

An Introduction to MATLAB

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1. Introduction

MATLAB is a software applications package for mathematics, engineering and technical computing. It has become an industry standard which you will meet in a number of modules throughout your course. These notes provide an introduction to the basic package and one of its add-ons, called Toolboxes, the Symbolic Mathematics Toolbox. You will use other Toolboxes in your engineering modules.

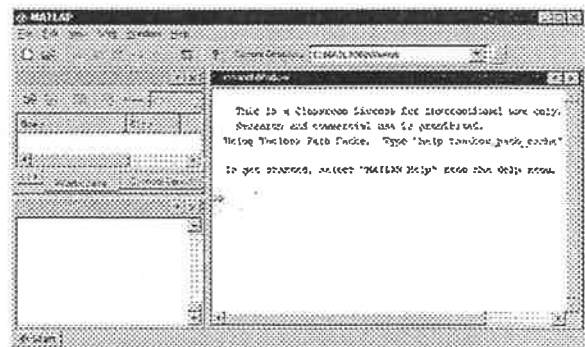
The first version of MATLAB was written by a numerical analyst called Cleve Moler in the 1970's and was designed as an interactive system for performing numerical computation, manipulating matrices (hence MATrix LABoratory) and drawing graphs. It has since been extensively developed and now includes symbolic algebra (such as for solving equations, differentiation and integration). You can also write programs to solve various problems - and much more besides through the various toolboxes.

You will learn the principles and techniques of the mathematics topics in the normal lectures and tutorials and use MATLAB to solve much bigger or more complicated problems than you would wish to do by hand.

1.1 Running a MATLAB session

1.1.1 Starting a session

Start every session by double clicking on the MATLAB icon on your desktop.



A MATLAB Command Window and two other windows appear and you can start using MATLAB immediately at the prompt `>>`. You do not need to select Help.

1.1.2 Mistakes

You will find that you will make plenty of mistakes when typing commands and will often want to make minor changes to previous lines. Note that

1. you must input at the **current** » prompt; you cannot go back to previous lines
2. mistakes in the current line can be corrected using the cursor keys → and ← to move backwards and forwards along a line. Delete or insert characters as required.
3. you can recall previous MATLAB commands by using the ↑ and ↓ cursor keys. Repeatedly pressing ↑ will review previous commands, starting with the most recent. To re-execute the command press the return key, editing it first if necessary. This is useful when you want to execute a command that is very similar to one used previously.

1.1.3 Copying to a Word file.

At the end of a session you can copy the working commands and responses to a Word file in the usual way. That is, open a Word file, highlight the material in the Matlab file to be copied, Edit Copy (or Ctrl C), click on the Word file at the point of insertion and Edit Paste (or Ctrl V).

Graphs can also be transferred; see section 2.2.

1.2 MATLAB as a calculator

The basic arithmetic operators are

addition	subtraction	multiplication	division	powers
+	-	*	/	^

and these are used with brackets if necessary.

You should type in the commands following the prompt » and press the enter key to end your line. These are shown in **boldface** to distinguish them from the response. The result appears on the next line. For example,

```
» 5+6+7
ans =
    18.00
» 1+4*3
ans =
    13.00
» 24/8-1
ans =
     2.00
```

Note that we can do the three calculations at the same time:

```
» 5+6+7,1+4*3,24/8-1
ans =
    18
ans =
    13
ans =
     2
```

MATLAB works according to the following priorities:

- | | | |
|----|---------------------------------|-----------------------|
| 1. | quantities in brackets | $2+(7-3) = 2 + 4 = 6$ |
| 2. | powers | $3+4^2 = 3 + 16 = 19$ |
| 3. | *, / working from left to right | $2*6/4 = 12/4 = 3$ |
| 4. | +, - working from left to right | $1+7 - 3 = 8 - 3 = 5$ |

Examples:

```
» 3+(4-2)^2*5, 3+6/2+1, (3+6)/(2+1)
ans =
    23.00
ans =
     7.00
ans =
     3.00
```

1.3 Formats

Numbers are calculated correct to 15 significant figures, but how they are printed is controlled by the Format command. The default is to print numbers between 0.001 and 1000 correct to 4 decimal places and other numbers in exponential form, so

```
» 200/3, 20000/3, 0.0004/3
ans =
    66.6667
ans =
    6.6667e+003           this means  $6.6667 \times 10^3 = 6666.7$ 
ans =
    1.3333e-004           this means  $1.3333 \times 10^{-4} = 0.00013333$ 
```

Type `» help format` to see a list of format commands.

Obtain the following – remember to type *format* commands in lower case.

```
» format bank
» 200/3
ans =
    66.67
» format long
» 200/3
ans =
    66.66666666666667
```

`format compact` reduces the number of blank lines (the print outs in these notes were produced using `format compact`).

To return to the default format:

```
» format
```

1.4 Variables

```
» 8+2^3
ans =
    16
» 3*ans
ans =
    48
```

The result of the first calculation is called *ans* and is used in the second calculation, after which its value is 48. It could be used in further calculations. *ans* is an example of a variable.

We can use our own names to store numbers. For example:

```
» a = 3;b = 5;
» x = 3*a+2*b,y = a^2 + b^2
x =
    19
y =
    34
```

(Note that the use of a semi-colon ; in the first line stops the values of *a* and *b* being printed in the succeeding lines)

The values of *a*, *b*, *x* and *y* can be used in subsequent calculations. They are called **variables** and the expressions such as *a* = 3 are examples of **assignment statements**.

Variable names can be any combination of letters and digits, starting with a letter. MATLAB distinguishes between upper and lower case, so *n* and *N* are different variables.

The following are valid variable names: *a*, *x*, *y1*, *max*, *Temp*, *cost*, *R2D2*

but the following are **not** allowable: *2x*, *Max-temp*, *#ab*, *a%*, *(ABC)*

Use names that reflect what they represent. For example, *v* for velocity, *t* for time and *A* or *area* for area.

You should avoid certain names as they are already in use. For example, *pi* is used for $\pi = 3.14159\dots$ and *i* and *j* for $\sqrt{-1}$. Also, do not use *ans*.

Variables can represent text as well as numbers. For example

```
>> c='apple', name='george', temp='temperature'
c =
    apple
name =
    george
temp =
    temperature
```

If you start a new section or problem it is a good idea to start with

```
» clear all
» format short
```

This clears all variables, preventing you from using a previous value erroneously. For example:

```
» qq=24
qq =
    24
» clear all
» qq
??? Undefined function or variable 'qq'.
```

1.5 Built-in Functions

The most useful of these are:

Function	calculates	condition/comment
$\text{sqrt}(x)$	$\sqrt{x}$	$x \geq 0$
$\log(x)$	$\ln x = \log_e x$	$x > 0$
$\log_{10}(x)$	$\log_{10} x$	$x > 0$
$\exp(x)$	e^x	
$\text{round}(x)$		rounds to nearest integer
$\sin(x)$	$\sin x$	x in radians
$\cos(x)$	$\cos x$	x in radians
$\tan(x)$	$\tan x$	x in radians
$\text{asin}(x)$	$\sin^{-1} x$	returns value in range $-\frac{\pi}{2}$ to $\frac{\pi}{2}$
$\text{acos}(x)$	$\cos^{-1} x$	returns value in range 0 to π
$\text{atan}(x)$	$\tan^{-1} x$	returns value in range $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

Obtain the following and check with your calculator:

```
»sqrt(7),log(10),exp(1.52),sin(0.841),acos(-1),
                                     round(200/3)
ans =
    2.6458
ans =
    2.3026
ans =
    4.5722
ans =
    0.7453
ans =
    3.1416
ans =
    67
```

To work in degrees rather than radians in the trigonometric functions the argument must be converted using π radians = 180° . Type pi to obtain π . For example, to find (i) $\cos 43^\circ$ and (ii) $\tan^{-1} 1.59$ in degrees:

```
» cos(43*pi/180),atan(1.59)*180/pi
ans =
    0.7314
ans =
   57.8329
```

Functions can be nested, so to find $\sin \sqrt{0.63}$ and $\sqrt{1 + e^{\cos^{-1} 0.25}}$:

```
» x = sin(sqrt(0.63)),y = sqrt(1+exp(acos(0.25)))
x =
    0.7130
y =
    2.1763
```

Note You can see a list of available functions in Help. Click on Help on the toolbar, MATLAB Help and then, in the right hand window, either "MATLAB Functions Listed by Category" or "MATLAB Functions Listed Alphabetically". Clicking on a function tells you how to use it and gives some examples.

1.6 Comments

You can describe what is going on with comments. These are preceded with % and MATLAB ignores anything on the line after %. Examples are

```
» a = base*height/2      % find the area of a triangle
» % draw the graphs of sin x and cos x
» % x takes values from 1 to 5 in steps of 0.01
```

These will be included where appropriate from here on.

1.7 Keeping a record

You can keep a record of your Matlab session, and save any graphs drawn, on a floppy in A drive as follows.

1.7.1 To save in a Word file.

Use this if all you want is a hard copy of your commands, results and graphs. Open a Word file and, at the end of your Matlab session, use the mouse to highlight everything in the Command Window. Click on *Edit* and then *Copy*. Go to your Word document, click on *Edit*, then *Paste*, and everything will be copied to the Word file.

Graphs can be copied as well. In the Figure window click on *Edit*, then *Copy Figure*. In the Word file put the cursor at the correct point and click on *Edit*, then *Paste*.

You can edit your work to remove mistakes and improve the layout.

Use *Edit*, then *Save As...* to save the file on your floppy in A: drive.

1.7.2. To save a graph to a floppy.

In the Figure window click on *File*, then *Save As...*

Next to *Save in:* find $3\frac{1}{2}$ *Floppy (A:)*

Type a name in the *File name* box, for example *Graph 1*.

The *Save as type:* box should say *Figures(*.fig)*

Click on *Save*

Your graph will now be saved, and can be opened again during a later Matlab session if you wish.

Exercise 1

1. Evaluate each of the following using MATLAB. Try to predict the answer first.
 $1 + 2 + 6$, $6 - (4 - 2)$, $6 \times 5 - 12 / 2$, $(2 + 3)^2 - 4 \times 2$
2. Predict the results of the following, and check.
 $24 / (2 \times 3)$, $24 / 2 \times 3$, $24 / 2 / 3$.
 What is the value of $1000 / 2 / 4 / 5$? How else could it be written?
3. Calculate $\frac{10}{7}$ correct to (i) 2D (ii) 4D (iii) 15D.
4. Use the default format to find (i) π (ii) 1000π (iii) $\frac{\pi}{100}$ (iv) $\frac{\pi}{10000}$.
 (Type pi to obtain π)
5. Find $\sqrt{174}$, $\ln 30$, $\log_{10} 21.6$, $e^{2.07}$ correct to 4D.
 Round the first two of these to the nearest integer.
6. (i) Find $\sin 0.8$, $\cos 1.26$, $\tan 2.35$, $\sin^{-1} 0.763$, $\cos^{-1}(-0.47)$, $\tan^{-1}(-3.17)$,
 where all angles are in radians.
 (ii) Find $\sin 27^\circ$, $\cos 138.5^\circ$, $\tan(-67^\circ)$, and find, to the nearest degree,
 $\sin^{-1} 0.721$, $\cos^{-1}(-0.0488)$, $\tan^{-1} 1.233$.
7. Find $\frac{1}{\sqrt{31.493}}$, $e^{\sqrt{1+\ln 4}}$, $\sin(2 \tan^{-1} 1)$, $(\log_{10}(3 \sin^{-1} 0.7))^2$.
 Show that $e^{\sqrt{\ln 162 \times \cos 54^\circ}} = 6$, to the nearest integer.
8. Which of the following are valid variable names?
 p , $5q$, PQr , *henry*, *jan20*, *jan_20*, *20jan*, *Area*, *£pounds*, *Minimum*
9. Set variables $p = 5$, $q = 3.7$. Calculate $r = 3p + 5q$, $s = \sqrt{p^2 + q^2}$,
 $t = \ln(rs)$, $u = \sqrt{p + q}$, $v = \frac{u^3}{t}$. You should obtain $v = 4.8061$.

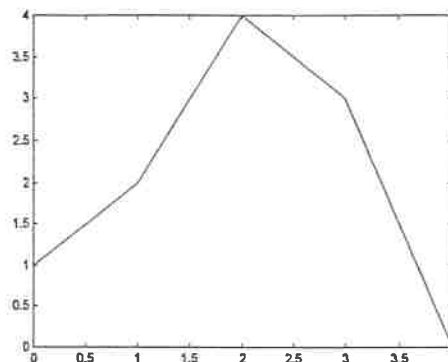
2 Graph Plotting

2.1 Graphs from data

We can plot graphs from lists of numbers (called vectors) of equal length. For example:

```
» clear all
» x=[0 1 2 3 4];
» y=[1 2 4 3 0];
» plot(x,y)
```

The plot command takes these to be the points (0, 1), (1, 2), (2, 4), (3, 3) and (4, 0), plots them and joins them with straight lines, resulting in the graph below.



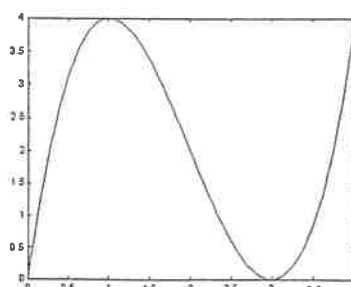
2.2 Graphs of Functions.

Suppose we wish to plot the graph of $y = x^3 - 6x^2 + 9x$ from $x = 0$ to $x = 4$. MATLAB does this by calculating the y values at a large number of equally spaced values of x , plotting these points and joining them up with straight lines (as for the data in Section 2.1). Two ways of doing this are using the *fplot* and *plot* commands:

2.2.1 using *fplot*

Here the function is given in single quotes, together with the range of x values. MATLAB automatically chooses the points within this range at which to plot the function. The commands to plot $y = x^3 - 6x^2 + 9x$ are

```
» clear all
» fplot('x^3-6*x^2+9*x', [0,4])
```



2.2.2 using *plot*

We specify the x values at which the function is to be evaluated, and the function itself. Where necessary we must use the array multiplication, division and power operators `.*`, `./` and `.^`, rather than `*`, `/` and `^`. (See Section 3.6 for an explanation of these). For example, x^2 must be typed as `x.^2`.

If too few x values, and hence points plotted, are used the graph will not be smooth. Using a step size of 0.1 is usually sufficient.

To plot $y = x^3 - 6x^2 + 9x$ for values of x from 0 to 4 we will first set up the vector x , which will have values 0, 0.1, 0.2, 0.3, ..., 3.9, 4, calculate y at these points and then plot the graph.

```
>> x=0:0.1:4;           % x goes from 0 to 4 in steps of 0.1
>> y=x.^3-6*x.^2+9*x;    % calculates the y values
>> plot(x,y)
```

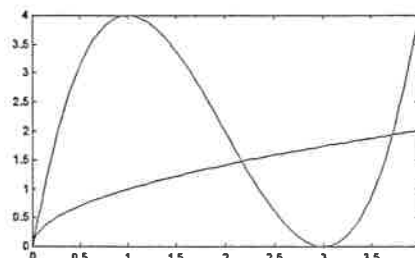
This produces the same graph as above (using *fplot*).

2.3 Hold

Each time an *fplot* command is used previous graphs are lost. This is not convenient if extra graphs need to be added later. To stop a window being cleared use *hold*.

```
» clear all
» fplot('x^3-6*x^2+9*x',[0,4])
» hold
Current plot held
» fplot('sqrt(x)',[0,4])
```

These two, and subsequent, graphs are plotted in the same window.



Disable the hold command with

```
» hold off
```

A window can be cleared with

```
» clf
```

We can, if we wish, specify the y limits as well as the x limits. Consider, for example,

$y = \frac{1}{x}$ and note that y gets very large when x approaches 0.

