = 2$.

$\therefore$ gradient of the curve at $(2, 4)$ is

$$\frac{dy}{dx} = 2^2 \ln 2 = 4 \ln 2 = \ln 2^4 = \ln 16$$

36 a $P = P_0(1.09)^t$

Take natural logs of both sides:

$$\begin{aligned} \ln P &= \ln(P_0(1.09)^t) \\ &= \ln P_0 + \ln(1.09)^t \\ &= \ln P_0 + t \ln 1.09 \end{aligned}$$

$$\therefore t \ln 1.09 = \ln P - \ln P_0$$

$$t = \frac{\ln P - \ln P_0}{\ln 1.09} \quad \text{or} \quad \frac{\ln\left(\frac{P}{P_0}\right)}{\ln 1.09}$$

b When $t = T$, $P = 2P_0$.

Substituting these into the expression in part **a** gives

$$T = \frac{\ln 2}{\ln 1.09} = 8.04 \quad (3 \text{ s.f.})$$

$$36 \text{ c } \frac{dP}{dt} = P_0(1.09)'(\ln 1.09)$$

$$\text{When } t = T, P = 2P_0 \text{ so } (1.09)^T = 2$$

$$\begin{aligned} \text{Hence } \frac{dP}{dt} &= P_0(1.09)^T(\ln 1.09) \\ &= P_0 \times 2 \times \ln 1.09 \\ &= 0.172P_0 \text{ (3 s.f.)} \end{aligned}$$

$$37 \text{ a } y = \ln(\sin x)$$

$$\frac{dy}{dx} = \cos x \times \frac{1}{\sin x} = \cot x$$

$$\text{At a stationary point } \frac{dy}{dx} = 0$$

$$\cot x = 0 \Rightarrow x = \frac{\pi}{2}$$

$$(\text{in the interval } 0 < x < \pi)$$

$$\text{When } x = \frac{\pi}{2}, y = \ln\left(\sin \frac{\pi}{2}\right) = \ln 1 = 0$$

$$\therefore \text{stationary point is at } \left(\frac{\pi}{2}, 0\right).$$

$$b \quad \frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$$

$$\operatorname{cosec}^2 x = \frac{1}{\sin^2 x} > 0 \text{ for all } 0 < x < \pi$$

$$\therefore \frac{d^2y}{dx^2} < 0 \text{ for all } 0 < x < \pi$$

Hence the curve C is concave for all values of x in its domain.

$$38 \text{ a } m = 40e^{-0.244t}$$

After 9 months, $t = 0.75$, so

$$m = 40e^{-0.244 \times 0.75} = 40e^{-0.183} = 33.31 \dots$$

$$b \quad \frac{dm}{dt} = -0.244 \times 40e^{-0.244t} = -9.76e^{-0.244t}$$

c The negative sign indicates that the mass is decreasing.

$$39 \text{ a } f(x) = \frac{\cos 2x}{e^x}$$

$$f'(x) = \frac{-2e^x \sin 2x - e^x \cos 2x}{e^{2x}}$$

$$= -\frac{2 \sin 2x + \cos 2x}{e^x}$$

$$\text{At } A \text{ and } B, f'(x) = 0$$

$$2 \sin 2x + \cos 2x = 0$$

$$2 \tan 2x + 1 = 0$$

$$\tan 2x = -0.5$$

$$2x = 2.678 \text{ or } 5.820$$

$$x = 1.339 \text{ or } 2.910$$

$$(\text{in the interval } 0 \leq x \leq \pi)$$

$$x = 1.339 \Rightarrow y = f(x) = -0.2344$$

$$x = 2.910 \Rightarrow y = f(x) = 0.04874$$

Therefore, to 3 significant figures:
coordinates of A are $(1.34, -0.234)$;
coordinates of B are $(2.91, 0.0487)$.

b The curve of $y = 2 + 4f(x - 4)$ is a transformation of $f(x)$, obtained via a translation of 4 units to the right, a stretch by a factor of 4 in the y -direction, and then a translation of 2 units upwards.

Turning points are:

minimum $(1.34 + 4, -0.234 \times 4 + 2)$ and

maximum $(2.91 + 4, 0.0487 \times 4 + 2)$,

i.e. minimum $(5.34, 1.06)$

and maximum $(6.91, 2.19)$.

$$c \quad f''(x)$$

$$= -\frac{e^x(4 \cos 2x - 2 \sin 2x) - e^x(2 \sin 2x + \cos 2x)}{e^{2x}}$$

$$= \frac{4 \sin 2x - 3 \cos 2x}{e^x}$$

$$f(x) \text{ is concave when } f''(x) \leq 0$$

$$f''(x) = 0 \text{ when}$$

$$4 \sin 2x - 3 \cos 2x = 0$$

$$\tan 2x = \frac{3}{4}$$

$$2x = 0.644 \text{ or } 3.785$$

$$x = 0.322 \text{ or } 1.893$$

The curve has a minimum point and hence is convex between these values, so it is concave for

$$0 \leq x \leq 0.322 \text{ and } 1.892 \leq x \leq \pi.$$

Challenge

a $y = 2 \sin 2t, x = 5 \cos\left(t + \frac{\pi}{12}\right)$

$$\frac{dy}{dt} = 4 \cos 2t, \frac{dx}{dt} = -5 \sin\left(t + \frac{\pi}{12}\right)$$

