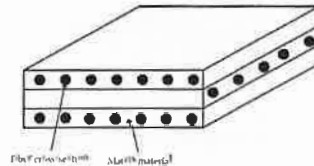


Macromechanical analysis of a lamina

An alternative to the Micromechanical approach is to use the Macromechanical approach to understanding composite materials. Macromechanical approaches can be applied to either a Lamina or a Laminate. A Lamina is 1 ply and is typically ~0.125mm thick. A Laminate is constructed by stacking a number of Laminae.



The above shows a Laminate made up of 3 Laminae. In order to understand the Laminate, we must first understand the Lamina.

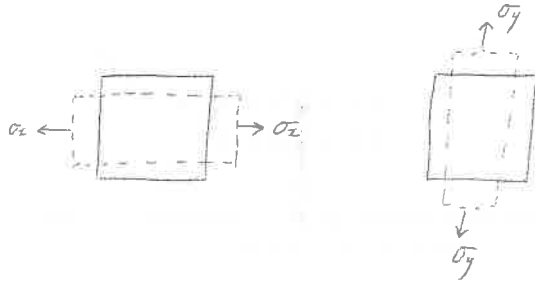
As a reminder consider the following definitions when applied to materials:

Homogeneous:	uniform composition throughout.
Isotropic:	identical values of a property in all directions.
Anisotropic:	different properties in different directions.
Orthotropic:	properties that differ along three mutually-orthogonal axes

The Lamina is not a homogeneous material, however macromechanical analysis is based on average properties based on the individual mechanical properties of the fibre and the matrix as discussed in the micromechanical section. However, even with the homogenization of a Lamina it behaves different from a homogeneous isotropic material.

Stiffness and Compliance Matrices

Consider a Linear Elastic material stressed in x and y;



The Strain in x is given by

$$\epsilon_x = \frac{1}{E} \sigma_x - \frac{\nu}{E} \sigma_y$$

similarly

$$\epsilon_y = \frac{1}{E} \sigma_y - \frac{\nu}{E} \sigma_x$$

If we considered this in 3 dimensions we would have the following

$$\epsilon_x = \frac{1}{E} \sigma_x - \frac{\nu}{E} \sigma_y - \frac{\nu}{E} \sigma_z \quad [1]$$

$$\epsilon_y = -\frac{\nu}{E} \sigma_x + \frac{1}{E} \sigma_y - \frac{\nu}{E} \sigma_z \quad [2]$$

$$\epsilon_z = -\frac{\nu}{E} \sigma_x - \frac{\nu}{E} \sigma_y + \frac{1}{E} \sigma_z \quad [3]$$

If this material were also subjected to shear, since $G = \frac{\tau}{\gamma}$, then we would also have

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \quad [4]$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \quad [5]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad [6]$$

If we write equations 1-6 in matrix form we thus get;

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \quad [7]$$

Equation 7 is called the 'compliance matrix', [S] of an isotropic material. If this were to be inverted we would get the 'stiffness matrix', [C]. Assuming that the composite material is Linear and Elastic IS acceptable, however, assuming that the material is isotropic IS NOT acceptable. Therefore, the constants relating stress and strain are more than those seen in equation 7. In the most generalized form for an isotropic material we would see the below (using 1,2,3 instead of x,y,z)

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} \quad [8]$$

Thus for an Isotropic material

$$S_{11} = \frac{1}{E} = S_{22} = S_{33}$$

$$S_{12} = -\frac{\nu}{E} = S_{13} = S_{21} = S_{23} = S_{31} = S_{32}$$

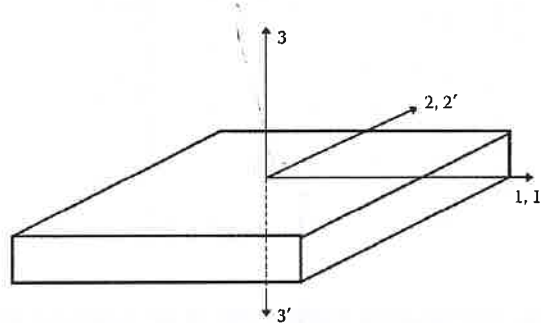
$$S_{44} = \frac{1}{G} = S_{55} = S_{66}$$

The compliance matrix [8] appears to have 36 unknowns, however, due to the fact that this matrix is symmetrical, $S_{61}=S_{16}$ etc so this matrix reduces to the following with only 21 unknowns.

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} \quad [9]$$

Monoclinic Composites

If the material has one plane of symmetry (i.e. it is MONOCLINIC) such as in the following figure in which the structure is symmetrical about the 3 plane then you can simplify the compliance matrix;



The **direct strains** are not related to the shear stresses in the 23 or the 31 planes

- ε_1 is not related to τ_{23} , therefore $S_{14}=0$ ✓
- ε_1 is not related to τ_{31} , therefore $S_{15}=0$ ✓
- ε_2 is not related to τ_{23} , therefore $S_{24}=0$ ✓
- ε_2 is not related to τ_{31} , therefore $S_{25}=0$ ✓
- ε_3 is not related to τ_{23} , therefore $S_{34}=0$ ✓
- ε_3 is not related to τ_{31} , therefore $S_{35}=0$ ✓

The direct stresses are not related to the **shear strains in the 23 plane**

- γ_{23} is not related to σ_1 , therefore $S_{41}=0$ ✓
- γ_{23} is not related to σ_2 , therefore $S_{42}=0$ ✓
- γ_{23} is not related to σ_3 , therefore $S_{43}=0$ ✓

The **shear strain in the 23 plane**, is not related to the shear stress in the 12 plane

- γ_{23} is not related to τ_{12} , therefore $S_{46}=0$ ✓

Similarly, for the **shear strains in the 31 plane**

- γ_{31} is not related to σ_1 , therefore $S_{51}=0$ ✓
- γ_{31} is not related to σ_2 , therefore $S_{52}=0$ ✓
- γ_{31} is not related to σ_3 , therefore $S_{53}=0$ ✓
- γ_{31} is not related to τ_{12} , therefore $S_{56}=0$ ✓

Finally, the **shear strain in the 12 plane** is not related to the shear stress in the 23 plane or the 31 planes

- γ_{12} is not related to τ_{23} , therefore $S_{64}=0$ ✓
- γ_{12} is not related to τ_{31} , therefore $S_{65}=0$ ✓

The direction perpendicular to the plane of symmetry is the **principal direction**, and the 21 unknowns has now reduced to 13 unknowns, thus the compliance matrix is now;

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \quad [10]$$

Feldspar is an example of a monoclinic material.