The commands

```
>> clf
>> fplot('1/x', [-4,4])
```

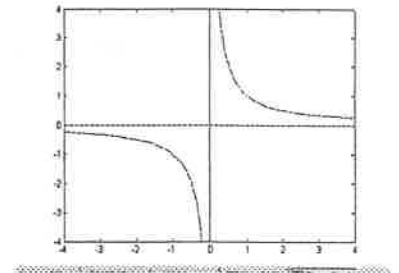
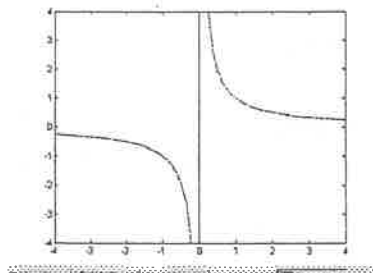
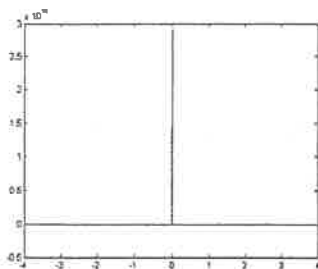
produce the left hand graph below. This is clearly unsatisfactory. We can put in limits for y to rectify this:

```
>> clf
>> fplot('1/x', [-4,4,-4,4])
```

This produces the middle graph below.

Another improvement is to draw the x axis (the line $y = 0$) as well.

```
>> hold
Current plot held
>> fplot('0', [-4,4])
```



2.4 Copying graphs to a Word file

To copy the contents of a graphics window to a Word file:

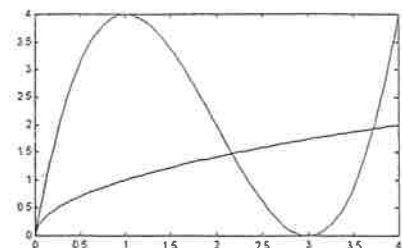
1. open a Word document
2. in the graphics window click on Edit, then Copy Figure.
3. in the Word file put the cursor on the correct point and then click on Paste in the Edit menu (or type Ctrl V). (To get a white, rather than grey, background use Edit Copy Options first).

2.5 Multiplots and zooming

We will draw the graphs of $y = x^3 - 6x^2 + 9x$ and $y = \sqrt{x}$, and find their point of intersection closest to $x = 2$ correct to 2 decimal places (2D).

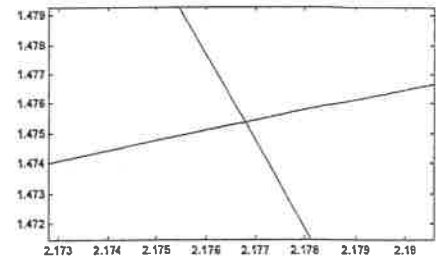
As before, plot both graphs together with the commands

```
>> clf
>> fplot('x^3-6*x^2+9*x', [0,4])
>> hold
>> fplot('sqrt(x)', [0,4])
```



From the graphs, the curves intersect near $x = 2.2$, $y = 1.4$.

On the toolbar in the graphing window click on the zoom-in icon (third from the right). Use the mouse to move the arrow close to the point of intersection and click the left mouse to zoom in by a factor of 2. Repeat until the graph shown alongside is obtained. (You can zoom out by a factor of 2 by first clicking the second icon from the right, the zoom-out icon, on the toolbar.)



Looking at the scales on the axes, the x coordinate of the point of intersection lies between 2.176 and 2.177 and so is 2.18 correct to 2D. Similarly, the y coordinate is 1.48 to 2D.

Hence $y = x^3 - 6x^2 + 9x$ and $y = \sqrt{x}$, intersect at (2.18, 1.48) correct to 2D.

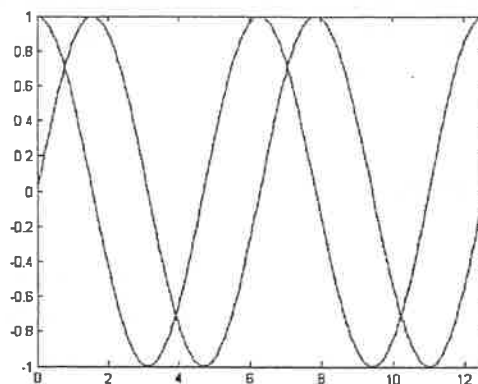
2.6 Improving the look

The presentation of graphs can be improved using different styles and colours for lines, adding a title to the graph, labeling the axes and using a legend to distinguish between graphs.

The use of these is illustrated in the following example.

We will plot $y = \sin x$ and $y = \cos x$ over two periods, from $x = 0$ to $x = 4\pi$.

```
>> clf
>> fplot('sin(x)', [0, 4*pi])
>> hold
Current plot held
>> fplot('cos(x)', [0, 4*pi])
```



The default is to plot solid lines. However, there is another argument in the *fplot* command to enable you to choose the line style and colour if you wish. If the sine and cosine graphs were to be printed, different styles would enable you to distinguish between them more easily.

We can specify the colour and style as follows

colour		style	
blue	<i>b</i>	-	solid
cyan	<i>c</i>	--	dashed
green	<i>g</i>	.	point
black	<i>k</i>	:	dotted
magenta	<i>m</i>	+	plus
red	<i>r</i>	x	cross
yellow	<i>y</i>	o	circle

Your choice is typed after the range for x , between single quotes.

For example,

```
fplot('2*x-1', [-2,6], 'm-')
```

plots $y = 2x - 1$ from -2 to 6
with magenta solid lines

```
fplot('sin(x)', [-pi,3*pi], 'bo')
```

plots $y = \sin x$ from $-\pi$ to 3π
with blue circles

```
fplot('x^2+1', [0,2], 'kx-')
```

plots $y = x^2 + 1$ from 0 to 2 with
black crosses joined with solid lines

Plot the sine and cosine curves with

```
>> fplot('sin(x)', [0,4*pi], 'r-')
```

```
>> hold
```

Current plot held

```
>> fplot('cos(x)', [0,4*pi], 'bx')
```

We will now add a title, labels for the axes and a legend.

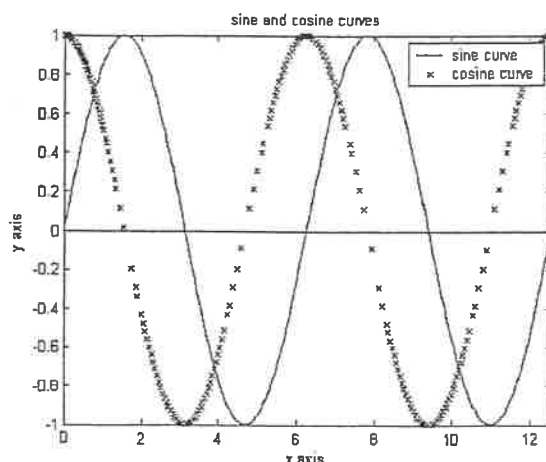
Title: Go to the Insert menu in the graphing window, click on Title and type "sine and cosine curves".

Labels: Click on Insert, then X label and type "x axis", followed by Insert, then Y label and type "y axis".

Legend: Click on Insert, then Legend. Double click on "data 1" and replace with "sin x". Similarly replace "data 2" with "cos x".

It is also useful to draw the x and y axes. Do this by clicking on Insert in the toolbar, then Line. Position the cursor $+$ at the origin, hold the left mouse down, move the cursor horizontally to the right hand side of the graph, and release to draw the x axis. You could repeat for the y axis (not necessary here).

Your final graph should be similar to:



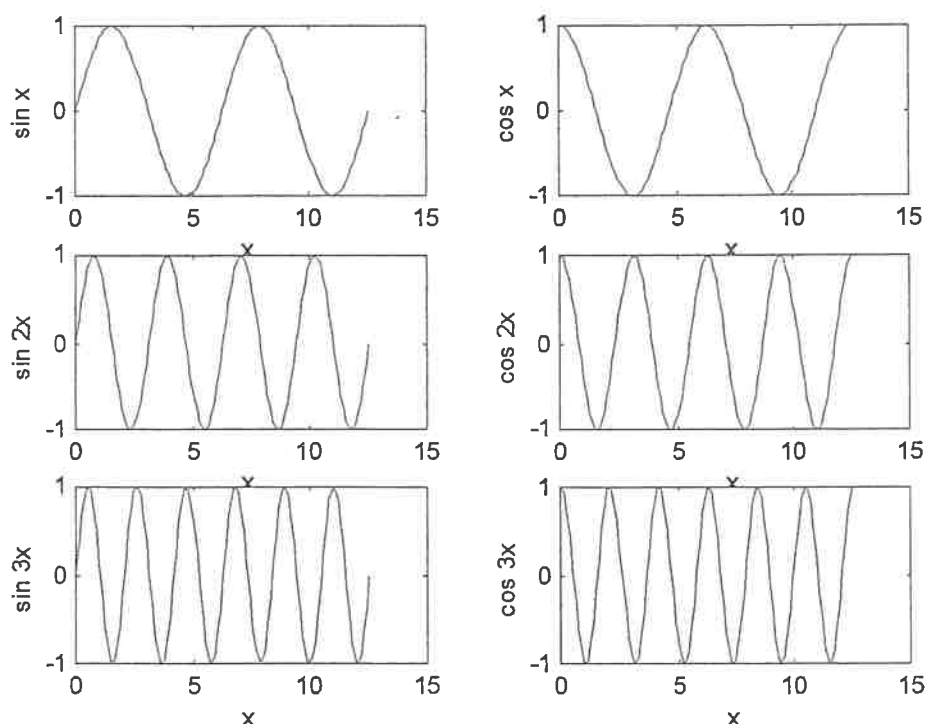
2.7 Subplot

The graphics window can be split into a rectangular array of smaller windows into each of which one or more graphs can be plotted.

The windows are counted in rows from the top left. For example, if we wanted 3 rows of 2 windows, the first row would be windows 1, 2, the second row windows 3, 4 and the last row windows 5, 6. The subplot command takes the form *subplot(abc)*, where *a* is the number of rows, *b* the number in each row and *c* the window number. Hence *subplot(325)* refers to window 5 of an array of 6 windows in 3 rows and 2 columns.

The following commands plot the graphs of $y = \sin x$, $y = \cos x$, $y = \sin 2x$, $y = \cos 2x$, $y = \sin 3x$ and $y = \cos 3x$ from $x = 0$ to $x = 4\pi$.

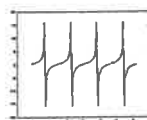
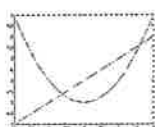
```
>> subplot(321),fplot('sin(x)',[0,4*pi])
>> % click on Insert, X label and type 'x', then
>> % Insert, Y label and type 'sin x'
>> subplot(322),fplot('cos(x)',[0,4*pi])
>> % add labels 'x' and 'cos x'
>> subplot(323),fplot('sin(2*x)',[0,4*pi])
>> % add labels 'x' and 'sin 2x'
>> subplot(324),fplot('cos(2*x)',[0,4*pi])
>> % add labels 'x' and 'cos 2x'
>> subplot(325),fplot('sin(3*x)',[0,4*pi])
>> % add labels 'x' and 'sin 3x'
>> subplot(326),fplot('cos(3*x)',[0,4*pi])
>> % add labels 'x' and 'cos 3x'
```



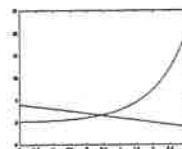
Exercise 2

1. Plot straight line segments for
- | x | 0 | 1 | 2 | 3 |
|-----|-----|-----|-----|-----|
| y | 0.8 | 3.2 | 4.9 | 7.1 |

2. Plot, in separate windows, the graphs of
- $x+1$ and x^2 from $x=-2$ to $x=2$
 - $\ln x$ from $x=1$ to $x=10$
 - $\tan x$ from $x=-2\pi$ to $x=2\pi$.



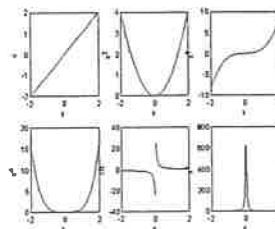
3. Plot, in the same window, e^x and $2-x$.
Use zoom to show that the point of intersection, correct to 3D, is (0.443, 1.557).



4. Plot on the same graph, from $x=-2\pi$ to $x=4\pi$, $y = \cos x$, $y = 1 + \cos x$, $y = 1.5 \cos x$, $y = \cos(x + \frac{\pi}{4})$ and $y = \cos 2x$. Use a legend and different styles or colours to distinguish them.

5. Plot on the same graph, from $x=-2\pi$ to $x=4\pi$, $y = \sin x$, $y = \sin x - 1$, $y = 1.5 \sin x$, $y = \sin(x - \frac{\pi}{2})$ and $y = \sin 2x$. Use a legend and different styles or colours to distinguish them.

6. Use *subplot* to plot x , x^2 , x^3 , x^4 , $\frac{1}{x}$ and $\frac{1}{x^2}$ from $x=-3$ to $x=3$.



7. Plot
- | x | 1 | 2 | 3 | 4 | 5 |
|-----|-----|-----|-----|-----|-----|
| y | 6.5 | 2.7 | 1.9 | 2.9 | 6.2 |
- and show that $y = x^2 - 6x + 11$ is a good approximation.

8. Plot $y = x^3 - 6x^2 + 13$ and $y = 0$. Confirm that the equation $x^3 - 6x^2 + 13 = 0$ has three roots and that the largest root is 5.58, correct to 2D.

9. Plot, in the same window with different styles and colours, and a legend, the graphs of 1.5^x , 2^x , 3^x , 4^x and e^x . Show that e^x lies between 2^x and 3^x and very close to 2.7^x .
10. Show that the graphs of $\sin x + \cos x$, $\sqrt{2} \sin(x + \frac{\pi}{4})$ and $\sqrt{2} \cos(x - \frac{\pi}{4})$ are identical. Find the smallest positive angle that satisfies the equation $\sin x + \cos x = 0.7$ and show that, to the nearest degree, this is 105° .
11. For the data in question 1, show that the line $y = 2x + 1$ lies reasonably close to the points.
12. The temperature y of a cup of coffee at time t was taken:
- | | | | | | |
|-----------------|-----|----|----|----|----|
| t (minutes) | 0 | 5 | 10 | 20 | 30 |
| y (degrees C) | 100 | 74 | 56 | 36 | 27 |
- Plot these points and show that $y = 20 + 80e^{-0.08t}$ is a good approximation to them.

3. Row and column vectors, and graph plotting

In this section we will see how to tabulate functions in order to draw their graphs. To do this we need to use new procedures called array multiplication, array division and array power.

A row vector is a list of numbers in a row.

A column vector is a list of numbers in a column.

Row and column vectors are special forms of matrices. See Section 4.

$a = [1\ 2\ 6\ 4\ 9]$ is a row vector and $b = \begin{bmatrix} 0 \\ 6 \\ 2 \\ 1 \end{bmatrix}$ is a column vector.

3.1 Entering vectors

Enter a and b as follows

```
» clear all
» a=[1 2 6 4 9],b=[0;6;2;1]
a =
     1     2     6     4     9
b =
     0
     6
     2
     1
```

Type a space between numbers in a row. Rows are separated by semi-colons.

