

De Moivre's Theorem

Aim

To explain what De Moivre's Theorem is and to demonstrate its usage.

Learning Outcomes

At the end of this section you will be able to:

- Multiply two complex numbers of the form $r(\cos \theta + i \sin \theta)$,
- Understand what De Moivre's theorem is,
- Simplify complex numbers using De Moivre's Theorem.

Consider multiplying the two complex numbers $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$.

$$\begin{aligned} z_1 z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2), \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)], \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]. \end{aligned}$$

If we were to let $\theta_1 = \theta_2 = \theta$ then the above result would give,

$$(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta.$$

From this we can deduce the following,

$$\begin{aligned} (\cos \theta + i \sin \theta)^3 &= (\cos \theta + i \sin \theta)^2 (\cos \theta + i \sin \theta), \\ &= (\cos 2\theta + i \sin 2\theta)(\cos \theta + i \sin \theta), \\ &= \cos 3\theta + i \sin 3\theta. \end{aligned}$$

Similarly it can be shown that

$$(\cos \theta + i \sin \theta)^4 = \cos 4\theta + i \sin 4\theta.$$

This suggests the general formula

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

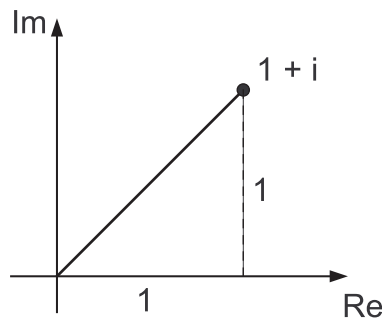
This is known as *De Moivre's Theorem*.

Example

Express $(1 + i)$ in modulus-argument form and hence express $(1 + i)^9$ in the form $a + ib$.

From the diagram below we can see that

$$|1 + i| = \sqrt{2} \quad \text{and} \quad \tan^{-1} 1 = \frac{\pi}{4}.$$



$$\begin{aligned} 1 + i &= \sqrt{2} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]. \\ \Rightarrow (1 + i)^9 &= \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^9, \\ &= (\sqrt{2})^9 \left(\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4} \right) = 2^{\frac{9}{2}} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right), \\ &= 2^{\frac{9}{2}} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right), \\ &= 16\sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right), \\ &= 16 + i16. \end{aligned}$$

Related Reading

Morris, O.D., P. Cooke. 1992. *Text & Tests 4*. The Celtic Press.

Stroud, K.A. 2001. *Engineering Mathematics*. 5th Edition. PALGRAVE.