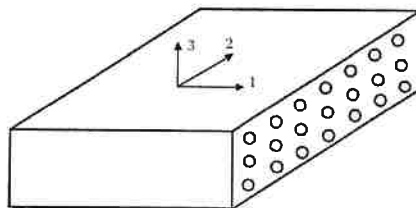
Orthotropic Composites

If the material has three mutually perpendicular planes of symmetry, i.e. it is Orthotropic then

- ε_1 is not related to τ_{12} , therefore $S_{16}=0$ ✓
- ε_2 is not related to τ_{12} , therefore $S_{26}=0$ ✓
- ε_3 is not related to τ_{12} , therefore $S_{36}=0$ ✓
- γ_{23} is not related to τ_{31} , therefore $S_{45}=0$ ✓
- γ_{31} is not related to τ_{23} , therefore $S_{45}=0$ ✓
- γ_{12} is not related to σ_1 , therefore $S_{16}=0$ ✓
- γ_{12} is not related to σ_2 , therefore $S_{26}=0$ ✓
- γ_{13} is not related to σ_3 , therefore $S_{36}=0$ ✓

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \quad [11]$$

Note that the number of unknowns has now reduced to 9. An example of an orthotropic material would be a single lamina of continuous fiber composite arranged in a rectangular array;



A unidirectional 2D lamina

Since $\sigma_3 = 0$, $\tau_{23} = 0$ and $\tau_{31} = 0$ then the problem of understanding the material behavior reduces to a 2D problem;

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{bmatrix}$$

Since γ_{23} and γ_{31} are clearly zero, these can be omitted thus

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 \\ S_{12} & S_{22} & S_{23} & 0 \\ S_{13} & S_{23} & S_{33} & 0 \\ 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ \tau_{12} \end{bmatrix}$$

Since ε_3 is related to ε_1 and ε_2 , this can also be omitted, i.e.

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Where [S] is the reduced compliance matrix and

$$S_{11} = \frac{1}{E_1}; S_{22} = \frac{1}{E_2}; S_{12} = -\frac{\nu_{12}}{E_1} \text{ and } S_{66} = \frac{1}{G_{12}}$$

$$\varepsilon_3 = S_{13} \sigma_1 + S_{23} \sigma_2$$

Examples

1)

- a) Write the compliance matrix in terms of the 9 engineering constants for an orthotropic material ~~if it is subjected to a stress in the 1 direction only.~~
- b) Hence find the compliance matrix for a Graphite/Epoxy lamina whose material properties are as follows;

$$\begin{aligned}E_1 &= 181 \text{ GPa} \\E_2 &= 10.3 \text{ GPa} \\E_3 &= 10.3 \text{ GPa} \\v_{12} &= 0.28 \\v_{23} &= 0.60 \\v_{13} &= 0.27 \\G_{12} &= 7.17 \text{ GPa} \\G_{23} &= 3.0 \text{ GPa} \\G_{31} &= 7.0 \text{ GPa}\end{aligned}$$

2) For a Graphite/Epoxy unidirectional lamina, find

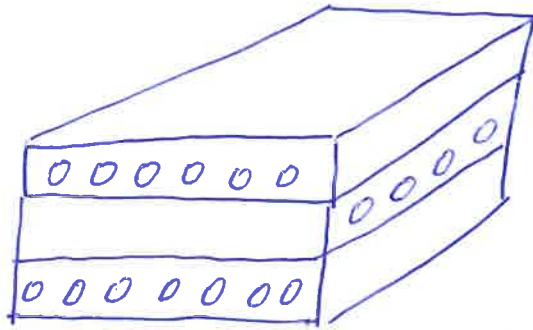
- a) The reduced compliance matrix
- b) Strains in the 1-2 co-ordinate axis if the following stresses are applied.
 $\sigma_1 = 2 \text{ MPa}$, $\sigma_2 = -3 \text{ MPa}$, $\tau_{12} = 4 \text{ MPa}$.

The material properties are as follows;

$$\begin{aligned}E_1 &= 181 \text{ GPa} \\E_2 &= 10.3 \text{ GPa} \\v_{12} &= 0.28 \\G_{12} &= 7.17 \text{ GPa}\end{aligned}$$

Notes adapted from/further reading:
Mechanics of composite materials, 2nd edition, Autar K Kaw

- ONE PLY $\sim 0.125\text{mm}$ = LAMINA
- MANY PLYS = LAMINATE



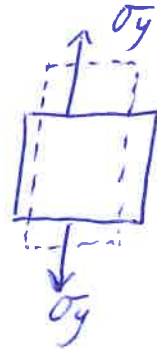
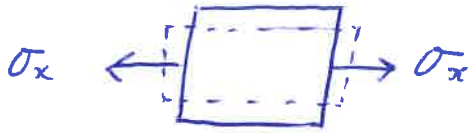
A LAMINATE MADE OF 3 LAMINAE

TO UNDERSTAND A LAMINATE WE MUST FIRST UNDERSTAND THE LAMINA.