A particular element of a vector can be referred to. For example

```
>> a(3)      % a(3) is the third element of a
ans =
     6
```

We can manipulate the elements. For example,

```
>> x=a(1)+a(4),y=a(2)*a(5)/a(3)
x =
     5
y =
     3
```

We can also have a text vector, and refer to any of them individually. Note the curly, rather than square, brackets.

```
>> names={'ken' 'fred' 'alan' 'jackie' 'chrissie'}
names =
'ken'    'fred'    'alan'    'jackie'    'chrissie'
>> names(4)
ans =
'jackie'
```

3.2 Length of a vector

The number of entries is known as the length of the vector.

```
» length(a),length(b)
```

```
ans =  
      5  
ans =  
      4
```

3.3 Arithmetic operations

Row and column vectors can be multiplied by a scalar, and two or more vectors of the same length can be added or subtracted.

```
» 2*a,10*b
```

```
ans =  
      2      4     12      8     18  
ans =  
      0  
      60  
      20  
      10
```

Note that each element of a has been multiplied by 2, and each element of b by 10.

```
» c=[7 -1 0 2 6];a+c,2*a-3*c
```

```
ans =  
      8      1      6      6     15  
ans =  
    -19      7     12      2      0
```

Elements can be sorted into ascending order:

```
» a1=sort(a),c1=sort(c)
```

```
a1 =  
      1      2      4      6      9  
c1 =  
     -1      0      2      6      7
```

3.4 Colon notation

This is a short cut for producing row or column vectors whose elements increase uniformly.

```
» d=[1:5],e=[-2:4]      % the square brackets are optional
```

```
d =  
      1      2      3      4      5  
e =  
     -2     -1      0      1      2      3      4
```

The elements in d and e go up in steps (increments) of 1. We can have any increment we wish.

```
» f=[2:0.5:5],g=[10:-1:0]
% the elements of f start at 2 and increase by 0.5 up to
5
% the elements of g decrease from 10 to 0 in steps of -1
f =
    2.0000    2.5000    3.0000    3.5000    4.0000
    4.5000    5.0000

g =
    10     9     8     7     6     5     4     3     2
     1     0
```

The general form is $[a:b:c]$. The values start at a and increase by b until c is reached.

3.5 Transpose of a vector

Transposing a row vector gives a column vector with the same elements.

Similarly, transposing a column vector gives a row vector.

The transpose of a is called a' . Use the single quotes on the same key as @.

```
» h=[1 4 9];k=[6;8;1];
» h,h',k,k'
h =
     1     4     9
ans =
     1
     4
     9
k =
     6
     8
     1
ans =
     6     8     1
```

3.6 Array product, division and power

If a and b are row (or column) vectors of the same length then the array product is $a.*b$.

The elements of $a.*b$ are the products of the corresponding elements of a and b .

```
» a=[1 4 12 7];b=[1 2 3 -1];
» a.*b
ans =
     1     8    36    -7
```

Similarly, the array division $a./b$ gives element by element division.

```
» a./b
ans =
     1     2     4    -7
```

Array power raises each element to the power given.

```
» a.^2,b.^3
ans =
     1    16   144    49
ans =
     1     8    27    -1
```

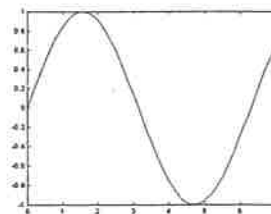
3.7 Tabulating functions and graph plotting

To tabulate $y = 4x - 2$ for values of x from -2 to 4 , calculate the x values as a column vector and then the corresponding y values.

```
» x=[-2:4]';y=4*x-2
y =
    -10
     -6
     -2
      2
      6
     10
     14
```

To draw the graph of $y = \sin x$ from $x = 0$ to $x = 7$ we first need to calculate y at a large number of values of x (see Section 2.2). If we want to have an increment of 0.1 there will be 70 x and 70 y values. There is no need to print these so use a semi-colon:

```
» x=[0:0.1:7]';y=sin(x);
» plot(x,y)
```



Suppose that we wish to obtain a table of values of $y = xe^x$ for $x = 0, 0.25, 0.5, \dots, 1$. To find y we must multiply each element of the vector x by the corresponding element of the vector e^x .

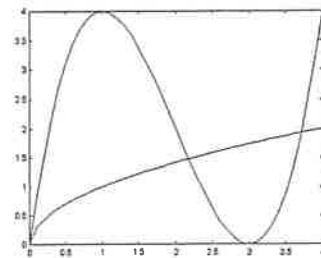
x	$\times$	e^x	$=$	xe^x
0	$\times$	1	$=$	0
0.25	$\times$	1.2840	$=$	0.3210
0.5	$\times$	1.6487	$=$	0.8244
0.75	$\times$	2.1170	$=$	1.5878
1	$\times$	2.7183	$=$	2.7183

In MATLAB we must use array multiplication.

```
» x=[0:0.25:1]';
» y=x.*exp(x)
y =
    0
    0.3210
    0.8244
    1.5878
    2.7183
```

To plot of $y = x^3 - 6x^2 + 9x$ and $y = \sqrt{x}$ from $x = 0$ to $x = 4$ we need array power to obtain x^3 and x^2 .

```
» x=[0:0.1:4];
» y1=x.^3-6*x.^2+9*x;
» y2=sqrt(x);
» plot(x,y1),hold,plot(x,y2)
Current plot held
```



These graphs were also produced in Section 2.3.

Exercise 3

- Enter $a = [6 \ -2 \ 1 \ 3]$, $b = [20 \ 4 \ -1 \ 7]$ and $c = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}$.
Find $3b$, $4c$ and $5a - 2b$.
Sort a and c into ascending order.
- Enter $d = [4 \ 5 \ 6 \ 7 \ 8]$, $e = [2 \ 2.2 \ 2.4 \ \dots \ 4]$ and $f = [100 \ 95 \ 90 \ \dots \ 50]$
- Find the vector whose elements are the

(i) products of those of a and b	(ii) quotients of those of a and b
(iii) quotients of those of b and a	(iv) the squares of those of d
(v) the cubes of those of e .	
(vi) twice the cube minus the square of those of f .	
- Tabulate $y = x + 1 - \cos x$ for values of x from 0 to 4 in steps of 0.5.
- Plot the graphs of

(i) $y = x \cos x$	from $x = 0$ to $x = 100$	
(ii) $y = e^{-x} \sin 10x$	from $x = 0$ to $x = 5$.	

4. Solution of equations in one variable

In this section we will see how MATLAB solves equations in one unknown, which is usually called x . Examples of such equations are

$$\begin{array}{lll} (1) & 2x - 5 = 7 & (2) \quad x^2 - 3x + 2 = 0 \quad (3) \quad 3x^2 = 1 - 5x \\ (4) & x + \frac{1}{x} = 3 & (5) \quad \sin t = 2 - t \quad (6) \quad y = 3 \ln(5x - 1) \end{array}$$

The command we will use is *solve*.

Algebraic or symbolic calculations are carried out using the Symbolic Maths Toolbox. We must first declare all variables that will be used as symbolic variables using the command *syms*.

The equation must be written in the form $f(x) = 0$. Of the equations above, (2) is already in this form but (1) and (3) must be written $2x - 5 - 7 = 0$ and $3x^2 + 5x - 1 = 0$ respectively. Similarly for the others.

Example 1

We will solve the equations (1), (2) and (3) above.

```
>> syms x                % defines x as a symbolic variable
>> solve(2*x-5-7)        % notice no " =0 "
ans =
6
>> solve(x^2-3*x+2)
ans =
[ 1]
[ 2]
>> solve(3*x^2+5*x-1)
ans =
[ -5/6+1/6*37^(1/2) ]
[ -5/6-1/6*37^(1/2) ]
```

So the equation	$2x - 5 = 7$	has solution $x = 6$
	$x^2 - 3x + 2 = 0$	has solutions $x = 1$ and $x = 2$
	$3x^2 + 5x - 1 = 0$	has solutions $x = \frac{-5 \pm \sqrt{37}}{6}$

The solution to the last equation is given exactly, but we can use the command *double* to put it in decimal form.

```
>> double(ans)
ans =
0.1805
-1.8471
```

The results are given as the elements of a vector. If we want to refer to the solutions individually in further work we can do so. To call the roots x_1 and x_2 :

```
ans =
    0.1805
   -1.8471
>> x1=ans(1),x2=ans(2)
x1 =
    0.1805
x2 =
   -1.8471
```

Example 2

In Section 2.2 we solved, graphically, the equation $x^3 - 6x^2 + 9x = \sqrt{x}$. We can now do it with *solve*.

```
>> solve(x^3-6*x^2+9*x-sqrt(x))
ans =
[ .12554518059198278430819923790695e-1]
[ 2.1767162463141172673883633905468]
[ 3.7200502387885013371644022991649]
[ 0]
```

MATLAB has given us all four answers correct to 32 significant figures. The one we found previously was the second of these and rounds to 2.18 correct to 2D, as before.

Example 3

This example illustrates how MATLAB can solve an equation symbolically.

The solution of $x^2 + bx + 3 = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 12}}{2}$. In MATLAB:

```
>> syms x b % space between x and b
>> solve(x^2+b*x+3)
ans =
[ -1/2*b+1/2*(b^2-12)^(1/2) ]
[ -1/2*b-1/2*(b^2-12)^(1/2) ]
```

Example 4

Solve the equations $x + \frac{1}{x} = 3$ and $\sin t = 2 - t$.

```
>> solve(x+1/x-3)
ans =
[ 3/2+1/2*5^(1/2) ]
[ 3/2-1/2*5^(1/2) ]
>> double(ans)
```

```

ans =
    2.6180
    0.3820
>> syms t
>> double(solve(sin(t)-2+t)) % commands can be nested
ans =
    1.1061

```

The solutions of $x + \frac{1}{x} = 3$ are $x = 2.6180, 0.3820$.

The solution of $\sin t = 2 - t$ is $t = 1.1061$.

Example 5

This example illustrates how we can solve for one variable in terms of another (rearrange an equation).

Rearrange the equations $y = 3x - 9$, $y = 2\ln(5x - 4)$ and $y = e^{3x} + 1$ as $x = \dots$

```

>> syms x1 x2 x3 y
>> x1=solve(y-3*x1+9) , x2=solve(y-2*log(5*x2-4)) ,
>> x3=solve(y-exp(3*x3)-1)
x1 =
1/3*y+3
x=2
1/5*exp(1/2*y)+4/5
x=3
1/3*log(y-1)

```

Exercise 4

1. Solve the following equations.

$$\begin{array}{lll}
 \text{(i)} & 1 + 5x = 2x + 16 & \text{(ii)} \quad x^2 - 5x + 6 = 0 \quad \text{(iii)} \quad 2x^2 - 7x + 2 = 0 \quad \text{(iv)} \\
 e^{x^2-4} = 100 & \text{(v)} \quad \sin x = 1 - x & \text{(vi)} \quad e^x = 8 - 3x
 \end{array}$$

2. Solve the following equations in terms of α .

$$\text{(i)} \quad x^3 + 4\alpha x^2 + x = 0 \quad \text{(ii)} \quad \sin(\alpha x) = 0.5$$

3. Rearrange the following equations for $x = \dots$

$$\begin{array}{ll}
 \text{(i)} & y = 2\cos\left(x + \frac{\pi}{2}\right) \quad \text{(ii)} \quad y = \tan^{-1}(3x - 1) \\
 \text{(iii)} & y = 20 + 80e^{-0.02x} \quad \text{(iv)} \quad y = 2x^2 + x^4
 \end{array}$$

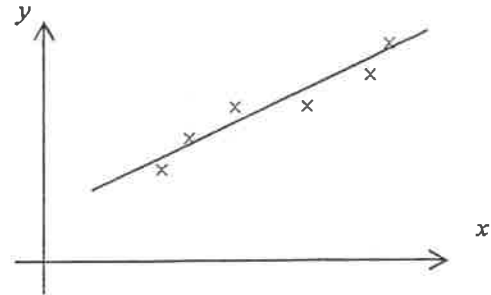
4. (i) Show that the sum of the two roots of the equation $x^2 = x + 1$ is 1.
(ii) Show that the sum of the three roots of the equation $x^3 - 6x^2 + 13 = 0$ is 6 and the product is -13 .

5. Least Squares Linear Approximations

Given a set of experimental (x, y) values, we want to find the best straight line that approximates the data.

That is find m and c so that the line $y = mx + c$ fits the points as well as possible.

The technique used to find m and c is called Least Squares.



Example 1

Find the least squares straight line approximation to

x	0.5	1.3	2.5	3.8
y	1.8	4.1	4.6	7.9

1. input the x and y values as row vectors

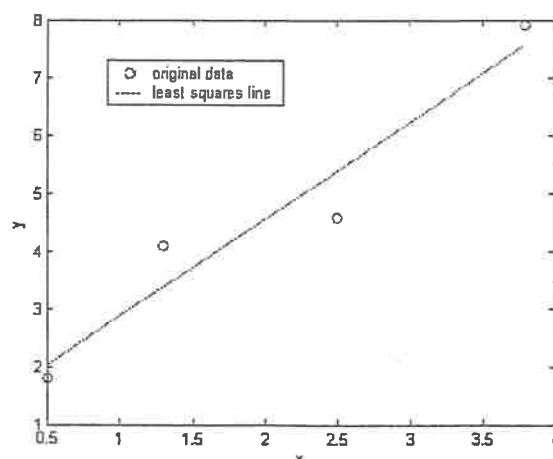
```
>> x=[0.5 1.3 2.5 3.8];  
>> y=[1.8 4.1 4.6 7.9];
```
2. Use the command *polyfit* to find m and c . The command below finds m and c as the elements of a row vector p .

```
>> p=polyfit(x,y,1)  
p =  
    1.6845    1.1890
```

so the equation of the line is $y = 1.6845x + 1.1890$.

3. Use *polyval* to calculate the y values according to the least squares line (called *yfit*) and draw the graphs of the original points and the line.

```
>> yfit=polyval(p,x)  
yfit =  
    2.0312    3.3788    5.4001    7.5899  
>> plot(x,y,'o',x,yfit)  
>> % use Insert menu for labels and legend
```



4. We can use the line to estimate y at different values of x .

```
>> y1=polyval(p,2)      % finds y at x=2
y1 =
    4.5579
>> y2=polyval(p,3.2)    % finds y at x=3.2
y2 =
    6.5792
so y(2) ≈ 4.5579 and y(3.2) ≈ 6.5792.
```

5. We can compare the actual and computed values of y :

```
>> diff=y-yfit
diff =
   -0.2312    0.7212   -0.8001    0.3101
```

6. We can use the line to estimate x for a given y value.

```
>> syms x
>> solve(1.6845*x+1.1890-6) % to find x when y=6
ans =
    9622/3369
>> double(ans)
ans =
    2.8560
```

so when $y = 6$, $x = 2.8560$.

If the equation connecting x and y is not linear, it must first be converted to straight line form. The next example illustrates this.

Example 2

The temperature θ °C at time t minutes of a cup of coffee above room temperature is given by the formula $\theta = ae^{bt}$. θ was measured at different times t .

t	0	5	10	20	30
θ	80	50	30	10	5

Find a and b using least squares.

Take logs to convert $\theta = ae^{bt}$ to straight line form.

Then $\ln \theta = \ln a + \ln(e^{bt}) = \ln a + bt = bt + \ln a$ and so the graph of $\ln \theta$ against t will be a straight line.

Comparing $\ln \theta = bt + \ln a$ with $y = mx + c$ we need $y = \ln \theta$ and $x = t$.

The gradient of the least squares line is $m = b$ and the intercept $c = \ln a \Rightarrow a = e^c$.

```

>> % input given data and prepare new x y data
>> % original data
>> t=[0 5 10 20 30];
>> theta=[80 50 30 10 5];

>> % new data
>> x=t % no change in t needed
x =
     0     5    10    20    30
>> y=log(theta)
y =
     4.3820     3.9120     3.4012     2.3026     1.6094

>> % find least squares line
>> p=polyfit(x,y,1)
p =
    -0.0948     4.3539
>> % p=[p(1) p(2)]=[c m]

>> % find a and b
>> b=p(1),a=exp(p(2))
b =
    -0.0948
a =
    77.7839

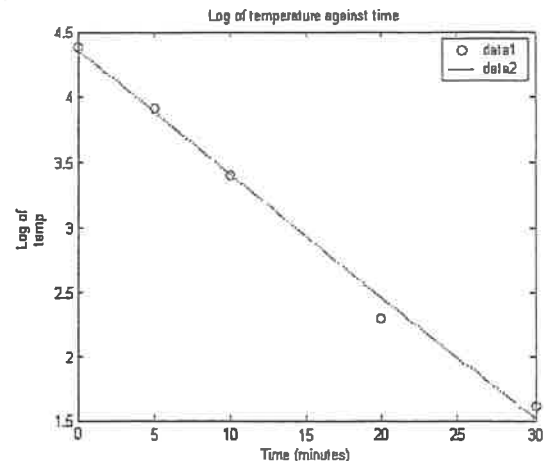
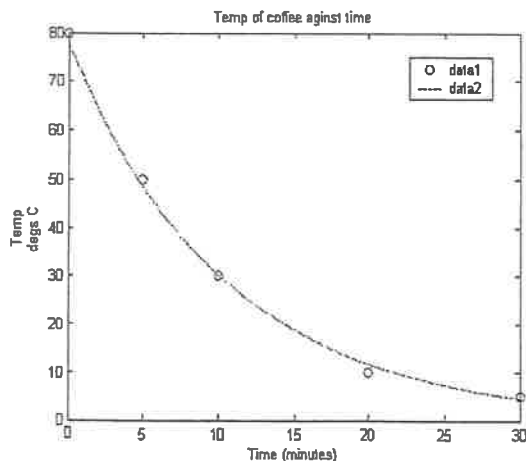
```

Rounding to 2S, the equation is $\theta = 78e^{-0.095t}$

```

>> % plot data and theta against t
>> % to get smooth curve define t1 (300 points)
>> t1=0:0.1:30;
>> plot(t,theta,'o',t1,a*exp(b*t1))
>> % see left graph below
>> % plot straight line forms of data and curve
>> plot(t,log(theta),'o',t1,p(1)*t1+p(2))
>> % see right hand graph below

```



Exercise 5

1. The table gives the speed v of a car at time t .

t	1	2	3	4	5
v	33	38	42	46	49

v and t are related by the equation $v = at + b$, where a and b are constants.