$$\therefore \frac{dy}{dx} = -\frac{4 \cos 2t}{5 \sin\left(t + \frac{\pi}{12}\right)}$$

b $\frac{dy}{dx} = 0$ when $4 \cos 2t = 0$

$$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \text{ or } \frac{7\pi}{2}$$

$$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

(in the interval $0 \leq x \leq 2\pi$)

$$t = \frac{\pi}{4} \Rightarrow x = 5 \cos\left(\frac{\pi}{3}\right) = \frac{5}{2}$$

$$\text{and } y = 2 \sin \frac{\pi}{2} = 2, \text{ i.e. point } \left(\frac{5}{2}, 2\right)$$

$$t = \frac{3\pi}{4} \Rightarrow x = 5 \cos\left(\frac{5\pi}{6}\right) = -\frac{5\sqrt{3}}{2}$$

$$\text{and } y = 2 \sin \frac{3\pi}{2} = -2, \text{ i.e. point } \left(-\frac{5\sqrt{3}}{2}, -2\right)$$

$$t = \frac{5\pi}{4} \Rightarrow x = 5 \cos\left(\frac{4\pi}{3}\right) = -\frac{5}{2}$$

$$\text{and } y = 2 \sin \frac{5\pi}{2} = 2, \text{ i.e. point } \left(-\frac{5}{2}, 2\right)$$

$$t = \frac{7\pi}{4} \Rightarrow x = 5 \cos\left(\frac{11\pi}{6}\right) = \frac{5\sqrt{3}}{2}$$

$$\text{and } y = 2 \sin \frac{7\pi}{2} = -2, \text{ i.e. point } \left(\frac{5\sqrt{3}}{2}, -2\right)$$

c The curve cuts the x-axis when $y = 0$,
i.e. when $2 \sin 2t = 0$

$$2t = 0, \pi, 2\pi, 3\pi, 4\pi$$

$$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

$$t = 0 \Rightarrow x = 5 \cos \frac{\pi}{12} = 4.83, \text{ i.e. } (4.83, 0)$$

$$\text{with gradient } \frac{dy}{dx} = \frac{-4}{5 \sin \frac{\pi}{12}} = -3.09$$

$$t = \frac{\pi}{2} \Rightarrow x = 5 \cos \frac{7\pi}{12} = -1.29, \text{ i.e. } (-1.29, 0)$$

$$\text{with gradient } \frac{dy}{dx} = \frac{4}{5 \sin \frac{7\pi}{12}} = 0.828$$

$$t = \pi \Rightarrow x = 5 \cos \frac{13\pi}{12} = -4.83, \text{ i.e. } (-4.83, 0)$$

$$\text{with gradient } \frac{dy}{dx} = \frac{-4}{5 \sin \frac{13\pi}{12}} = 3.09$$

$$t = \frac{3\pi}{2} \Rightarrow x = 5 \cos \frac{19\pi}{12} = 1.29, \text{ i.e. } (1.29, 0)$$

$$\text{with gradient } \frac{dy}{dx} = \frac{4}{5 \sin \frac{19\pi}{12}} = -0.828$$

The curve cuts the y-axis when $x = 0$.

$$\text{i.e. when } 5 \cos\left(t + \frac{\pi}{12}\right) = 0$$

$$t + \frac{\pi}{12} = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$t = \frac{5\pi}{12}, \frac{17\pi}{12}$$

$$t = \frac{5\pi}{12} \Rightarrow y = 2 \sin \frac{5\pi}{6} = 1, \text{ i.e. } (0, 1)$$

$$\text{with gradient } \frac{dy}{dx} = \frac{-4 \cos \frac{5\pi}{6}}{5 \sin \frac{\pi}{2}} = 0.693$$

$$t = \frac{17\pi}{12} \Rightarrow y = 2 \sin \frac{17\pi}{6} = 1, \text{ i.e. } (0, 1)$$

$$\text{with gradient } \frac{dy}{dx} = \frac{-4 \cos \frac{17\pi}{6}}{5 \sin \frac{3\pi}{2}} = -0.693$$

So the curve cuts the y-axis twice at $(0, 1)$
with gradients 0.693 and -0.693 .

$$\mathbf{d} \quad \frac{dx}{dy} = -\frac{5 \sin\left(t + \frac{\pi}{12}\right)}{4 \cos 2t}$$

$$\frac{dx}{dy} = 0 \text{ when } \sin\left(t + \frac{\pi}{12}\right) = 0$$

$$t + \frac{\pi}{12} = \pi, 2\pi$$

$$t = \frac{11\pi}{12}, \frac{23\pi}{12}$$

$$t = \frac{11\pi}{12} \Rightarrow y = 2 \sin \frac{11\pi}{6} = -1$$

$$\text{and } x = 5 \cos\left(\frac{11\pi}{12} + \frac{\pi}{12}\right) = -5$$

$$t = \frac{23\pi}{12} \Rightarrow y = 2 \sin \frac{23\pi}{6} = -1$$

$$\text{and } x = 5 \cos\left(\frac{23\pi}{12} + \frac{\pi}{12}\right) = 5$$

So points where curve is vertical are $(-5, -1)$ and $(5, -1)$.

e