- UNLIKE AN ISOTROPIC HOMOGENEOUS MATERIAL
- ANISOTROPIC — NOT ISOTROPIC
- ORTHOTROPIC — 3 $\perp$ DIRECTIONS
- "MACRO" — TO SIMPLIFY THINGS THE LAMINATE IS BASED ON THE 'AVERAGE' OF THE REINFORCEMENT / FIBRE PROPERTIES & IS THUS CONSIDERED 'HOMOGENEOUS'

CONSIDER A LINEAR ELASTIC MATERIAL

STRESSED IN x & y
NORMALLY



STRAIN IN x $\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu \sigma_y}{E}$

$$\epsilon_x = \frac{1}{E} \sigma_x - \frac{\nu}{E} \sigma_y$$

SIMILARLY

$$\epsilon_y = \frac{1}{E} \sigma_y - \frac{\nu}{E} \sigma_x$$

IF WE CONSIDER THE z DIRECTION WE WOULD ALSO HAVE

$$\epsilon_x = \frac{1}{E} \sigma_x - \frac{\nu}{E} \sigma_y - \frac{\nu}{E} \sigma_z \quad (1)$$

$$\epsilon_y = -\frac{\nu}{E} \sigma_x + \frac{1}{E} \sigma_y - \frac{\nu}{E} \sigma_z \quad (2)$$

$$\epsilon_z = -\frac{\nu}{E} \sigma_x - \frac{\nu}{E} \sigma_y + \frac{1}{E} \sigma_z \quad (3)$$

IF WE CONSIDER THAT THIS IS ALSO UNDER SHEAR WE WOULD HAVE

$$G = \frac{\tau}{\gamma} \quad \text{ie.} \quad \gamma_{yz} = \frac{\tau_{yz}}{G} \quad (4)$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \quad (5)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad (6)$$

WRITING EQUATIONS ①-⑥ IN MATRIX FORM WE HAVE ③

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}$$

THE 6×6 MATRIX IS CALLED THE "COMPLIANCE" MATRIX. IF THIS IS INVERTED WE GET THE "STIFFNESS" MATRIX

• ASSUMING LINEAR & ELASTIC BEHAVIOUR FOR A COMPOSITE IS ACCEPTABLE - HOWEVER ~~IF~~ ASSUMING IT IS ISOTROPIC IS NOT ... WE HAVE COSINE 1, 2, & 3 INSTEAD OF x, y, z)

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

• THUS FOR AN ISOTROPIC MATERIAL

$$S_{11} = \frac{1}{E} = S_{22} = S_{33}$$

$$S_{12} = -\frac{\nu}{E} = S_{13} = S_{21} = S_{23} = S_{31} = S_{32}$$

$$S_{44} = \frac{1}{G} = S_{55} = S_{66}$$

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NOTE THAT

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

HAS 36 UNKNOWN IN THE COMPLIANCE MATRIX, HOWEVER

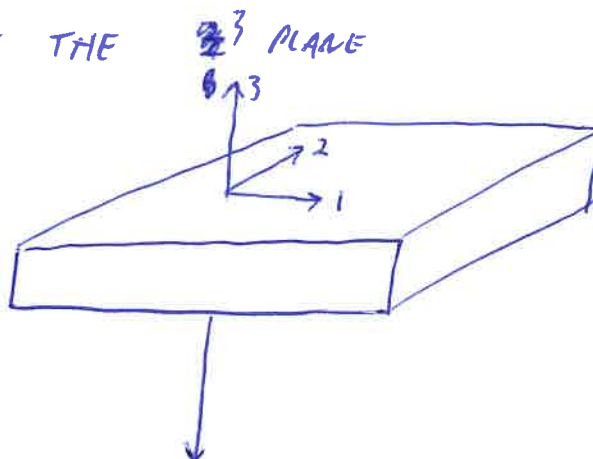
$S_{11} = S_{16}$ ETC, THIS REDUCES TO 21 UNKNOWN

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11}^{(1)} & S_{12}^{(2)} & S_{13}^{(3)} & S_{14}^{(4)} & S_{15}^{(14)} & S_{16}^{(11)} \\ S_{12} & S_{22}^{(5)} & S_{23}^{(12)} & S_{24}^{(13)} & S_{25}^{(15)} & S_{26}^{(15)} \\ S_{13} & S_{23} & S_{33}^{(8)} & S_{34}^{(16)} & S_{35}^{(17)} & S_{36}^{(15)} \\ S_{14} & S_{24} & S_{34} & S_{44}^{(6)} & S_{45}^{(18)} & S_{46}^{(23)} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55}^{(7)} & S_{56}^{(21)} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66}^{(9)} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

↑ ANISOTROPIC MATERIAL : 21 CONSTANTS!

BUT THIS CAN BE SIMPLIFIED

IF THE MATERIAL HAS ONE PLANE OF SYMMETRY
(i.e. IT IS MONOCLINIC) SUCH AS THE FOLLOWING IN WHICH IT
IS SYMMETRICAL ABOUT THE x_3 PLANE



$$\gamma_{23} = 0$$

$$\gamma_{13} = 0$$

$$\gamma_{31} = 0$$

$$\gamma_{23} = 0$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & \cancel{S_{14}} & \cancel{S_{15}} & S_{16} \\ S_{21} & S_{22} & S_{23} & \cancel{S_{24}} & \cancel{S_{25}} & S_{26} \\ S_{31} & S_{32} & S_{33} & \cancel{S_{34}} & \cancel{S_{35}} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

BECOMES ...

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

5

IF $\tau_{23} = 0$

$\tau_{13} = 0$

$\gamma_{31} = 0$

$\gamma_{23} = 0$

THE STRAINS IN 1, 2 & 3 ($\epsilon_1, \epsilon_2, \epsilon_3$) ARE NOT RELATED TO THE SHEAR STRESSES IN THE 23 OR THE 13 PLANES (τ_{23}, τ_{31}) AS THESE CANCEL OUT *

* & THE SHEAR STRAIN IN THE 12 PLANE (γ_{12})

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11}^{(1)} & S_{12}^{(7)} & S_{13}^{(8)} & 0 & 0 & S_{16}^{(9)} \\ S_{12} & S_{22}^{(2)} & S_{23}^{(10)} & 0 & 0 & S_{26}^{(11)} \\ S_{13} & S_{23} & S_{33}^{(3)} & 0 & 0 & S_{36}^{(12)} \\ 0 & 0 & 0 & S_{44}^{(4)} & S_{45}^{(13)} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55}^{(5)} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66}^{(6)} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

THE SHEAR STRAINS IN THE 23 & THE 13 PLANES (γ_{23}, γ_{31}) ARE ONLY DEPENDENT ON SHEAR STRESSES ON THESE PLANES

• THE DIRECTION ⊥ TO THE PLANE OF SYMMETRY IS CALLED THE PRINCIPAL DIRECTION (6)

NOTE THAT OUR 36 UNKNOWN THAT REDUCED TO 21 HAS NOW REDUCED TO (13)