- Input the data and use *polyfit* to find the best fit values of a and b .
[$v = 29.6 + 4t$]
- Use *polyval* to find the estimated values of v .
- Draw a graph showing the data and the least squares straight line.

2. The table gives the distance s travelled by the car in question 1 in terms of the time t taken.

t	1	2	3	4	5
s	32	67	107	150	198

s and t are related by the equation $s = at + bt^2$, where a and b are constants.

- Input the data and plot s against t . You will not get a straight line as $s = at + bt^2$ is a parabola.
- Divide by t to convert the equation to straight line form.
Then $\frac{s}{t} = a + bt$. This is of the form $y = mx + c$.
- Find the transformed data using $x = t$ and $y = \frac{s}{t}$ (remember to use array division, that is type $s./t$) and use *polyfit* to find a and b .
[$s = 29.89t + 1.92t^2$]
- Use *polyval* to find the estimated values of y and s (*yfit* and *sfit*) and plot
 - the straight line forms of data and curve.
 - the actual data and fitted parabola.

3. The surface area s of rust on a metal plate at time t was measured.

t	2	3	4	5	6	7	8
s	2.34	2.75	3.16	3.63	4.16	4.79	5.62

s and t are related by $s = ae^{bt}$. Estimate a and b and draw suitable graphs of the data and approximation.

6. Matrices

6.1 Entering matrices

Enter the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ as follows (and remember that A and a are different!)

```
» clear all
» A=[3 5;1 2]
A =
     3     5
     1     2
```

Type a space after each number. Rows are separated by semi-colons.

In the same way enter $B = \begin{bmatrix} 1 & 6 & 2 \\ 5 & 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$.

```
» B=[1 6 2;5 3 4],C=[2 -1 3;0 4 2]
B =
     1     6     2
     5     3     4
C =
     2    -1     3
     0     4     2
```

A has 2 rows and 2 columns and so its size is (2×2) . B and C are of size (2×3) .

This can be obtained using the *size* command:

```
» size(A),size(B)
ans =
     2     2
ans =
     2     3
```

If we need to, we can refer to a particular element of a matrix. For example, to obtain the element in the first row and second column of A enter

```
» A(1,2)
ans =
     5
```

To add the element in the second row, third column of B to this:

```
» A(1,2)+B(2,3)
ans =
     9
```

6.2 Addition and Subtraction

Find $B + C$ and $B - C$:

```
» B+C,B-C
ans =
     3     5     5
     5     7     6
ans =
    -1     7    -1
     5    -1     2
```

Check that the answers are as you would expect.

Matrices can only be added or subtracted if they are of the same size, so

```
» A+B
??? Error using ==> +
Matrix dimensions must agree.
```

6.3 Multiplication of a matrix by a scalar

The elements of $2A$ are all twice the elements of A .

```
» A,TWOTIMESA=2*A
A =
     3     5
     1     2
TWOTIMESA =
     6    10
     2     4
```

We can find $2B + 3C$ and $5B - 4C$ as follows:

```
» 2*B+3*C,5*B-4*C
ans =
     8     9    13
    10    18    14
ans =
    -3    34    -2
    25    -1    12
```

6.4 Multiplication of matrices

Matrices can be multiplied when the number of columns in the first matrix is the same as the number of rows in the second matrix, so the only products between A , B and C that we can find are AB and AC .

```
» A*B,A*C
ans =
    28    33    26
    11    12    10
ans =
     6    17    19
     2     7     7
```

If you try any other combination an error message is given:

```
» B*C
??? Error using ==> *
Inner matrix dimensions must agree.
```

Find $Q = FG$ and $R = GF$, where $F = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $G = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$.

```
» F=[1 3;2 4];G=[2 5;1 4];Q=F*G,R=G*F
Q =
     5    17
     8    26
R =
    12    26
     9    19
```

This illustrates the fact that even when we can multiply two matrices in either order, the result may not be the same.

We can find powers of square matrices. To find Q^2 and R^3 :

```
» FSQ=F^2,GCUB=G^3
FSQ =
     7    15
    10    22
GCUB =
    48    165
    33    114
```

6.5 Transpose of a matrix

To transposes of A and C are usually called A^T and C^T , but A' and C' in MATLAB. Use the single quotes on the same key as @:

```
» A',C'
ans =
     3     1
     5     2
ans =
     2     0
    -1     4
     3     2
```

6.6 Diagonal and Unit matrices

To input the **diagonal** matrix $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ we can proceed as in Section 4.1 or,

more conveniently:

```
» H=diag([1,2,3])
H =
     1     0     0
     0     2     0
     0     0     3
```

To input the **unit** (identity) matrices $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ either

use *diag* or

```
» I2=eye(2), I3=eye(3)
I2 =
     1     0
     0     1
I3 =
     1     0     0
     0     1     0
     0     0     1
```

Multiplying a matrix by a unit matrix leaves it unchanged.

```
» A,A*I2,I2*A,B,I2*B,B*I3
A =
     3     5
     1     2
```

```

ans =
     3     5
     1     2
ans =
     3     5
     1     2
B =
     1     6     2
     5     3     4
ans =
     1     6     2
     5     3     4
ans =
     1     6     2
     5     3     4

```

(What would happen if we tried to find $B \times I_2$ or $I_3 \times B$?)

6.7 The determinant of a square matrix

This is found using *det*. The determinants of $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 3 & -4 & 2 \\ 2 & -7 & 9 \\ 4 & -2 & -3 \end{bmatrix}$ are

```

» P=[3 -4 2;2 -7 9;4 -2 -3];
» detA=det(A),detP=det(P)
detA =
     1
detP =
    -3

```

You can only find the determinant of a square matrix:

```

» det(B)
??? Error using ==> det
Matrix must be square.

```

6.8 The inverse of a square matrix (A^{-1})

This is found using *inv*.

To find the inverses of A and P and verify that $A \times A^{-1} = I_2$ and $P \times P^{-1} = I_3$.

```

» IA=inv(A),A*IA,IP=inv(P),P*IP
IA =
     2.0000    -5.0000
    -1.0000     3.0000
ans =
     1.0000         0
    -0.0000     1.0000

```

```

IP =
   -13.0000    5.3333    7.3333
   -14.0000    5.6667    7.6667
    -8.0000    3.3333    4.3333
ans =
    1.0000    0.0000    0
         0    1.0000    0.0000
         0         0    1.0000

```

6.9 Solving systems of linear equations

The equations $4x + y = 13$
 $6x + 2y = 20$ can be written in matrix form as

$$\begin{bmatrix} 4 & 1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \end{bmatrix}, \text{ or } AX = B.$$

The solution is $X = A^{-1}B$, so:

```

» A=[4 1;6 2];    % note that this overwrites the previous
A
» B=[13;20];
» solution=inv(A)*B
solution =
    3.0000
    1.0000

```

so the solution is $x = 3, y = 1$.

$$3x - 4y + 2z = 1$$

To find the solution of $2x - 7y + 9z = 15$ note that the matrix of coefficients is the

$$4x - 2y - 3z = -9$$

matrix P defined earlier, with inverse called IP , so

```

» solution=IP*[1;15;-9]
solution =
    1.0000
    2.0000
    3.0000

```

so $x = 1, y = 2$ and $z = 3$.

MATLAB can solve these equations a second way, using backslash \:

```

» solution1=A\B,solution2=P\[1;15;-9]
solution1 =
    3.0000
    1.0000

```

```
solution2 =
1
2
3
```

The second method, using \, is more efficient and should always be used for large sets of equations.

Exercise 6

1. Input the matrices $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 0 \\ -1 & 6 \\ 3 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 7 \\ 0 & 1 & 5 \end{bmatrix}$.

Find, if possible,

- (i) $A + B$, $2A - 3B$, $C - D$
 - (ii) AB , BA , AC , CB , DC
 - (iii) A^T , C^T
 - (iv) $\det A$, $\det C$ and $\det D$
 - (v) A^{-1} , B^{-1} , C^{-1} and D^{-1}
 - (vi) A^2 , B^3 and D^4 .
2. Find $AB - BA$. Is $AB = BA$?
3. Find $(A + B)^2$ and $A^2 + 2AB + B^2$. Are they equal?
Show that $(A + B)^2 = A^2 + AB + BA + B^2$.
4. Find $(AB)^2$ and A^2B^2 . Are they equal? Show that $(AB)^2 = ABAB$.
5. Show that $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1} A^{-1}$.
6. Show that
- (i) A satisfies the equation $A^2 - 4A + 3I = 0$
 - (ii) D satisfies the equation $A^3 - 6A^2 - 4A + 24I = 0$.
7. If $E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ show that $E^{-1} = E^T$ and $F^{-1} = F^T$.
 E and F are called orthogonal matrices.
8. If $Q = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ show that $(QDQ^{-1})^4 = QD^4Q^{-1} = \begin{bmatrix} -44 & 90 \\ -30 & 61 \end{bmatrix}$.
Can you explain this result?
9. Solve the following sets of equations by matrix inversion.

(i) $2x + y = 7$ $x + 2y = 2$	(ii) $2x + y + 3z = 7$ $-y + 7z = 9$ $y + 5z = 3$	(iii) $3x + 2y - 4z + w = 17$ $5x - 3y + 2z + 3w = 2$ $x + 4y + 5z + 3w = 12$ $2x + 5y + 6z - 4w = 9$
--	---	--

7. Calculus

MATLAB has commands in its Symbolic Maths Toolbox to perform differentiation, integration and other calculus operations. The engine behind the Symbolic Maths Toolbox is the computer package Maple.

When invoking calculus commands we must define the symbolic variables. For example, to start with a clean sheet of paper and use the variables x , y and t you should type

```
>> clear all
>> syms x y t % spaces, not commas, between x,y and t
```

7.1 Differentiation

Use the command *diff* to differentiate symbolic expressions once, twice or more.

Example 1

To differentiate x^2 :

```
>> diff(x^2)
```



```
ans =
```



```
2*x
```

so $\frac{d}{dx}(x^2) = 2x$.

Example 2

Differentiate $\sin x$ and $\ln x$.

```
>> diff(sin(x)), diff(log(x))
ans =
cos(x)
ans =
1/x
```

so $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Example 3

Find $\frac{dy}{dx}$ given (i) $y = x^3 \cos x$ (ii) $y = \frac{x^2 + 1}{x^2 - 1}$ (iii) $y = (2x + 1)^6$.

Note that, by hand, you would have to use the product rule for (i), the quotient rule for (ii) and the function of a function rule for (iii).

```
>> diff(x^3*cos(x))
ans =
3*x^2*cos(x) - x^3*sin(x)
>> diff((x^2+1)/(x^2-1))
ans =
2*x/(x^2-1) - 2*(x^2+1)/(x^2-1)^2*x
>> diff((2*x+1)^6)
ans =
12*(2*x+1)^5
```

We can also find higher derivatives by specifying the order in the *diff* command.

Example 4

Find (i) the second derivative of x^5 (ii) the third derivative of $\tan 2x$ (iii) the fourth derivative of x^n .

- (i)

```
>> diff(x^5,2)
ans =
    20*x^3
```
- (ii)

```
>> diff(tan(2*x),3)
ans =
    4*(2+2*tan(2*x)^2)^2+16*tan(2*x)^2*(2+2*tan(2*x)^2)
```
- (iii)

```
>> syms n
>> diff(x^n,4)
ans =
    x^n*n^4/x^4-6*x^n*n^3/x^4+11*x^n*n^2/x^4-6*x^n*n/x^4
```

We can factorize the answer with *factor*.

```
>> factor(ans)
ans =
    x^n*n*(n-1)*(n-2)*(n-3)/x^4
```

Example 5

Show that $y = \sin 2x$ satisfies the equation $\frac{d^2y}{dx^2} + 4y = 0$.

Here the first and second derivatives of y are called y_1 and y_2 .

```
>> y=sin(2*x);y1=diff(y),y2=diff(y,2)
y1 =
    2*cos(2*x)
y2 =
    -4*sin(2*x)
>> y2+4*y
ans =
    0
```

Example 6

If $y = e^{ax} \sin bx$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Show that y satisfies the equation $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0$.

Note the use of *simplify* to reduce a complicated result to zero.

```
>> syms x y a b
>> y=exp(a*x)*sin(b*x), y1=diff(y), y2=diff(y,2)
y =
exp(a*x)*sin(b*x)
y1 =
a*exp(a*x)*sin(b*x)+exp(a*x)*cos(b*x)*b
y2 =
a^2*exp(a*x)*sin(b*x)+2*a*exp(a*x)*cos(b*x)*b
-exp(a*x)*sin(b*x)*b^2

>> y2-2*a*y1+(a^2+b^2)*y      % LHS of equation
ans =
a^2*exp(a*x)*sin(b*x)+2*a*exp(a*x)*cos(b*x)*b
-exp(a*x)*sin(b*x)*b^2-2*a*(a*exp(a*x)*sin(b*x)+
exp(a*x)*cos(b*x)*b)+(a^2+b^2)*exp(a*x)*sin(b*x)

>> simplify(ans)
ans =
0
```

Exercise 7.1

- Find the first derivatives of x^7 , $x^2 \sin x$, $\frac{2x+1}{4x+1}$ and $\cos x^3$.
- Find the second derivative of $\sin 5x$, the third derivative of $(3x-2)^4$ and the fourth derivative of $\ln(1+e^x)$.
- Show that $y = \cos 4x$ satisfies the equations
 - $\frac{dy}{dx} + 4y \tan 4x = 0$
 - $\frac{d^2y}{dx^2} + 16y = 0$
 - $\frac{d^4y}{dx^4} - 256y = 0$.
- If $y = e^{ax} \cos bx$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
 Show that y satisfies the equation $\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0$.

7.2 Maximum and Minimum

The function $y = f(x)$ has a stationary point where $\frac{dy}{dx} = 0$.

If, at this point, $\frac{d^2y}{dx^2} < 0$ then the point is a local maximum.

If, at this point, $\frac{d^2y}{dx^2} > 0$ then the point is a local minimum.

If, at this point, $\frac{d^2y}{dx^2} = 0$ the test fails and the first derivative test must be used.

We will need the command *subs* to evaluate y at different values of x .

The form of *subs* is *subs(expression, old value, new value)*

For example, if $y = 2x - 1$, then to find y when $x = 3$:

```
>> y=subs(2*x-1,x,3)
y =
    5
```

Example 1

Find and classify the stationary points for $y = 2x^3 - 3x^2 - 6x + 10$.

```
>> statpts=solve(diff(2*x^3-3*x^2-6*x+10))
statpts =
    [ 1/2+1/2*5^(1/2)]
    [ 1/2-1/2*5^(1/2)]

>> % find the decimal form of these values
>> statpts=double(statpts)
statpts =
    1.6180
   -0.6180

>> % find the corresponding y values
>> y=subs(2*x^3-3*x^2-6*x+10,x,statpts)
y =
    0.9098
   12.0902
```

y has two stationary points, at (1.6180, 0.9098) and (-0.6180, 12.0902).

We will use the second derivative test to see what sort of point each is.

```
>> diff2=diff(2*x^3-3*x^2-6*x+10,2)
diff2 =
    12*x-6
>> yd2=subs(diff2,x,statpts)
```

$$\begin{aligned} y_{d2} &= \\ &13.4164 \\ &-13.4164 \end{aligned}$$

At $(1.6180, 0.9098)$ $\frac{d^2y}{dx^2} = 13.4164 > 0$ and so this point is a local minimum.

At $(-0.6180, 12.0902)$ $\frac{d^2y}{dx^2} = -13.4164 < 0$ and so this point is a local maximum.

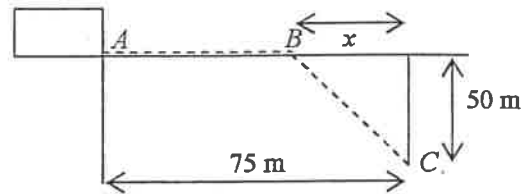
Example 2

In laying water to a building A the contractor finds he must connect to a point C on the main which lies under a car park.

The total cost, P , of laying the pipe is

$$P = 3600 - 48x + 80\sqrt{x^2 + 2500}$$

Find x so that P is a minimum and hence find the minimum cost of the work to the contractor.



```
>> syms x P
>> P=3600-48*x+80*sqrt(x^2+2500)
P =
    3600-48*x+80*(x^2+2500)^(1/2)

>> dP=diff(P)
dP =
    -48+80/(x^2+2500)^(1/2)*x

>> solve(dP)
ans =
    75/2
>> % stationary point at x=75/2=37.5

>> d2P=diff(P,2)
d2P =
    -80/(x^2+2500)^(3/2)*x^2+80/(x^2+2500)^(1/2)

>> d2Pst=subs(d2P,x,37.5)
d2Pst =
    0.8192

>> % 0.8192>0, so P is a min when x=37.5.
>> Pmin=subs(P,x,37.5)
Pmin =
    6800
```

So the lowest cost to the contractor is £6800, when $x = 37.5$ metres.

Exercise 7.2

1. Find and classify the stationary points of

(i) $y = x^2 + 4x + 5$

(ii) $y = x^3 - 3x^2 + 4$

(iii) $y = \frac{2x}{x^2 + 1}$

(iv) $y = \frac{2x-7}{(x-1)^2}$

(v) $y = x^4 e^x$

(vi) $y = e^{-x} \sin x$

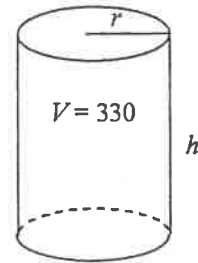
2. A standard drinks can holds 330 ml of liquid.

The total surface area S of the can is given by

$$S = 2\pi r^2 + \frac{660}{r} \text{ and the height } h \text{ by } h = \frac{330}{\pi r^2}.$$

Show that S is a minimum when $r \approx 3.745$

and $h \approx 7.49$ and that $S_{\min} \approx 266 \text{ cm}^2$.



7.3 Integration

MATLAB can compute definite and indefinite integrals.

Example 1

Find $\int \cos x \, dx$, $\int \frac{1}{x} \, dx$ and $\int x^2 \, dx$.

```
>> clear all;syms x
>> int(cos(x))
ans =
    sin(x)
>> % note no constant of integration

>> int(1/x)
ans =
    log(x)

>> int(x^2)
ans =
    1/3*x^3
```

Example 2

Find (i) $\int 6x\sqrt{1+x^2} \, dx$ (ii) $\int x^2 \sin x \, dx$

(By hand, (i) would be done by substitution and (ii) by parts.)

```
(i) >> int(6*x*sqrt(1+x^2))
ans =
    2*(1+x^2)^(3/2)
```

```
(ii) >> int(x^2*sin(x))
ans =
      -x^2*cos(x)+2*cos(x)+2*x*sin(x)
```

Example 3

Find the definite integrals (i) $\int_0^2 x^3 dx$ (ii) $\int_0^{\pi/2} \sin x dx$ (iii) $\int_1^2 e^x dx$.

The limits of integration are included in the *int* command.

```
>> int(x^3,0,2),int(sin(x),0,pi/2),int(exp(x),1,2)
ans =
      4
ans =
      1
ans =
      exp(2)-exp(1)

>> double(ans)      % to get decimal form of last answer
ans =
      4.6708
```

Example 4

Find (i) $\int_0^1 \frac{x}{1+x^2} dx$ (ii) $\int_0^{\pi/4} x^4 \cos 2x dx$.

```
>> double(int(x/(1+x^2),0,1))
ans =
      0.3466

>> double(int(x^4*cos(2*x),0,pi/4))
ans =
      0.0150
```

Example 5

Consider the curve $y = x^2$ from the point (0, 0) to (2, 4).

Find (i) the area under the curve
(ii) the volume V_1 of the solid formed when the curve is rotated about the x axis
(iii) the volume V_2 of the solid formed when the curve is rotated about the y axis

A , V_1 and V_2 are found from $A = \int_a^b y dx$, $V_1 = \pi \int_a^b y^2 dx$ and $V_2 = \pi \int_c^d x^2 dy$.

```

>> clear all
>> syms x y
>> % integrate y=x^2 to find A
>> A=int(x^2,0,2)
A =
      8/3
>> % integrate y=(x^2)^2=x^4 to find V1
>> V1=pi*int(x^4,0,2)
V1 =
    32/5*pi
>> % integrate x^2=y wrt y to find V2
>> V2=pi*int(y,0,4)
V2 =
      8*pi

>> % to obtain these exact answers in decimal form
>> A=double(A),V1=double(V1),V2=double(V2)
A =
    2.6667
V1 =
    20.1062
V2 =
    25.1327

```

Example 6

Consider $y = \frac{1}{1+x^3}$ from (0, 1) to (1, 0.5). Find A , V_1 and V_2 in decimal form.

(Note that the integrals here cannot be done analytically)

```

>> A=double(int(1/(1+x^3),0,1))
A =
    0.8356
>> V1=double(pi*int(1/(1+x^3)^2,0,1))
V1 =
    2.2738

>> % rearrange y=1/(1+x^3) for x=..
>> % limits for y are 1 and 0.5
>> % integrate x^2 =..... wrt y
>> V2=double(pi*int((1/y-1)^(2/3),0.5,1))
Warning: Explicit integral could not be found.
>In C:\MATLAB6p5\toolbox\symbolic\@sym\int.m atline 58
V2 =
    0.7763

```

Exercise 7.3

1. Find the following standard integrals

$$(i) \int x^4 dx \quad (ii) \int \tan x dx \quad (iii) \int_0^{\pi/6} \cos x dx \quad (iv) \int_1^2 \frac{1}{x} dx$$

2. Find the following integrals which, if done by hand, would need to be done using a substitution.

$$(i) \int 30x^2 (x^3 + 1)^4 dx \quad (ii) \int_0^1 x(1+x)^6 dx \quad (iii) \int \frac{\sqrt{x^2 - 4}}{x} dx \quad (iv) \int_0^1 \frac{x^3}{\sqrt{9+x^4}} dx$$

3. Find the following integrals which, if done by hand, would need to be done using partial fractions.

$$(i) \int \frac{x-8}{(x+1)(x-2)} dx \quad (ii) \int_1^3 \frac{2x^2-1}{x(x+1)^2} dx \quad (iii) \int_0^1 \frac{4x+3}{(x+2)(x^2+1)} dx$$

4. Find the following integrals which, if done by hand, would need to be done using integration by parts.

$$(i) \int_0^{\pi/4} x^5 \sin 2x dx \quad (ii) \int \sec^3 x dx \quad (iii) \int_1^e (\ln x)^2 dx$$

5. Find the area A under the curve and the volumes of revolution V_1 and V_2 for

$$(i) \quad y = \sqrt{x} \text{ from } x = 0 \text{ to } x = 4 \quad [5.33, 25.1, 8.34]$$

$$(ii) \quad y = \sqrt{\tan x} \text{ from } x = 0 \text{ to } x = \frac{\pi}{4}. \quad [0.4875, 1.0858, 0.9319]$$

6. A ball is thrown vertically upwards with speed 20 ms^{-1} . It experiences the forces due to gravity and air resistance. The greatest height h reached by the ball and the time t

$$\text{taken are given by } h = \int_0^{20} \frac{v}{10 + 0.01v^2} dv \quad \text{and} \quad t = \int_0^{20} \frac{1}{10 + 0.01v^2} dv.$$

Show that $h = 16.83$ and $t = 1.78$.

8. Statistics

MATLAB can produce different types of visual and numerical summaries of data. The visual summaries include pie charts, bar charts and histograms. The numerical summaries include median, mean, variance and standard deviation.

Note that MATLAB has many more options available than those illustrated in the examples below. See the Help file for details.

8.1 Pie charts

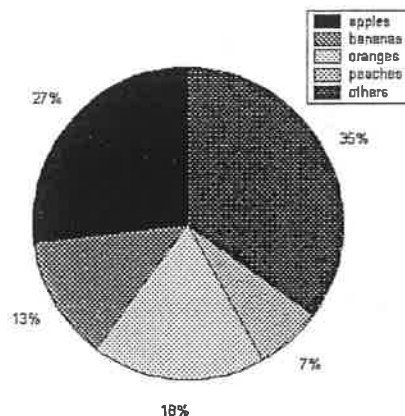
Example 1

The percentages of different types of fruit sold from a greengrocers were:

Apples	Bananas	Oranges	Peaches	Others
27%	13%	18%	7%	35%

To draw a pie chart for this data use the command *pie*,

```
>> % input data as a vector
>> g=[27 13 18 7 35];
>> pie(g)
>> % use Insert Legend in Figure No.1 window
```

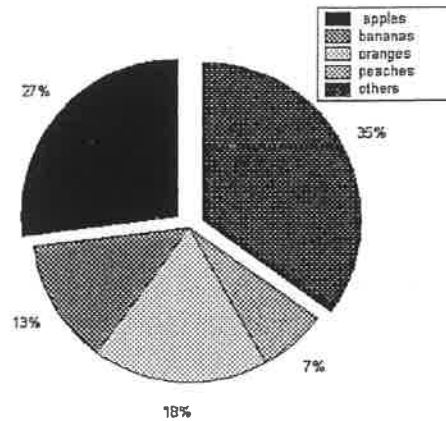


We can make sections of the pie stand out using *explode*. *explode* is a vector of 1's and 0's of the same length as the data. The elements of the data corresponding to the 1's will be *exploded*.

Example 2

Explode the two largest slices of the pie in Example 1.
These are the first and last elements of *g* and so

```
>> explode=[1 0 0 0 1];  
>> pie(g,explode)
```



8.2 Bar charts

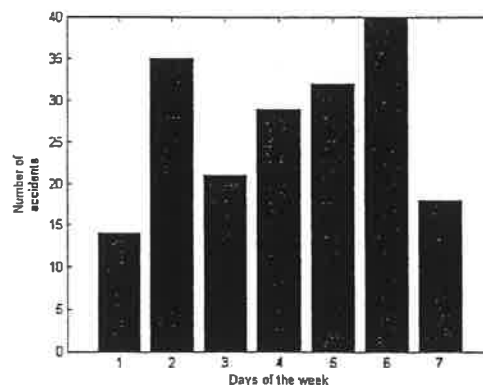
Example 3

The following frequency distribution shows the number of car accidents on one stretch of road on each day of the week over a year. Sunday = day1, Monday = day 2, . . . , Saturday = day 7.

Day <i>d</i>	1	2	3	4	5	6	7
No. of accidents <i>f</i>	14	35	21	29	32	40	18

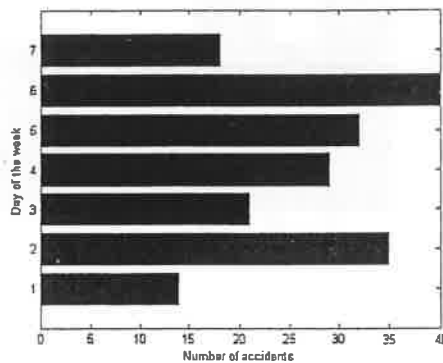
To draw a bar graph use the command *bar*.

```
>> % input days as elements of vector d  
>> d=[1 2 3 4 5 6 7];  
>> % input no. of accidents as vector f  
>> f=[14 35 21 29 32 40 18];  
>> % draw bar graph  
>> bar(d,f)
```



We can plot the bar chart horizontally if we wish:

```
>> barh(d,f)
```



8.3 Histograms from data

To draw histograms from data use the command *hist*.

The simplest form is $n = \text{hist}(y)$, where x is a vector of the data elements. *hist*(y) uses 10 class intervals. n is a vector containing the number of elements in each class.

More generally use the form $n = \text{hist}(y, k)$, where k is the number of classes required.

If a frequency distribution is required use $[n, xout] = \text{hist}(y, k)$. The elements of n are the frequencies and those of $xout$ the class mid points.

Example 4

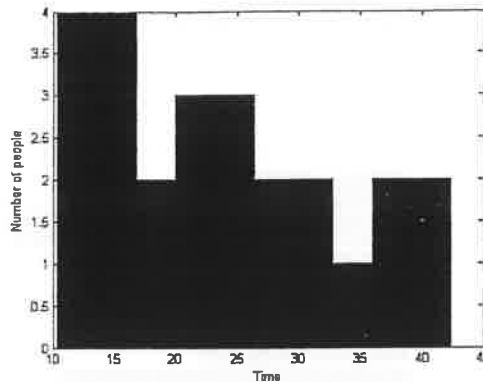
The table gives the time taken by 25 people to perform a particular task.

24.8	20.0	23.4	14.4	15.8	22.4	15.7	28.1	23.2	31.3
33.9	14.1	11.1	20.0	25.0	32.7	10.4	12.5	27.5	42.4
12.6	39.7	20.3	36.1	37.4					

Draw a histogram with (i) 10 (ii) 8 classes. Tabulate the class mid points in (ii).

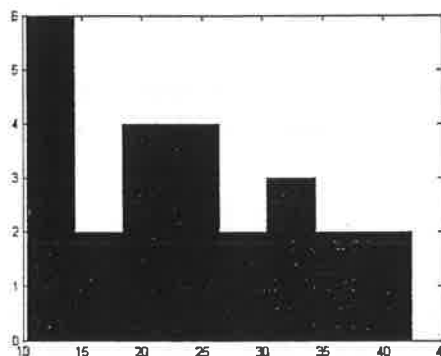
(i) using 10 intervals

```
>> % input the times as the elements of vector t
>> t=[24.8 20.0 23.4 14.4 15.8 22.4 15.7 28.1 23.2
31.3 33.9 14.1 11.1 20.0 25.0 32.7 10.4 12.5 27.5
42.4 12.6 39.7 20.3 36.1 37.4];
>> % find frequencies and draw histogram
>> n=hist(t) % 10 intervals used by default
n =
Columns 1 through 8
    4     4     2     3     3     2     2     1
Columns 9 through 10
    2     2
```



(ii) using 8 intervals

```
>> [n xout]=hist(t,8)
>> n, xout are the frequencies, class mid points
n =
     6     2     4     4     2     3     2     2
xout =
Columns 1 through 5
    12.4000    16.4000    20.4000    24.4000    28.4000
Columns 6 through 8
    32.4000    36.4000    40.4000
```



Notice how the number of intervals affects the histogram.

8.4 Histograms from frequency distributions

Input the x values as the elements of a vector x , the frequencies as the elements of a vector f and use `bar(x,f)` to draw the histogram.