• IN ADDITION FOR A MATERIAL THAT HAS 3 MUTUALLY ⊥ PLANES OF SYMMETRY & IT IS ORTHOTROPIC

THE DIRECT STRAINS IN 1, 2, & 3 ($\epsilon_1, \epsilon_2, \epsilon_3$)

ARE ONLY RELATED TO THE DIRECT STRESSES

& THE SHEAR STRAINS ARE ONLY RELATED TO THE SHEAR STRESSES IN THOSE PLANES

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11}^{(1)} & S_{12}^{(7)} & S_{13}^{(8)} & 0 & 0 & 0 \\ S_{12} & S_{22}^{(2)} & S_{23}^{(9)} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33}^{(3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44}^{(6)} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55}^{(5)} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66}^{(4)} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

NOTE THAT OUR UNKNOWN ARE NOW (9)

EXAMPLE 1a

(7)

- WRITE THE COMPLIANCE MATRIX IN TERMS OF THE 9 ELASTICITY CONSTANTS FOR AN ORTHOTROPIC MATERIAL

SOLUTION
IF MATERIAL SUBJECTED TO STRESS IN σ_1 ONLY.

i.e. $\sigma_1 \neq 0$

SOLUTION $\sigma_2 = \sigma_3 = \tau_{23} = \tau_{32} = \tau_{12} = 0$

NOW

TO WORK OUT

TO WORK OUT ETC

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Annotations:
 - Red box around S_{11}, S_{12}, S_{13} in the first row of the compliance matrix.
 - Green box around S_{12}, S_{22}, S_{23} in the second row of the compliance matrix.
 - Red box around $\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{31}, \tau_{12}$ in the stress vector.
 - Red arrow pointing to σ_2 with text "SET TO ZERO".
 - Green arrow pointing to τ_{23} with text "SET TO ZERO".
 - Asterisk (*) next to the stress vector.

$$\epsilon_1 = S_{11} \sigma_1 \quad (1)$$

$$\epsilon_2 = S_{12} \sigma_1 \quad (2)$$

$$\epsilon_3 = S_{13} \sigma_1 \quad (3)$$

$$\gamma_{23} = 0$$

$$\gamma_{31} = 0$$

$$\gamma_{12} = 0$$

FROM *

NOW $E_1 = \frac{\sigma_1}{\epsilon_1} = \frac{1}{S_{11}}$ FROM (1)

i.e. $S_{11} = \frac{1}{E_1}$ (1)

$$\nu_{12} = -\frac{\epsilon_2}{\epsilon_1} = -\frac{S_{12} \sigma_1}{S_{11} \sigma_1} = -\frac{S_{12}}{S_{11}} \quad \text{FROM (2) \& (3)}$$

i.e. $\nu_{12} = -\frac{S_{12}}{S_{11}} = -S_{12} \div \frac{1}{E_1} = -S_{12} \times E_1 \quad \therefore \nu_{12} = -S_{12} E_1 \quad \therefore S_{12} = \frac{-\nu_{12}}{E_1}$ (2)

$$\nu_{13} = -\frac{\epsilon_3}{\epsilon_1} = -\frac{S_{13} \sigma_1}{S_{11} \sigma_1} = -\frac{S_{13}}{S_{11}}$$

i.e. $\nu_{13} = -\frac{S_{13}}{S_{11}} = -S_{13} \div \frac{1}{E_1} = -S_{13} E_1 \quad \therefore S_{13} = \frac{-\nu_{13}}{E_1}$ (3)

NOW CONSIDER MATERIAL SUBJECTED TO σ_2 ONLY

$$\sigma_1 = \sigma_3 = \tau_{23} = \tau_{31} = \tau_{12} = 0$$

$$\sigma_2 \neq 0$$

$$\epsilon_1 = s_{12} \sigma_2$$

$$\epsilon_2 = s_{22} \sigma_2$$

$$\epsilon_3 = s_{23} \sigma_2$$

$$\gamma_{23} = 0$$

$$\gamma_{31} = 0$$

$$\gamma_{12} = 0$$

FROM *

NOW $E_2 = \frac{\sigma_2}{\epsilon_2} = \frac{1}{s_{22}} \quad \therefore s_{22} = \frac{1}{E_2} \quad (4)$

$$\nu_{21} = -\frac{\epsilon_1}{\epsilon_2} = -\frac{s_{12} \sigma_2}{s_{22} \sigma_2} = -\frac{s_{12}}{s_{22}}$$

ie $\nu_{21} = -\frac{s_{12}}{s_{22}} = -s_{12} \div \frac{1}{E_2} \quad \therefore \nu_{21} = -s_{12} E_2 \quad \therefore s_{12} = -\frac{\nu_{21}}{E_2}$

$$\nu_{32} = -\frac{\epsilon_3}{\epsilon_2} = -\frac{s_{23} \sigma_2}{s_{22} \sigma_2} = -\frac{s_{23}}{s_{22}}$$

ie $\nu_{32} = -\frac{s_{23}}{s_{22}} = -s_{23} \div \frac{1}{E_2} \quad \therefore \nu_{32} = -s_{23} E_2 \quad \therefore s_{23} = -\frac{\nu_{32}}{E_2} \quad (5)$

NOW CONSIDER SUBJECTED TO σ_3 ONLY:

(9)

$$\sigma_1 = \sigma_2 = \tau_{23} = \tau_{31} = \tau_{12} = 0 \quad \text{AND} \quad \sigma_3 \neq 0$$

$$\epsilon_1 = s_{13} \sigma_3$$

$$\epsilon_2 = s_{23} \sigma_3$$

$$\epsilon_3 = s_{33} \sigma_3$$

$$\gamma_{23} = 0$$

$$\gamma_{31} = 0$$

$$\gamma_{12} = 0$$

FROM *

$$\therefore E_3 = \frac{\sigma_3}{\epsilon_3} = \frac{1}{s_{33}}$$

$$s_{33} = \frac{1}{E_3} \quad (6)$$

$$\nu_{31} = -\frac{\epsilon_1}{\epsilon_3} = -\frac{s_{13} \sigma_3}{s_{33} \sigma_3} = -\frac{s_{13}}{s_{33}} \quad \text{ie } \nu_{31} = -\frac{s_{13}}{s_{33}} = -s_{13} E_3 \therefore s_{13} = -\frac{\nu_{31}}{E_3}$$

$$\nu_{32} = -\frac{\epsilon_2}{\epsilon_3} = -\frac{s_{23} \sigma_3}{s_{33} \sigma_3} = -\frac{s_{23}}{s_{33}} \quad \text{ie } \nu_{32} = -\frac{s_{23}}{s_{33}} = -s_{23} E_3 \therefore s_{23} = -\frac{\nu_{32}}{E_3}$$