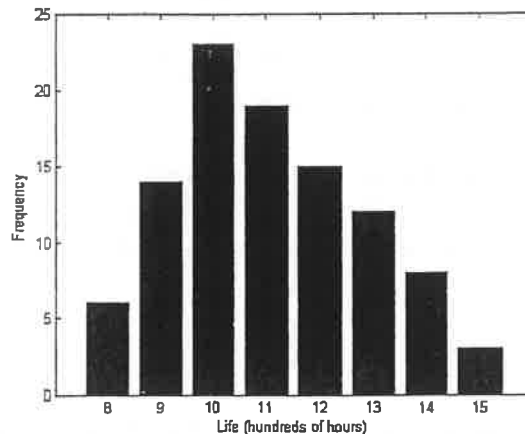
Example 5

A sample of 100 light bulbs was selected from a large production run and each was tested to failure to find its life time. The results, to the nearest 100 hours were:

Life (hundreds of hours)	8	9	10	11	12	13	14	15
Frequency	6	14	23	19	15	12	8	3

Draw a histogram of the data.

```
>> x=[8 9 10 11 12 13 14 15];
>> f=[6 14 23 19 15 12 8 3];
>> bar(x,f)
```



8.5 Numerical summaries

If the data values are the elements of a vector x then

$mean(x)$ finds the mean (average) value

$median(x)$ finds the median

$std(x,1)$ finds the standard deviation from the mean

The variance is the square of the standard deviation.

Example 6

Find the mean, median, standard deviation and variance of the data in Example 4.

```
>> t=[24.8 20.0 23.4 14.4 15.8 22.4 15.7 28.1 23.2
      31.3 33.9 14.1 11.1 20.0 25.0 32.7 10.4 12.5 27.5
      42.4 12.6 39.7 20.3 36.1 37.4];
>> xbar=mean(t),m=median(t),s=std(t,1),var=s^2
xbar =
    23.7920
m =
    23.2000
s =
     9.2882
var =
    86.2703
```

If the data is given in the form of a frequency distribution there is a bit more work to do to find the mean $\bar{x}$, variance σ^2 and standard deviation σ .

The formulae are $\bar{x} = \frac{\sum fx}{\sum f}$, $\sigma^2 = \frac{\sum fx^2}{\sum f} - \bar{x}^2$, $\sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$.

This is illustrated in the following example.

Example 7

A sample of 100 light bulbs was selected from a large production run and each was tested to failure to find its life time. The results, to the nearest 100 hours were:

Life (hundreds of hours)	8	9	10	11	12	13	14	15
Frequency	6	14	23	19	15	12	8	3

Find the mean, variance and standard deviation.

```
>> % input the data as vectors x and f
>> x=[8 9 10 11 12 13 14 15];f=[6 14 23 19 15 12 8 3];
>> % multiply x and f element by element to form fx
>> fx=f.*x
fx =
    48    126    230    209    180    156    112    45

>> % find the sums of vectors f and fx
>> sumf=sum(f),sumfx=sum(fx)
sumf =
    100
sumfx =
    1106
>> % find the mean
>> mean=sumfx/sumf
mean =
    11.0600
>> % find the f x squared by multiplying fx and x
>> fxsq=x.*fx
fxsq =
Columns 1 through 4
    384    1134    2300    2299
Columns 5 through 8
    2160    2028    1568    675
>> % find the sum of vector fxsq
>> sumfxsq=sum(fxsq)
sumfxsq =
    12548

>> % find the variance
>> var=sumfxsq/sumf-mean^2
var =
    3.1564
>> % find the standard deviation
>> stdev=sqrt(var)
stdev =
    1.7766
```

The mean is 11.06 hundreds of hours, or 1106 hours, with a standard deviation of about 178 hours.

Exercise 8

1. The areas of the continents, to the nearest million square miles, are:

Continent	Africa	America (N+S)	Antarctica	Asia	Australia	Europe
Area	12	16	5	17	3	4

Draw a pie chart for this data. *explode* the two largest continents.

2. Draw vertical and horizontal bar charts for the data in question 1.

3. The times taken by 35 people to perform a certain calculation were as follows (to the nearest second)

41	33	58	50	58	31	51
53	58	55	36	68	46	62
46	44	52	56	47	54	37
60	64	48	55	49	51	50
51	43	60	40	37	41	45

Draw a histogram with 10 intervals and tabulate the number of people in each interval.

4. The breaking strengths in N/m^2 of 50 metal specimens are given in the following table, with a class interval of 0.2 N/m^2 .

Centre of Interval x	32.0	32.2	32.4	32.6	32.8	33.0	33.2	33.4
No. of specimens f	1	4	7	12	13	9	3	1

Input the the centres of the intervals and the frequencies as the elements of vectors x and f and draw a histogram of the data.

5. Find the median, mean, standard deviation and variance of the data in question 3.
6. Find the mean, standard deviation and variance of the data in question 4.

9. m-files

Up to now MATLAB has been used in an interactive manner only. That is, you issue a command, see the result, issue a second command, see the result, and so on. This is often inconvenient for longer or more complex problems. With m-files we can store MATLAB commands in a separate file which can be executed any number of times. It can also be edited.

There are two types of m-files, called script m-files and function m-files; these are discussed below.

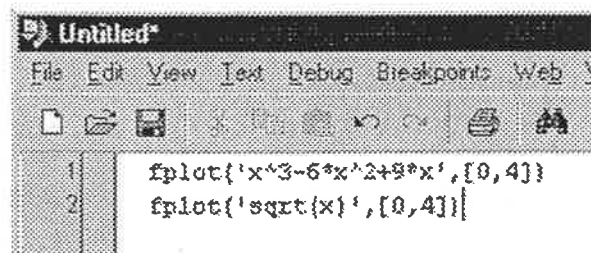
9.1 Script m-files

Two examples illustrate how script m-files are created and used.

Example 1

We will create a script m-file to draw the graphs of $y = x^3 - 6x^2 + 9x$ and $y = \sqrt{x}$ from $x = 0$ to $x = 4$. (These graphs were also produced in Section 2.2)

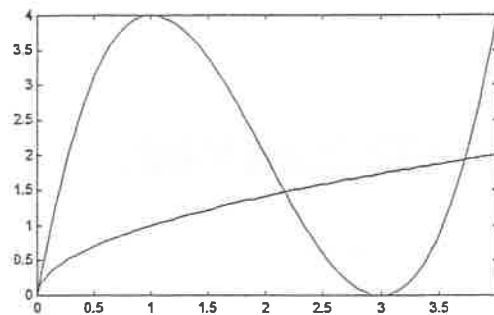
1. open a new m-file by clicking on the white file icon at the top left of the MATLAB window and then type in the commands, to produce



2. Save the file with a name of your choice.
Click on **File**, **Save As**. The filename will be Untitled.m. Change this to file1.m (you can choose any name you like rather than file1, but you must include the .m) and then **Save**.
(You can also save on your own floppy disk in A drive if you wish. In **Save in** scroll to 3 $\frac{1}{2}$ Floppy (A:) and click to select.)
3. To execute the file go to the Command Window. It is a good idea to start off with *clear all* and *close all* so that your work is not affected by anything you did previously.

```
>> clear all  
>> close all  
>> hold      % so that both graphs are shown  
Current plot held  
>> file1
```

The graphs appear as before.



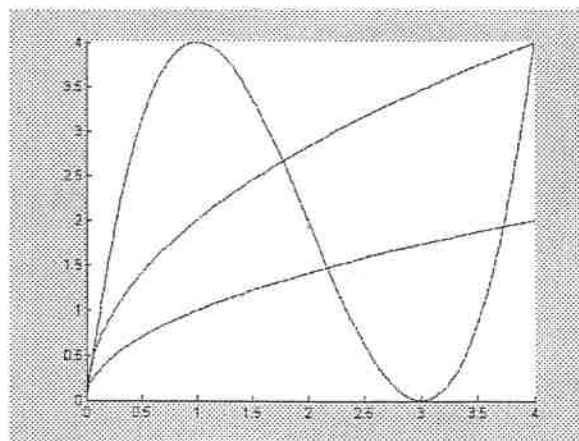
4. an m-file can be edited if required. Suppose we wanted to plot $y = 2\sqrt{x}$ as well as the other two. To do this click on File Open and then file1.m. Add an extra line and click on Save.

```
C:\MATLAB6p5\work\file1.m
File Edit View Text Debug Breakpoints Web Y
[Icons]
1 - fplot('x^3-6*x^2+9*x',[0,4])
2 - fplot('sqrt(x)', [0,4])
3 - fplot('2*sqrt(x)', [0,4])
```

In the Command Window

```
>> file1
```

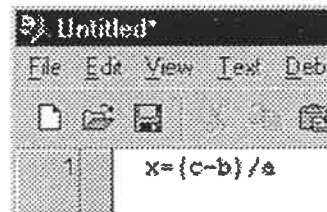
This produces



Example 2

We will create an m-file to solve a linear equation of the form $ax + b = c$, the solution of which is $x = \frac{c-b}{a}$.

1. open a new m-file and type



2. Save file as file2.m (or your choice of name)
3. In the Command Window give a , b and c values and then find x by running file2,

```
>> a=2;b=5;c=9;  
>> file2  
x =
```

2

(Check: $x = \frac{c-b}{a} = \frac{9-5}{2} = \frac{4}{2} = 2$.)

4. Run it again with different values of a , b and c .

```
>> a=3;b=-4;c=17;  
>> file2  
x =
```

7

Note.

1. To see all the m-files you have created

```
>> what
```
2. To see the contents of a file use *type*

```
>> type file1
```
3. The lines of the m-file are not printed when it is executed. If you do want them shown use the *echo* command.

```
>> echo on  
>> file2  
x=(c-b)/a  
x =
```

7

echo off reverts to the original situation.
4. To delete an m-file click on File Open, highlight the file to be deleted, click right mouse and then delete.

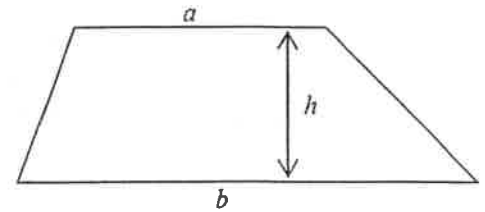
Exercise 9.1

1. Write and save a script m-file called *trap* to find the area of the trapezium shown.

The formula is $area = \frac{1}{2}h(a+b)$.

In the Command Window define $a = 6$, $b = 10$ and $h = 2$.
Execute *trap* and obtain $area = 16$.

Execute *trap* again with your own values of a , b and h ,



2. A car travelling with initial speed $u \text{ ms}^{-1}$ accelerates at $a \text{ ms}^{-2}$ for t seconds. It has then traveled a distance $s = ut + \frac{1}{2}at^2$ (1)

The speed v is related to t by $v = u + at$ (2)

and to s by $v^2 = u^2 + 2as$ (3)

Write a script m-file called *car* to find s from (1) and v from (2) and (3).

(Use $v_1 = u + at$ and $v_2 = \sqrt{u^2 + 2as}$)

In the Command Window define $u = 5$, $a = 2$, $t = 6$, execute *car* and obtain $s = 66$, $v_1 = v_2 = 17$.

Execute *car* again with your own values.

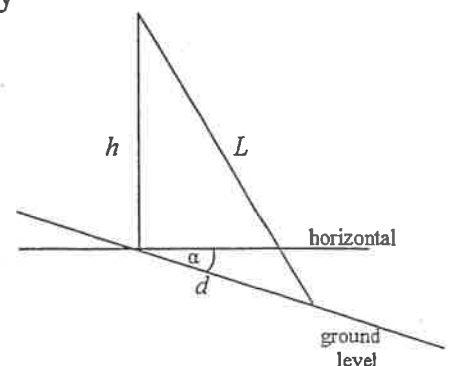
3. The length L of a guy rope to a pole of height h is given by $L = \sqrt{h^2 + d^2 + 2hd \sin \alpha}$.

Write a script m-file *guy* to find L .

Define $h = 10$, $d = 5$, $\alpha = 10^\circ$ and execute *guy* and obtain $L = 11.9317$.

If $h = 15$, $d = 6$, $\alpha = 15^\circ$ show that $L = 17.5382$.

(Hint: use *alpha* for α and convert the angle to radians)



9.2 Function m-files

As we saw in Section 1.5, MATLAB has a number of built in functions such as $\sin(x)$ and $\sqrt{x}$. A function m-file allows us to write our own functions for complicated or frequently used groups of expressions.

Two examples illustrate how function m-files are created and used.

Example 1

We will create a function m-file to calculate the function $y = \frac{\sqrt{x}}{x+1}$.

1. open a new m-file by clicking on the white file icon at the top left of the MATLAB window

2. type in the lines of the function

(i) decide on a name for your function – say *fred*

(ii) the first line of the file must have the format

function[list of outputs]=name(list of inputs)

In this example there is only one output, *y*, and one input, *x* so type

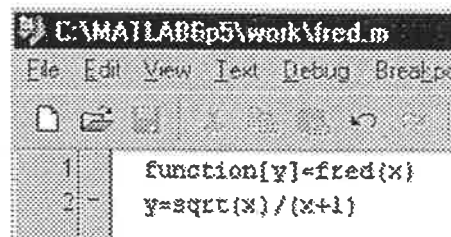
function[y]=fred(x)

(iii) enter the code that defines the function. In this example this is

y=sqrt(x)/(x+1)

(iv) for larger files explain what the function is doing using comment lines starting with %.

You should obtain



3. Save the file with File Save (or click on the save icon). The file will be given the name you chose in the first line (*fred*).

4. To execute a file go to the Command Window. Type the function name together with the value of x in brackets (no .m here). If $x = 4$ then $y = \frac{\sqrt{4}}{4+1} = 0.4$.

```
>> fred(4)
```

```
y =
```

```
0.4000
```

Run it again:

```
>> a=fred(1),b=fred(10),c=fred(100)
```

```
y =
```

```
0.5000
```

```
a =
```

```
0.5000
```

```
y =
```

```
0.2875
```

```
b =
```

```
0.2875
```

```
y =
```

```
0.0990
```

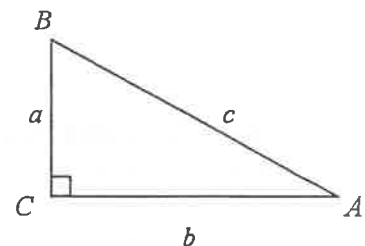
```
c =
```

```
0.0990
```

Example 2

We will write a function m-file to input two sides a and b of a right angled triangle and calculate the hypotenuse c , the angles A and B and the area of the triangle.

We will call the function *triangle*.



1. open a new m-file, type the following commands and Save. The inputs are a and b and the outputs are c , A , B and $area$.

```
C:\MATLAB5p5\work\triangle.m
File Edit View Test Debug Breakpoints Web Window
[Icons]
1 function[c,A,B,area]=triangle(a,b)
2 %find the hypotenuse
3 c=sqrt(a^2+b^2)
4 %find A and B in degrees
5 A=atan(a/b)*180/pi
6 B=90-A
7 %find the area
8 area=a*b/2
```

2. In the command widow run the function with $a = 3$ and $b = 4$

```
>> triangle(3,4)
```

```
c =
```

```
5
```

```
A =
```

```
36.8699
```

```
B =
```

```
53.1301
```

```
area =
```

```
6
```

Check these answers and run the function again with your own values of a and b .

3. Suppose that we wish to find the circumference of the triangle as well ($circ = a + b + c$).
Edit the function to do this, save the edited file and run it again. Make sure that it does what you expect!

9.3 A comparison of script and function m-files

Script m-files

Do not accept input arguments or return output arguments.

A file is executed by typing its name (for example, *file2*). The values used must have already been defined in the Command Window. (a , b , c in *file2*)

Function m-files

Can accept input arguments (a and b in Example 2)
Can return output arguments (c , A , B , *area*)

Executed by typing its name with the values of the input arguments in brackets. (for example *triangle*(3,4))

The variables used in a function are local to the function (that is, we could use a to represent something entirely different in the Command Window)

Exercise 9.2

1. The quadratic equation $ax^2 + bx + c = 0$ has roots
- $$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
- (i) Write a function m-file called *quadratic* to accept values of a , b and c and output the roots x_1 and x_2 .
- (ii) In the Command Window type *quadratic*(1,-6,5) to get $x_1 = 1$, $x_2 = 5$ and

quadratic(3,10,2) to get $x_1 = -3.1196$, $x_2 = -0.2137$.
Solve your own quadratic equations.

2. £ P invested for n years at r % grows to

$$S = P \left(1 + \frac{nr}{100} \right) \text{ with simple interest}$$

$$C = P \left(1 + \frac{r}{100} \right)^n \text{ if interest is added annually}$$

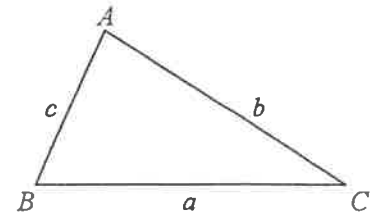
$$C_k = P \left(1 + \frac{r}{100k} \right)^{nk} \text{ if interest is added } k \text{ times per year.}$$

- (i) Write a function m-file called *interest* to accept values of P , n , r and k , and output S , C and C_k to the nearest penny.
- (ii) Execute *interest* for $P = 1000$, $n = 7$, $r = 5$ and $k = 12$ (monthly) and obtain $S = 1350.00$, $C = 1407.10$ and $C_k = 1418.04$.
Execute *interest* for your own values of S , C and C_k .

3. The angles A , B , C and area S of a triangle can be found, given the sides a , b and c , from

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$C = 180^\circ - A - B, \quad S = \frac{1}{2}ab \sin C$$



- (i) Write a function m-file *triangle2* that accepts a , b and c and finds A , B , C and S .
- (ii) Execute *triangle2* for the following values of a , b and c .
 $a = 11$, $b = 10$, $c = 8$
the equilateral triangle with $a = b = c = 2$.
the isosceles right angled triangle with $a = b = 1$, $c = \sqrt{2}$.
the right angled triangle with $a = 3$, $b = 4$ and $c = 5$.
the isosceles triangle with $a = b = 1$, $c = \sqrt{3}$.
You should get the following results.

a	b	c	A	B	C	S	
11	10	8	74.4	61.1	44.5	38.5	(1D)
2	2	2	60	60	60	1.7321	
1	1	$\sqrt{2}$	45	45	90	0.5	
3	4	5	37	53	90	6	
1	1	$\sqrt{3}$	30	30	120	0.4330	

10. MATLAB Programming

MATLAB has programming constructs similar to those found in most computer languages and computer algebra packages such as Maple and Mathcad. These allow the user to make decisions or perform a task a number of times. The following is an introduction to these constructs.

10.1 For loops

A FOR loop allows a statement, or group of statements, between `for` and `end` to be repeated a number of times. The general form is

```
for i = start:stepsize:finish
    statements
end
```

The statements inside the loop will be executed as the loop or count variable *i* runs from *start* to *finish* in steps of *stepsize*. If the *stepsize* is 1 it can be left out.

Example 1

Print the first 4 integers. (This could easily be the first 100 or 1000 integers, but there would be a lot of output!)

Solution

```
>> for i=1:4
    i
end
i =
    1
i =
    2
i =
    3
i =
    4
```

Example 2

Print the squares of the first 4 odd integers. (That is, $1^2 = 1$, $3^2 = 9$, $5^2 = 25$, $7^2 = 49$)

Solution

i must take the values 1,3,5 and 7; that is *i* must go from 1 to 7 in steps of 2, or $i = 1(2)7$. Then

```
>> for i=1:2:7
    i^2
end
ans =
    1
ans =
    9
ans =
   25
ans =
   49
```

Example 3

Print the cubes of the first 4 even integers.

Solution

```
>> for i=2:2:8
    i^3
end
```

You should see 8, 64, 216 and 512 printed.

We can generalize the above examples. In Example 1 the loop is executed 4 times. In the following code the loop is executed n times and we input n first.

Example 4

Input n and print the square roots of the first n integers. Run with $n = 5$.

Solution

```
>> n=5
n =
    5
>> for i=1:n
    sqrt(n)
end
ans =
    1
ans =
    1.4142
ans =
    1.7321
ans =
    2
ans =
    2.2361
```

Example 5

Input Fred's marks (67, 72, 40 and 53) in his 4 maths tests into a vector m and find his average mark.

Solution

```
>> m=[67 72 40 53];           % ; suppresses output of m
>> s=0;                       % s is the running total
>> for i=1:4
    s=s+m(i);                  % adds next mark m(i) to s
end
>> 'sum of marks',s
sum of marks
s =
    232
>> ave=s/4;
>> 'average mark',ave
average mark
ave =
    58
```

Note that the input and output are intermingled. We can avoid that, and use the program on another occasion, if we write it as a function m-file.

Example 6

Write a function m-file *marks* to input n exam marks and to find and output the average mark. The inputs to *marks* are the number n of marks, and the n marks as the elements of a vector m .

The output is the average mark.

Solution

With the same data as in Example 5:



```
>> n=4;
>> m=[67 72 40 53];
>> marks(n,m)
ave =
    58
```

Example 7

Write a function m-file *trapezium* to estimate the area under a curve using the Trapezium Rule. The rule is

$$\text{area} = \int_a^b y dx \approx \frac{1}{2} h ((y_1 + y_{n+1}) + 2(y_2 + y_3 + y_4 + \dots + y_n)).$$

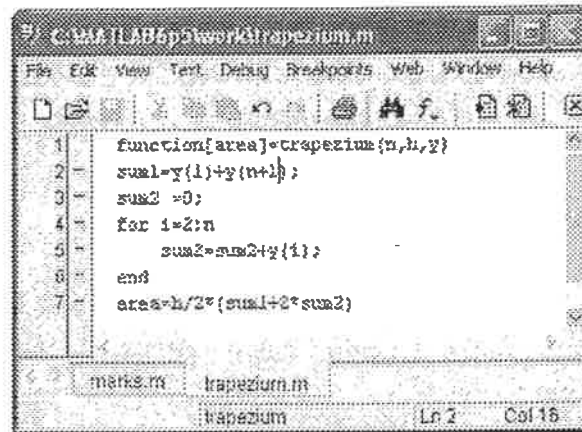
The inputs are the number of strips n , the width of each strip h , and the y values as the elements of a vector y .

The output is the estimate of the area.

Run the function with the following data, which represents the depth of a 24 metre wide river every 3 metres, so $n = 8$ and $h = 3$.

Distance from bank (metres)	0	3	6	9	12	15	18	21	24
Depth y metres	1.1	1.8	2.3	3.1	3.6	3.8	2.9	2.2	1.2

Solution



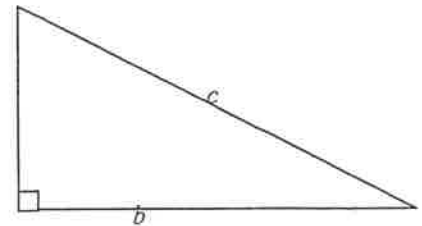
```
>> n=8;  
>> h=3;  
>> y=[1.1 1.8 2.3 3.1 3.6 3.8 2.9 2.2 1.2];  
>> trapezium(n,h,y)  
area =  
    62.5500
```

Note the indentation in the m-file; this makes it easier to read.

FOR loops can be nested. The following example illustrates this.

Example 8

By Pythagoras' Theorem $c = \sqrt{a^2 + b^2}$. Calculate c for values of a going from 1 to 5 in steps of 1 (that is, $a = 1(1)5$) and for values of b going from 2 to 8 in steps of 2. ($b = 2(2)8$)



Solution

Note that there are 5 values of a and 4 values of b , so we will find $4 \times 5 = 20$ values of c .

```
>> for a=1:5  
    for b=2:2:8  
        c=sqrt(a^2+b^2)  
    end  
end
```

% this is the end of the b loop
% this is the end of the a loop

You should see 20 values of c , starting with 2.2361 and ending with 9.4340.

Exercise 10.1

1. Print the squares of the first 10 integers.
2. Find $y = x^2 + 2x + 1$ for values of x going from 0 to 2 in steps of 0.25.

3. A temperature in degrees Celsius C is calculated from a temperature in degrees Fahrenheit F from $C = \frac{5}{9}(F - 32)$.
Find C for values of F going from 0 to 200 in steps of 10.
4. Write a function m-file *simpson* to estimate the area under a curve using Simpson's Rule. The rule is

$$\text{area} \approx \frac{1}{3}h((y_1 + y_{n+1}) + 4(y_2 + y_4 + \dots + y_n) + 2(y_3 + y_5 + \dots + y_{n-1})).$$
The inputs are the number of strips n , the width of each strip h , and the y values as the elements of a vector y .
The output is the estimate of the area.
Run the function with the same data as in Example 7 to get an answer of 63.5.
5. £ P invested for n years at $r\%$ grows to £ C , where $C = P\left(1 + \frac{r}{100}\right)^n$.
Write a function m-file to obtain a table of values of C for varying values of r and n .
 r varies from $r1$ to $r2$ in steps of $rstep$.
 n varies from $n1$ to $n2$ in steps of $nstep$.
Inputs to the file are $P, r1, r2, rstep, n1, n2, nstep$.
Outputs are r, n and C .
Run the file with $P = 1000, n = 1(1)5$ and $r = 3(0.5)7$.

10.2 The IF statement

An `if` statement allows us to branch and perform different sequences of commands in different cases. An `if` statement must have a matching `end` statement.

For example, suppose we wished to perform a set of commands only if the value of a variable called `fred` was positive. This would be achieved by

```
if fred > 0
    statements to be executed if fred is positive
end
```

The statements between `if` and `end` are only executed if `fred` is positive. If `fred` is negative or zero the instructions are ignored and the `if` block is exited.

As a second example, suppose we wanted to perform different sets of instructions depending on whether `fred` was positive or non-positive (that is, zero or negative). This would be achieved by

```
if fred > 0
    statements to be executed if fred is positive
else
    statements to be executed if fred is nonpositive
end
```

Finally, suppose we wanted to perform different sets of instructions in the three cases `fred` positive, `fred` zero and `fred` negative. Here's how we would do it.

```
if fred > 0
    statements to be executed if fred is positive
elseif fred = 0
    statements to be executed if fred is zero
else
    statements to be executed if fred is negative
end
```

The logical test is commonly a test for equality or inequality. MATLAB uses the following conventions:

<	less than	<=	less than or equal
>	greater than	>=	greater than or equal
==	equal to	~=	not equal to

As with `for` loops, note the indentation in m-files.

Example 1

From two numbers x and y find another number z which is found from $z = (\text{larger of } x \text{ and } y) + 3 \times (\text{smaller of } x \text{ and } y)$. Run with $x = 5, y = 1$ and $x = 3, y = 8$.

Solution

>> x=5;y=1;	>> x=3;y=8;
>> if x>y	>> if x>y
z=x+3*y	z=x+3*y
else	else
z=y+3*x	z=y+3*x
end	end
z =	z =
8	17

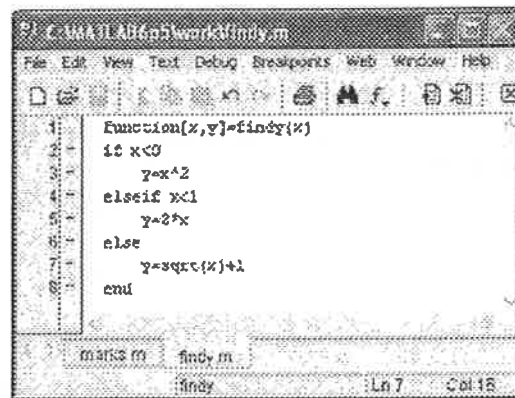
Example 2

y is calculated from x from $y = \begin{cases} x^2 & \text{if } x < 0 \\ 2x & \text{if } 0 \leq x < 1 \\ \sqrt{x} + 1 & \text{if } x \geq 1 \end{cases}$.

Write a function m-file *findy* to calculate y .

Run the function when (i) $x = -2$ (ii) $x = 0.5$ and (iii) $x = 9$.

Solution



```
>> findy(-2)
x =
    -2
y =
     4
>> findy(0.5)
x =
    0.5
y =
     1
>> findy(9)
x =
     9
y =
     4
```

Example 3

Find the largest in a list of positive numbers stored in a vector **x**.

Solution

```
>> x=[12 56 89 63 22 91 75 48 80 36];
>> n=length(x);      % n is the number of numbers
>> max=0;             % initial value of largest
>> for i=1:n          % to look at all numbers in turn
if x(i)>max            % compares number with largest so far
max=x(i);            % max is the largest so far
end                  % end of if
end                  % end of for loop
>> max                % largest number in the list
max =
    91
```

Example 4

Write a function m-file to find the average, smallest and largest of a set of exam marks in percentages.

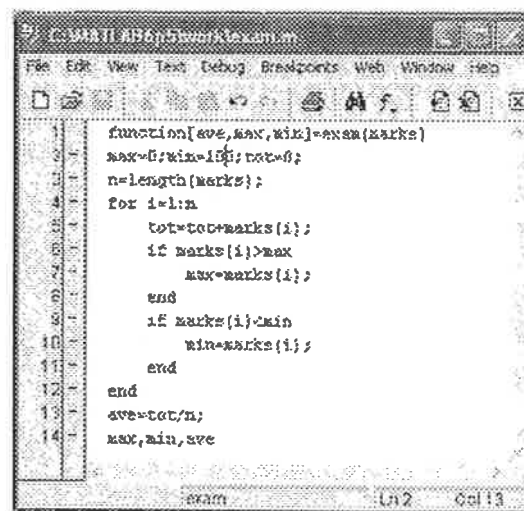
Solution

The number of marks n is found using *length*.

The marks are added together into *tot*.

The largest mark, *max*, is initially set at zero (which can't be the final answer) and compared with each mark in turn. If the current mark is bigger than *max* (the largest mark so far) then it is stored in *max*.

The smallest mark, *min*, is initially set at 100 (which can't be the final answer) and compared with each mark in turn. If the current mark is smaller than *min* (the smallest mark so far) then it is stored in *min*.



```
>> marks=[57 32 45 87 69 93 26 39 48 71 60 19 83 66 42];
>> exam(marks)
max =
    93
min =
    19
ave =
   55.8000
```

Exercise 10.2

- From two numbers a and b find another number c which is found from $c = (\text{larger of } a \text{ and } b)^2 + 2 \times (\text{smaller of } a \text{ and } b)$. Run with $a = 6$, $b = 2$ and $a = 3$, $b = 10$.

- Calculate y from x according to $y = \begin{cases} x^4 + 1 & \text{if } x < 1 \\ 3x^2 - 1 & \text{if } x \geq 1 \end{cases}$.
Run with $x = 0$, $x = 1$ and $x = 3$.

- A department store gives a discount for articles purchased according to the following scheme.

Amount spent:	< £100	≥ £100 but < £500	≥ £500 but < £1000	≥ £1000
Discount %	0	10	15	20

From the cost C of the items purchased calculate the price P to be paid after discount. Run with (i) $P = 65$ (ii) $P = 250$ (iii) $P = 800$ (iv) $P = 2000$.

- A beam of length 10 metres is simply supported at each end and carries a uniformly distributed load of 200 Nm^{-1} along its length. Loads of 250 N and 750 N are placed at distances of 2 m and 5 m from the left hand end respectively. The shearing force S and bending moment M equations are, with x measured from the left hand end,

$$\begin{aligned} S &= 200x - 1575, \quad M = 100x^2 - 1575x && \text{for } 0 \leq x < 2 \\ S &= 200x - 1325, \quad M = 100x^2 - 1325x - 500 && \text{for } 2 \leq x < 5 \\ S &= 200x - 575, \quad M = 100x^2 - 575x - 4250 && \text{for } 5 \leq x \leq 10 \end{aligned}$$

Write a function m-file to calculate S and M at 0.2 metre intervals along the beam.

- Write a function m-file to analyze a set of exam results. The file should accept as input a vector containing the students names and another with their marks as percentages.