NOW CONSIDER SUBJECTED TO τ_{23} ONLY

$$\sigma_1 = \sigma_2 = \sigma_3 = \tau_{31} = \tau_{12} = 0 \quad \text{AND} \quad \tau_{23} \neq 0$$

$$\epsilon_1 = 0$$

$$\epsilon_2 = 0$$

$$\epsilon_3 = 0$$

$$\gamma_{23} = s_{44} \tau_{23}$$

$$\gamma_{31} = 0$$

$$\gamma_{12} = 0$$

FROM *

$$\text{NOW} \quad G_{23} = \frac{\tau_{23}}{\gamma_{23}} = \frac{1}{s_{44}}$$

$$s_{44} = \frac{1}{G_{23}} \quad (7)$$

SIMILARLY, IF $\tau_{31} \neq 0$

$$G_{31} = \frac{1}{S_{55}}$$

$$S_{55} = \frac{1}{G_{31}} \quad (8)$$

AND IF $\tau_{12} \neq 0$

$$G_{12} = \frac{1}{S_{66}}$$

$$S_{66} = \frac{1}{G_{12}} \quad (9)$$

(10)

USING HIGHLIGHTED EQUATIONS ON P (7) - P (10)

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{E_1} (1) & -\nu_{12} (2) & -\nu_{13} (3) & 0 & 0 & 0 \\ -\nu_{12} (2) & \frac{1}{E_2} (4) & -\nu_{23} (5) & 0 & 0 & 0 \\ -\nu_{13} (3) & -\nu_{23} (5) & \frac{1}{E_3} (6) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} (7) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} (8) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} (9) \end{bmatrix}$$

EXAMPLE (2) (16)

(11)

NOW FIND THE COMPLIANCE MATRIX FOR A GRAPHITE/EPOXY LAMINA WHOSE MATERIAL PROPERTIES ARE

$$E_1 = 181 \text{ GPa}$$

$$E_2 = 10.3 \text{ GPa}$$

$$E_3 = 10.3 \text{ GPa}$$

$$\nu_{12} = 0.28$$

$$\nu_{23} = 0.60$$

$$\nu_{13} = 0.27$$

$$G_{12} = 7.17 \text{ GPa}$$

$$G_{23} = 3.0 \text{ GPa}$$

$$G_{31} = 7.0 \text{ GPa}$$

9 UNKNOWN!

SOLUTION

$$S_{11} = \frac{1}{E_1} = \frac{1}{181 \times 10^9} = 5.525 \times 10^{-12} \text{ Pa}^{-1}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.28}{181 \times 10^9} = -1.547 \times 10^{-12} \text{ Pa}^{-1}$$

$$S_{13} = -\frac{\nu_{13}}{E_1} = -\frac{0.27}{181 \times 10^9} = -1.492 \times 10^{-12} \text{ Pa}^{-1}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{10.3 \times 10^9} = 9.709 \times 10^{-11} \text{ Pa}^{-1}$$

$$S_{23} = -\frac{\nu_{23}}{E_2} = -\frac{0.60}{10.3 \times 10^9} = -5.825 \times 10^{-11} \text{ Pa}^{-1}$$

$$S_{33} = \frac{1}{E_3} = \frac{1}{10.3 \times 10^9} = 9.709 \times 10^{-11} \text{ Pa}^{-1}$$

$$S_{44} = \frac{1}{G_{23}} = \frac{1}{3.0 \times 10^9} = 3.333 \times 10^{-10} \text{ Pa}^{-1}$$

$$S_{55} = \frac{1}{G_{31}} = \frac{1}{7 \times 10^9} = 1.429 \times 10^{-10} \text{ Pa}^{-1}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17 \times 10^9} = 1.395 \times 10^{-10} \text{ Pa}^{-1}$$

~~$$S_{22} = \frac{1}{E_2} = \frac{1}{10.3 \times 10^9} = 9.709 \times 10^{-11} \text{ Pa}^{-1}$$~~

$$[S] = \begin{bmatrix} 5.525 \times 10^{-12} & -1.547 \times 10^{-12} & -1.492 \times 10^{-12} & 0 & 0 & 0 \\ -1.547 \times 10^{-12} & 9.709 \times 10^{-11} & 5.825 \times 10^{-11} & 0 & 0 & 0 \\ -1.492 \times 10^{-12} & -5.825 \times 10^{-11} & 9.709 \times 10^{-11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.333 \times 10^{-10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.429 \times 10^{-10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.395 \times 10^{-10} \end{bmatrix} p_a^{-1}$$

A UNIDIRECTIONAL 2D LAMINA

13

SINCE $\sigma_3 = 0$, $\tau_{23} = 0$ & $\tau_{31} = 0$ IT REDUCES TO A 2D PROBLEM (2a)

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{bmatrix}$$

SINCE γ_{23} & γ_{31} ARE CLEARLY ZERO THESE CAN BE OMITTED

$$\text{ie } \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 \\ S_{12} & S_{22} & S_{23} & 0 \\ S_{13} & S_{23} & S_{33} & 0 \\ 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ \tau_{12} \end{bmatrix}$$

SINCE ϵ_3 IS RELATED TO ϵ_1 & ϵ_2 THIS CAN ALSO BE OMITTED, ie.

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

WHERE
[S] IS THE
REDUCED COMPLIANCE
MATRIX.

$$\epsilon \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

WHERE

$$S_{11} = \frac{1}{E_1}, \quad S_{22} = \frac{1}{E_2}$$

$$S_{12} = -\frac{\nu_{12}}{E_1}, \quad S_{66} = \frac{1}{G_{12}}$$

FOR (26)

EXAMPLE: A GRAPHITE/EPoxy UNIDIRECTIONAL LAMINA FIND

- THE REDUCED COMPLIANCE MATRIX
- STRAINS IN THE 1-2 COORDINATE AXIS IF THE FOLLOWING STRESSES ARE APPLIED:

$$\sigma_1 = 2 \text{ MPa}$$

$$\sigma_2 = -3 \text{ MPa}$$

$$\tau_{12} = 4 \text{ MPa}$$

$$(E_1 = 181 \text{ GPa}, E_2 = 10.36 \text{ GPa}, \nu_{12} = 0.28, G_{12} = 7.17 \text{ GPa})$$

$$S_{11} = \frac{1}{E_1} = \frac{1}{181 \times 10^9} = 0.5525 \times 10^{-11} \text{ Pa}^{-1}$$

$$S_{12} = \frac{-\nu_{12}}{E_1} = \frac{-0.28}{181 \times 10^9} = -0.1547 \times 10^{-11} \text{ Pa}^{-1}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{10.3 \times 10^9} = 0.9709 \times 10^{-10} \text{ Pa}^{-1}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17 \times 10^9} = 0.1395 \times 10^{-9} \text{ Pa}^{-1}$$

$$[S] = \begin{bmatrix} 0.5525 \times 10^{-11} & -0.1547 \times 10^{-11} & 0 \\ -0.1547 \times 10^{-11} & 0.9709 \times 10^{-10} & 0 \\ 0 & 0 & 0.1395 \times 10^{-9} \end{bmatrix} \text{ Pa}^{-1}$$

$$\therefore \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 0.5525 \times 10^{-11} & -0.1547 \times 10^{-11} & 0 \\ -0.1547 \times 10^{-11} & 0.9709 \times 10^{-10} & 0 \\ 0 & 0 & 0.1395 \times 10^{-9} \end{bmatrix} \begin{bmatrix} 2 \times 10^6 \\ -3 \times 10^6 \\ 4 \times 10^6 \end{bmatrix}$$

$$= \begin{bmatrix} 15.69 \\ -294.4 \\ 557.9 \end{bmatrix} (10^{-6})$$

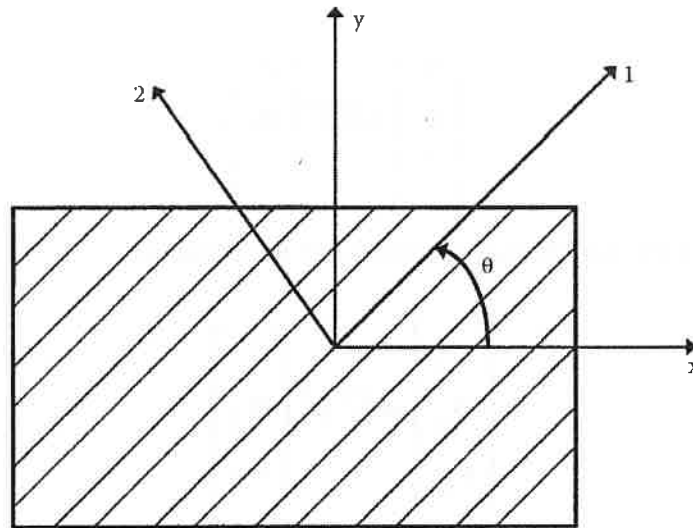
$$\therefore \varepsilon_1 = 15.69 \mu$$

$$\varepsilon_2 = -294.4 \mu$$

$$\gamma_{12} = 557.9 \mu$$

Macromechanical analysis of a lamina – Angle Plys

We have looked at calculating stress and strain for a uni-directional lamina. However due to the fact that the transverse stiffness is generally much lower in most laminates, some of the lamina are placed at an angle. As such it is necessary to develop the stress-strain relationship for an angle lamina. Consider the below lamina where the global axis are in x and y and the fibers are aligned at an angle theta such that the 1 direction is in the fiber orientation and the 2 direction is perpendicular to the fibers.



It can be shown that the stresses in the local axis relative to the global axis are given by;

$$\sigma_1 = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta \quad [1]$$

$$\sigma_2 = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2\tau_{xy} \sin \theta \cos \theta \quad [2]$$

$$\tau_{12} = -\sigma_x \cos \theta \sin \theta + \sigma_y \cos \theta \sin \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \quad [3]$$

If equations 1-3 are combined and written in matrix for we get the following;

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Where $c = \cos \theta$ and $s = \sin \theta$

This can also be written as;

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix},$$

Where $[T]$ is called the transformation matrix and is defined as;

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix},$$

Since

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

If we want to convert this to a local co-ordinate axis, we simply include the transformation matrix;

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Examples

For a 60° angle lamina of Graphite/Epoxy find

- a) The reduced compliance matrix
- b) The local stresses
- c) The local strains

The material properties and the loading conditions are as follows;