Each student falls into one of the following categories:

mark m	result
$m < 25$	fail, no resit
$25 \leq m < 40$	fail, resit
$40 \leq m < 65$	pass
$65 \leq m < 80$	pass with merit
$m \geq 80$	pass with distinction

The file should output each students name, mark and result, together with the total number of students in each category and the average exam mark. Test the file with your own data.

10.3 While loops

When using a FOR loop we specify in advance the number of times to run the loop (FOR $i = 1:10$ will execute the loop 10 times). A while loop keeps looping until a particular condition is satisfied.

```
The general form is  while  logical test
                      commands to be executed when
                      the condition is satisfied
                      end
```

The following examples illustrate this.

Example 1

Find the average height of a number of people. Terminate the list of heights with -1 .

Solution

We will store the list of heights in the vector h . The running sum of the heights is s and n counts the number of heights.

```
>> h=[1.48 1.71 1.64 1.82 1.59 1.63 -1];
>> s=0;n=1;
>> while h(n)>0
s=s+h(n);n=n+1;
end
>> ave=s/(n-1)
ave =
    1.6450
```

Example 2

Find the smallest integer n such that $n^3 > 10000$.

Solution

```
>> n=1;
>> while n^3<10000
n=n+1;
end
>> n,n^3
n =
    22
ans =
   10648
```

Example 3

The temperature of a cup of coffee falls by 0.1°C per minute for each $^{\circ}\text{C}$ above room temperature at the start of the minute. That is, $\text{fall} = 0.1(\text{coffeetemp} - \text{roomtemp})$.

Given the initial temperature of the coffee, the room temperature and the minimum drinkable temperature print the temperature every minute, stopping when the coffee temperature is less than the minimum drinkable temperature.

Assume the initial temperatures of the coffee and room are 100° and 20° , and that the minimum drinkable temperature is 40°C .

Solution

```
>> coffeetemp=100;roomtemp=20;mindrinktemp=40;time=0;
>> while coffeetemp>mindrinktemp
    fall=0.1*(coffeetemp-roomtemp);
    coffeetemp=coffeetemp-fall;
    time=time+1;
    time,coffeetemp
end
```

You should find that it takes 14 minutes for the temperature of the coffee to fall below 40°C (to 38.3014°C)

Exercise 10.3

1. Find the smallest integer n such that $n^4 > 12345$.
2. £1000 is invested at 5%, compounded annually. Its value after n years is £ C , where $C = 1000 \times 1.05^n$. How many years will it take for C to exceed £1500?
3. Find the greatest value of n such that $1^2 + 2^2 + 3^2 + \dots + n^2$ is less than 300.
4. A baby is defined as a person under 3 years of age, a child as one who is at least 3 but under 12, an adolescent as one who is at least 12 but under 18 and an adult one who is at least 18.

Write a function m-file to analyze the ages of a group of people and find how many are in each of the above groups.

Inputs to the file are the ages of the people in a vector *ages*, the last element of which is -1 to show that all have been processed.

The output should be the total number of people in each category.

Run the file for

```
ages = [24.3 10.4 53.6 16.0 2.1 8.7 57.3 13.8 17.2 1.1 7.9 62.8 2.7 -1]
```

11 Case Studies

11.1 Oscillations and Vibrations

Oscillations and vibrations occur in many physical systems. For example,

- (i) a child on a swing oscillates backwards and forwards
- (ii) a ship causes waves which die out as they move further away from the ship
- (iii) a seismograph vibrates in response to an earthquake
- (iv) a bridge vibrates due to traffic or wind.

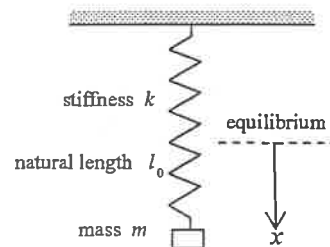
Many situations can be modelled by a system consisting of a mass, a spring and a damper. (A damper produces a resistive force representing friction or air resistance.)

These models lead to differential equations. You will see how differential equations are solved in your second year; here we will see how the mathematical solution describes what happens.

11.1.1 Simple Harmonic Motion

A particle is attached to the ceiling by a spring. When it is pulled down and released it will oscillate vertically. If all frictional forces are ignored the displacement of the particle from its equilibrium position is

$$x = A \cos\left(\sqrt{\frac{k}{m}}t + \phi\right), \text{ where } A \text{ and } \phi \text{ are constants.}$$



- (i) Plot the graph of x against t for $0 \leq t \leq 1$ when $A = 2$, $m = 1$, $k = 256$ and $\phi = 0$. What are the amplitude and period of the oscillations?
- (ii) Plot other graphs of x against t as m changes (larger or smaller). What happens to the amplitude as m changes? What happens to the period as m changes?
- (iii) Plot other graphs of x against t as k changes (larger or smaller). What happens to the amplitude as k changes? What happens to the period as k changes?
- (iv) With $m = 1$, what value of k is needed so that the period is
(a) less than 1 second (b) more than 2 seconds.

[39, 10]

11.1.2 Damped vibrations

Simple Harmonic Motion predicts that the particle will continue to oscillate indefinitely, but in practice the oscillations will eventually die away due to the presence of a resistive force such as friction or air resistance. In this case (and provided there is not too much friction – see 11.1.3) the solution takes the form

$$x = Ae^{\alpha t} \sin(\beta t + \phi), \text{ where } \alpha = -\frac{r}{2m} \text{ and } \beta = \frac{\sqrt{4mk - r^2}}{2m}.$$

r , called the damping constant, is a measure of the amount of resistance present in the system.

- (i) Plot the graph of x against t for $0 \leq t \leq 10$ when $m = 0.25, k = 25, r = 0.25, A = 2$ and $\phi = 0$.
Plot x against t and show that it takes about 6 seconds for the amplitude of the oscillations to reduce to 0.1.
- (ii) Investigate the effect on the solution of increasing or decreasing m and/or k .
- (iii) If the amount of resistance in the system is increased then r is increased. What happens to the oscillations as r increases? What happens when r increases towards, but does not reach, 5?

11.1.3 Strong damping

If there is sufficient resistance in the system there will be no oscillations and x decays to zero exponentially.

If $r > 2\sqrt{mk}$ the solution becomes $x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$, where $\lambda_1, \lambda_2 = \frac{-r \pm \sqrt{r^2 - 4mk}}{2m}$.

- (i) Plot the graph of x against t for when $m = 0.25, k = 25, r = 10, A = 1, B = 1$.
- (ii) What happens if A and/or B are changed. (the values of A and B are determined by the initial conditions; that is, how the system is set in motion.)
- (iii) What happens as r is increased further?

11.1.4 Forced vibrations and resonance

The body of a car is forced to oscillate when it is driven over a bumpy road – what is the effect on the passengers?

A seismograph is forced to vibrate by earthquake waves in the earth's crust and hence measures the strength of the earthquake.

The motion of the particle depends on the amplitude and period of the forcing term and, for systems with little or no damping, the motion of the particle can vary dramatically.

Resonance occurs when the amplitude of the oscillations of the particle become very large in comparison with that of the forcing function. Resonance is often something to be avoided, such as in car suspension systems, buildings or aircraft, where large vibrations due to a bumpy road or wind can do a great deal of damage. Sometimes, however, resonance is desirable, such as when tuning in a radio or when pushing a child's swing, when the best way to make it go higher is to push it in time with the oscillations.

Two well known examples of resonance having disastrous effects were the collapse of bridges at Broughton, near Manchester, in 1831 and Tacoma Narrows in Washington, USA, in 1940. You may have seen the famous film of the Tacoma Narrows collapse.

A spring/mass/damper system subjected to a periodic forcing term of the form $\cos \alpha t$ has

$$\text{steady state solution } x = \frac{9721924(\alpha \sin \alpha t - (\alpha^2 - 100) \cos \alpha t)}{(1559\alpha^2 - 31141\alpha + 155900)(1559\alpha^2 + 31141\alpha + 155900)}$$

- (i) Plot x against t for $\alpha = 4, 8$ and 12 . Note the similar amplitudes.
- (ii) Plot x against t for $\alpha = 9, 10$ and 11 . Note the amplitudes.
- (iii) The way x is related to r and α is given by
$$R = \frac{100}{\sqrt{\alpha^4 + 8\alpha^2(2r^2 - 25) + 10000}}.$$

(R is called the relative response)

When $r = 0.25$ plot R against α and show that R has a maximum value when $\alpha \approx 10$. Use zoom to show that its value is 9.97 correct to 2D.

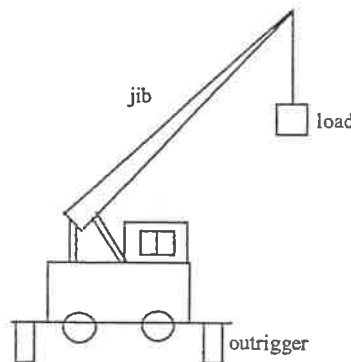
- (iv) Plot, on the same graph, R against α when $r = 0.1, 0.25, 0.5, 1$, and 2 .

Notice that as the amount of resistance (r) is increased the maximum relative response decreases, so that eventually there is no resonance.

Systems which are subject to an external forcing influence should be sufficiently heavily damped if resonance is to be avoided. The bridge collapses at Broughton and Washington were due to the periodic forcing term, caused by soldiers marching over the bridge in time in the first case and by wind in the second, being close to one of the natural frequencies of the bridge, causing resonance.

11.2 Cranes

A mobile crane is designed to operate with the weight of its body, jib, and load supported on four outriggers.

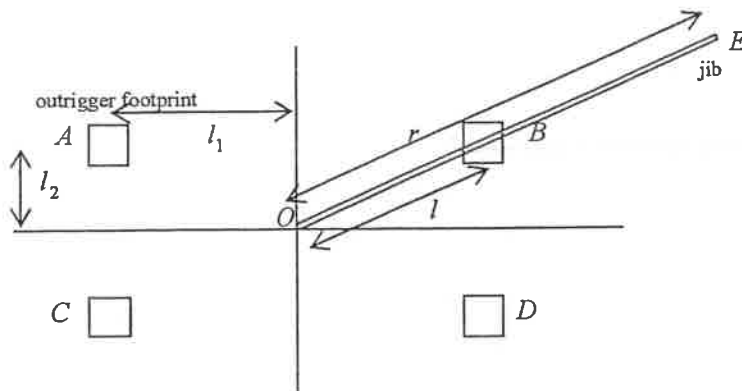


We will consider two problems.

1. If the load is too high, and/or the jib too long, the crane could tip over. If this is to be avoided what restrictions will there be on the greatest load that can be lifted, and how far out from the centre can the jib be used?
2. Is the ground strong enough to support the total load? If not, what strengthening is needed?

The greatest force on an outrigger occurs when the jib is directly over an outrigger.

A plan view of this is shown below.



Distance between front and back outriggers = $2l_1$.

Distance between the pair of front, or back, outriggers = $2l_2$.

Distance from centre O to an outrigger is $l = \sqrt{l_1^2 + l_2^2}$ (Pythagoras)

Horizontal distance from centre to the end of the jib = r . (all distances in metres)

Mass of crane = M kg

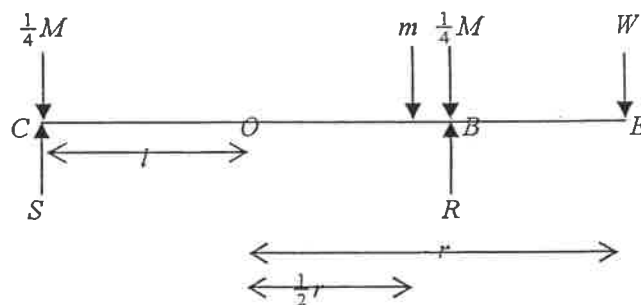
Mass of jib = m kg

Mass of load = W kg

Maximum pressure the ground can withstand = P kilonewtons per square metre
(KN m^{-2})

Length and breadth of square base plate of outrigger = d metres

The forces acting along the line BC are shown in the diagram below. S and R are the reactions at C and B respectively.



Some results:

By resolving forces and taking moments it can be shown that

$$R = \frac{1}{4}(M + 2m + 2W) + \frac{r}{4l}(m + 2W) \quad (1)$$

$$S = \frac{1}{4}(M + 2m + 2W) - \frac{r}{4l}(m + 2W) \quad (2)$$

For a given load W kg the largest value of r for the crane not to tip over is

$$r = \frac{(M + 2m + 2W)l}{m + 2W} \quad (3)$$

If the crane has a maximum radius r then the greatest load W kg it can lift

$$\text{without toppling over is } W = \frac{Ml + 2ml - mr}{2(r - l)} \quad (4)$$

For the ground to be able to support the outrigger the minimum value

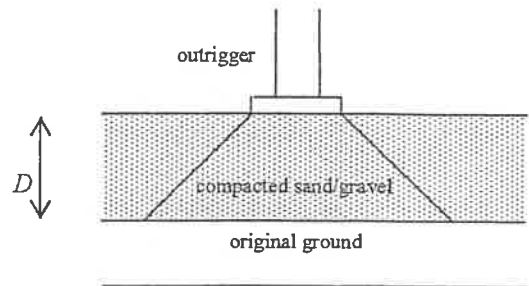
of d is
$$d = \sqrt{\frac{(M + 2m + 2W)g}{2000P}} \quad (5)$$

To support a crane with given d the minimum value of P is

$$P = \frac{(M + 2m + 2W)g}{2000d^2} \quad (6)$$

If this minimum value of P is too large for the ground being used, the outriggers can be supported on a stronger base, such as compacted sand and gravel, which can support a pressure of $P_{\max}$. The required depth of compacted sand and gravel is D metres, where

$$D = \frac{1}{2} \left(\sqrt{\frac{(M + 2m + 2W)g}{2000P_{\max}}} - d \right) \quad (7)$$



(Note that, in practice, safety margins will have to be built into all these results)

Use Matlab to solve the following problems. You should start with a *syms* statement and use equations (1) to (7) as necessary.

Problems

For a particular crane $M = 15000$, $m = 0.5$, $l_1 = 5$, $l_2 = 4$ and $d = 1$.

1. (i) A load of (i) 30000 kg (ii) 10000 kg must be lifted.
What is the largest value of the jib radius r that can be used?
[(a) 8.00 (b) 11.20]
- (ii) What is the greatest load W that the crane can lift if the maximum radius of the jib is (a) 10 (b) 20 metres.
[(a) 13352 (b) 3532]
- (iii) The load to be lifted is $W = 10000$ kg.
 - (a) Can the crane be supported by topsoil, for which $P = 85 \text{ KNm}^{-2}$?
[No]
 - (b) Can the crane be supported by compacted sand, for which $P = 300 \text{ KNm}^{-2}$?
[Yes]
 - (c) What is the minimum outrigger base plate size d so that the crane can be supported by loose gravel, for which $P = 200 \text{ KNm}^{-2}$?
[0.93]
 - (d) The crane is situated on topsoil, for which $P = 85$, and the outriggers are to be supported by compacted sand and gravel ($P = 800$). What minimum depth of sand/gravel is needed?
[0.21]

2. Plot the graph of W against r for values of r going from 7 (just more than the distance $OB = l$) to 25.
Use *zoom* to show that if $r = 10$ the maximum load that can be lifted is 1330 kg, and if a load of 3000 kg is to be lifted the value of r must be just over 8 metres.
3. Plot a graph of d against P for $P = 100$ to 1000. $W = 10000$.
What minimum value of d is required on soft limestone, for which $P = 600$?
[0.29]
If the outrigger base has size $d = 0.5$ can the crane and load be supported by
(i) sand ($P = 200$) or (ii) compacted gravel ($P = 600$). [No, Yes]
Investigate the requirements for other loads W .
4. Plot a graph of D against P for values of $P = 100$ to 1000. $W = 80000$.
Show that for clay ($P = 300$) d must be at least 0.35, while if $d = 0.6$ that value of P must not exceed 180.
Investigate the requirements for other loads W .
5. A crane on topsoil ($P = 85$) is to be supported by compacted sand and gravel of depth D . Plot a graph of D against d for values of d going from 0.1 to 1.
Show that if $d = 0.5$, the depth required is about $D = 0.47$.