$$E_1 = 181 \text{ GPa}$$

$$E_2 = 10.3 \text{ GPa}$$

$$\nu_{12} = 0.28$$

$$G_{12} = 7.17 \text{ GPa}$$

$$\sigma_x = 2 \text{ MPa}$$

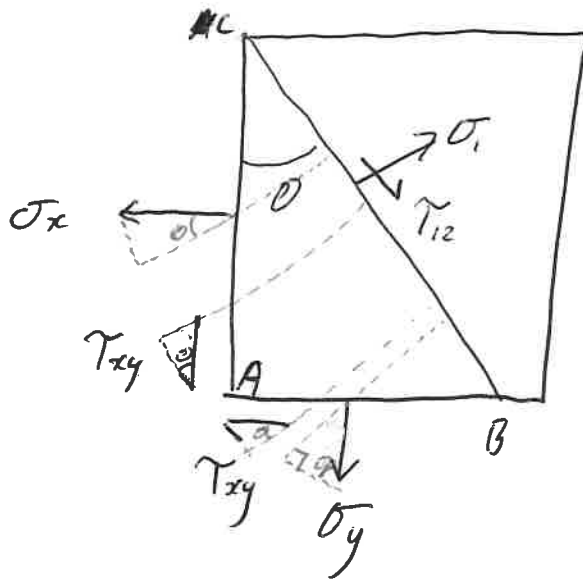
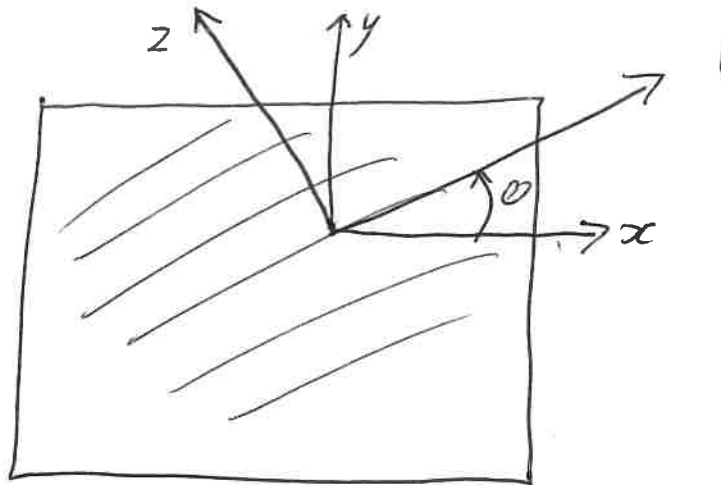
$$\sigma_y = -3 \text{ MPa}$$

$$\tau_{xy} = 4 \text{ MPa}$$

Notes adapted from/further reading:

Mechanics of composite materials, 2nd edition, Autar K Kaw

TRANSFORMATION OF STRESSES



$$\sum F_i = 0$$

$$\text{FORCE} = \text{STRESS} \times \text{AREA}$$

$$\sigma_1 BC - \tau_{xy} AB \cos \theta - \sigma_y AB \sin \theta - \tau_{xy} AC \sin \theta - \sigma_x AC \cos \theta = 0$$

$$\Sigma F \parallel TO \quad \text{NAVE} = 0$$

(17)

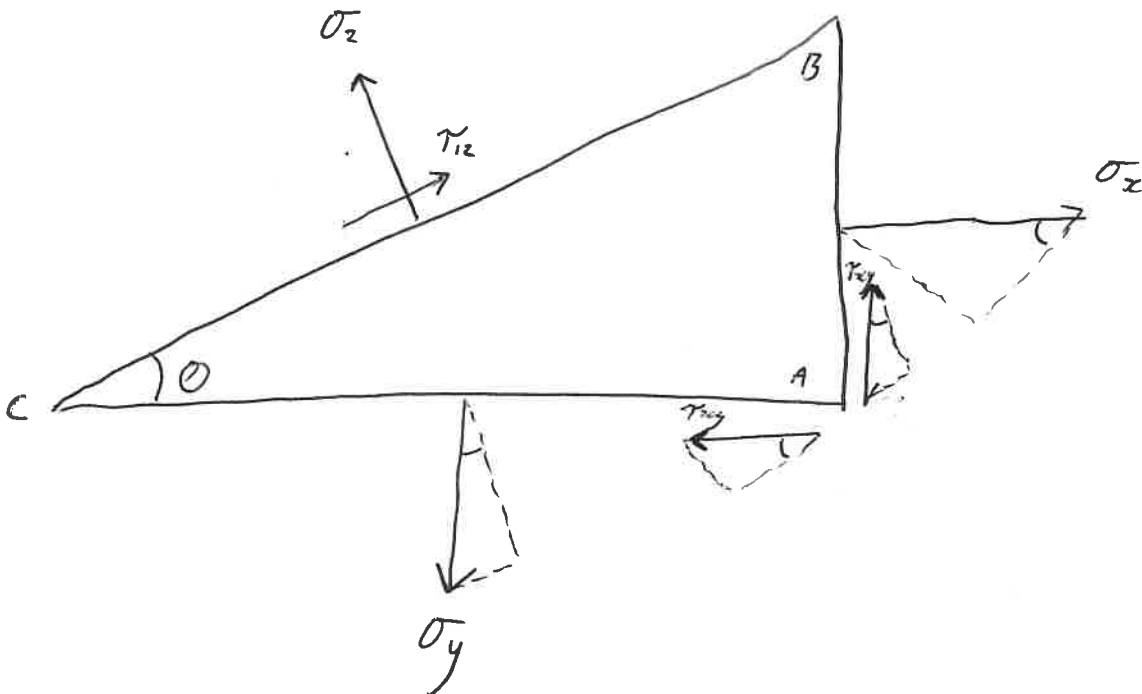
$$+ \sigma_y \cos \theta \overline{AB} + \tau_{xy} AC \cos \theta = \tau_{xy} \overline{AB} \sin \theta + \sigma_x AC \sin \theta + \tau_{12} \overline{BC}$$

$$\tau_{12} = -\tau_{xy} \frac{\overline{AB} \sin \theta}{\overline{BC}} - \sigma_x \frac{AC \sin \theta}{\overline{BC}} + \sigma_y \cos \theta \frac{\overline{AB}}{\overline{BC}} + \tau_{xy} AC \frac{\cos \theta}{\overline{BC}}$$

$$= -\tau_{xy} \sin \theta \sin \theta - \sigma_x \cos \theta \sin \theta + \sigma_y \cos \theta \sin \theta + \tau_{xy} \cos \theta \cos \theta$$

$$= \sigma_y - \sigma_x \cos \theta \sin \theta + \sigma_y \cos \theta \sin \theta - \tau_{xy} \sin^2 \theta + \tau_{xy} \cos^2 \theta$$

$$\tau_{12} = -\sigma_x \cos \theta \sin \theta + \sigma_y \cos \theta \sin \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \quad (2)$$



$$\Sigma F_2 = 0$$

$$\overline{BC} \sigma_2 + \tau_{xy} \overline{AB} \cos \theta - \sigma_y \overline{AC} \cos \theta - \sigma_x \overline{AB} \sin \theta + \tau_{xy} \overline{AC} \sin \theta = 0$$

$$\sigma_2 = -\tau_{xy} \frac{\overline{AB}}{\overline{BC}} \cos \theta + \sigma_y \frac{\overline{AC}}{\overline{BC}} \cos \theta + \sigma_x \frac{\overline{AB}}{\overline{BC}} \sin \theta - \tau_{xy} \frac{\overline{AC}}{\overline{BC}} \sin \theta$$

$$= -\tau_{xy} \sin \theta \cos \theta + \sigma_y \cos \theta \cos \theta + \sigma_x \sin \theta \sin \theta - \tau_{xy} \cos \theta \sin \theta$$

$$\sigma_2 = \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2 \tau_{xy} \sin \theta \cos \theta \quad (3)$$

WRITING EQUATIONS ①-③ IN MATRIX FORM
GIVES THE TRANSFORMATION MATRIX $[T]$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{TO CONVERT LOCAL} \\ \leftarrow \text{TO GLOBAL STRESSES} \end{array}$$

WHERE $c = \cos \theta$ & $s = \sin \theta$

NOW

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{TO CONVERT} \\ \text{GLOBAL STRESSES} \\ \text{TO LOCAL} \\ \text{STRAINS} \end{array}$$

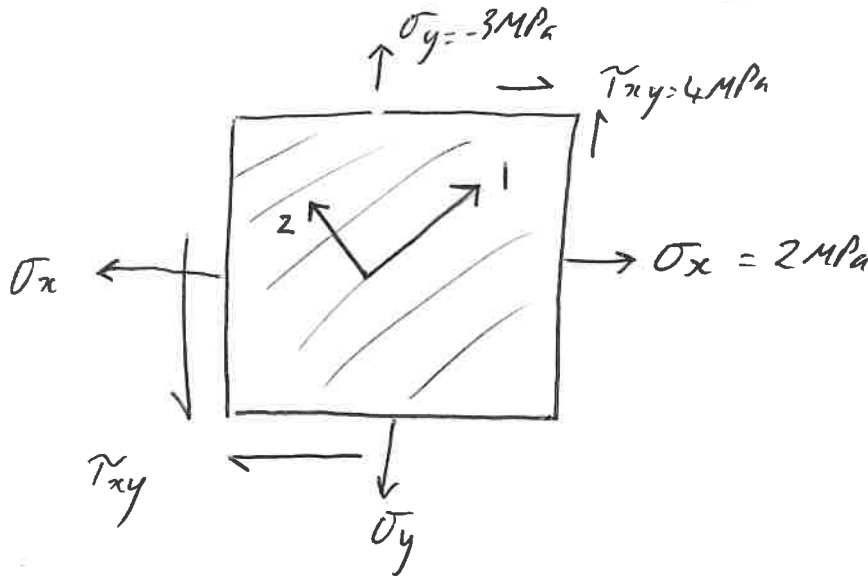
EXAMPLE

FOR A 60° ANGLE LAMINA OF GRAPHITE/EPOXY

FIND

REDUCED COMPLIANCE MATRIX

- (a) - THE
 (b) - THE LOCAL STRESSES
 (c) - THE LOCAL STRAINS



$$E_1 = 181 \text{ GPa}$$

$$E_2 = 10.36 \text{ GPa}$$

$$\nu_{12} = 0.28$$

$$G_{12} = 7.17 \text{ GPa}$$

(a)

$$[S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}$$

$$\text{Now } S_{11} = \frac{1}{E_1} = \frac{1}{181 \times 10^9} = \underline{\underline{0.5525 \times 10^{-11} \text{ Pa}^{-1}}}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{10.3 \times 10^9} = \underline{\underline{0.9709 \times 10^{-10} \text{ Pa}^{-1}}}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.28}{181 \times 10^9} = \underline{\underline{-0.1547 \times 10^{-11} \text{ Pa}^{-1}}}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17 \times 10^9} = \underline{\underline{0.1395 \times 10^{-9} \text{ Pa}^{-1}}}$$

$$\therefore [S] = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} = \begin{bmatrix} 0.5525 \times 10^{-11} & -0.1547 \times 10^{-11} & 0 \\ -0.1547 \times 10^{-11} & 0.9709 \times 10^{-10} & 0 \\ 0 & 0 & 0.1395 \times 10^{-9} \end{bmatrix} \text{ Pa}^{-1}$$

b)

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$= \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$$\begin{aligned} \sigma_1 &= 2 \times \cos^2 60 - 3 \times \sin^2 60 + 4 \times 2 \times \sin 60 \cos 60 \\ &= 0.5 - 2.25 + 3.46 \\ &= \underline{\underline{1.71 \text{ MPa}}} \end{aligned}$$

$$\begin{aligned} \sigma_2 &= 2 \times \sin^2 60 - 3 \times \cos^2 60 - 4 \times 2 \times \sin 60 \cos 60 \\ &= 1.5 - 0.75 - 3.46 \\ &= \underline{\underline{-2.71 \text{ MPa}}} \end{aligned}$$

$$\begin{aligned} \tau_{12} &= -\sin 60 \cos 60 \times 2 - \sin 60 \cos 60 \times 3 + 4(\cos^2 60 - \sin^2 60) \\ &= -0.87 - 1.30 + 4(0.25 - 0.75) \\ &= \underline{\underline{-4.17 \text{ MPa}}} \end{aligned}$$

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$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 0.5525 \times 10^{-11} & -0.1547 \times 10^{-11} & 0 \\ -0.1547 \times 10^{-11} & 0.9709 \times 10^{-10} & 0 \\ 0 & 0 & 0.1395 \times 10^{-9} \end{bmatrix} \begin{bmatrix} 1.71 \\ -2.71 \\ -4.17 \end{bmatrix} \times 10^6$$

$$\begin{aligned} \epsilon_1 &= 0.5525 \times 10^{-11} \times 1.71 \times 10^6 - 0.1547 \times 10^{-11} \times 2.71 \times 10^6 + 0 \\ &= \underline{\underline{0.136 \times 10^{-4}}} \end{aligned}$$

$$\begin{aligned} \epsilon_2 &= -0.1547 \times 10^{-11} \times 1.71 \times 10^6 - 0.9709 \times 10^{-10} \times 2.71 \times 10^6 + 0 \\ &= \underline{\underline{-0.266 \times 10^{-3}}} \end{aligned}$$

$$\begin{aligned} \gamma_{12} &= 0 + 0 + 0.1395 \times 10^{-9} \times -4.17 \times 10^6 \\ &= \underline{\underline{-0.582 \times 10^{-3}}} \end{aligned}$$