

Bending Of Beams

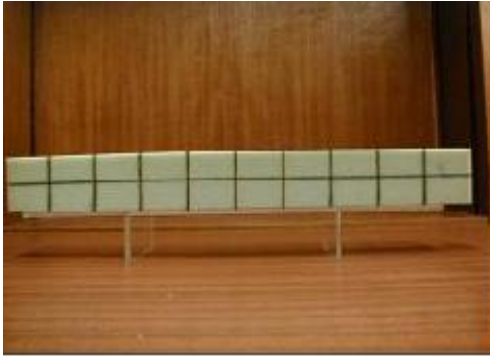
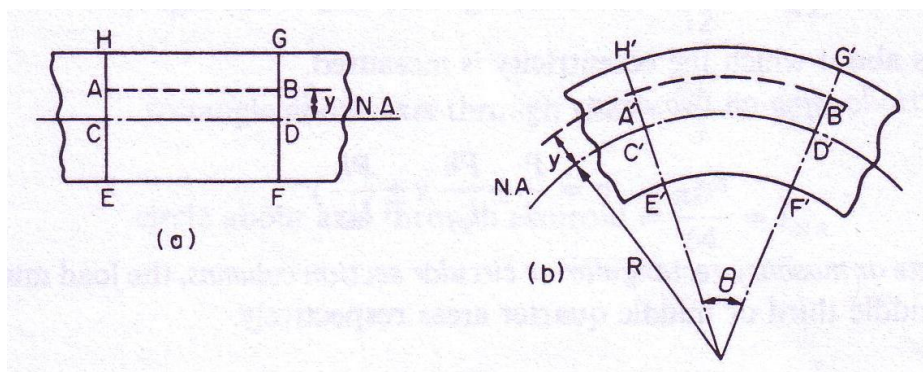


Figure 1: A straight beam



Figure 2: Beam in bending

Consider the beam shown above, we can see the bottom of the beam has gone into compression and the top of the beam is in tension. The line in the middle represents the **NEUTRAL AXIS**. Let's consider a small section of this beam;



$$\text{Strain} = \frac{\text{Change in length}}{\text{Original length}} = \frac{A'B' - AB}{AB} = \frac{A'B' - C'D'}{C'D'} = \frac{y}{R}$$

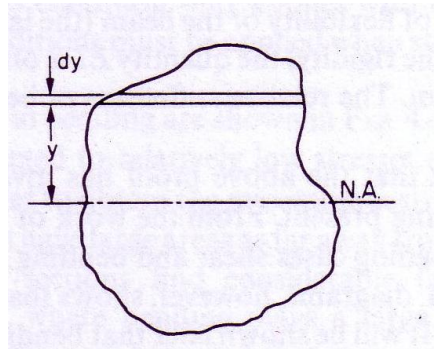
Also,

$$\text{Strain} = \frac{\text{Stress}}{\text{Young's Modulus}} = \frac{\sigma}{E}$$

$$\therefore \frac{\sigma}{y} = \frac{E}{R}$$

[1]

Now let's consider a section through a beam of generic section that is subject to bending.



$Force = Stress \times Area$

$$F = \sigma \delta A$$

$$\therefore F = \frac{E}{R} y \delta A$$

This has a moment of;

$$Fy = \frac{E}{R} y^2 \delta A$$

$$\therefore M = \sum \frac{E}{R} y^2 \delta A = \frac{E}{R} \sum y^2 \delta A$$

$\sum y^2 \delta A$ is termed the second moment of area I ;

$$\therefore M = \frac{E}{R} I$$

$$\text{Or, } \frac{M}{I} = \frac{E}{R} \quad [2]$$

Combining [1] and [2] we get the **simple theory of elastic bending**;

$$\boxed{\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}}$$

Notice that if the beam is a uniform section, the material is homogeneous and the moment is constant, I , E and M are all constants, therefore R must also be constant. i.e. the curvature of the beam is perfectly circular.

The simple theory of elastic bending states that;

$$\sigma = \frac{M}{I} y$$

i.e. the stress in our bent beam is directly proportional to the distance from the neutral axis.

Force on our beam element

Consider again our small element in our generic section that is subject to bending. We have shown that on the element;

$$Force = Stress \times Area$$

$$F = \sigma \delta A$$

Thus the force over the entire section is given by;

$$\therefore F = \int \sigma dA$$

If we assume that the beam is in equilibrium then;

$$\int \sigma dA = 0$$

$$\int \frac{E}{R} y dA = 0$$

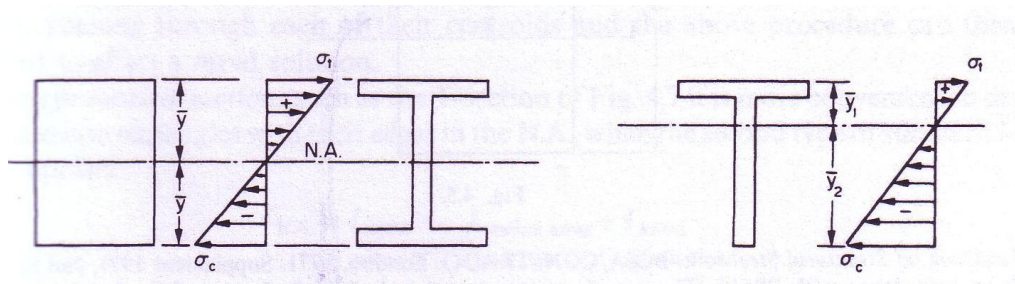
$$\frac{E}{R} \int y dA = 0$$

Where $\int y dA$ is termed the first moment of area, thus;

$\int y dA$ is termed the first moment of area

$\int y^2 dA$ is termed the second moment of area

By definition, $y=0$ at the neutral axis, thus the first moment of area=0 at the neutral axis, therefore, the **neutral axis must pass through the centroid**.



Typical bending stress distributions are shown above. Notice that stress is zero at the neutral axis and a maximum when y is greatest. Hence the use of beams with large areas away from the neutral axis ('I' section beams for example).

We have shown that;

$$\sigma = \frac{M}{I} y$$

$$\therefore \sigma_{\max} = \frac{M}{I} y_{\max}$$

Therefore the maximum moment that the beam can carry is given by;

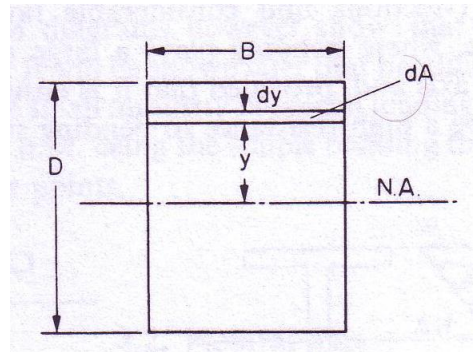
$$\therefore M = \frac{I}{y_{\max}} \sigma_{\max} = Z \sigma_{\max}$$

where $Z = \frac{I}{y_{\max}}$ and is called the section modulus

For easy comparisons between beams the **section modulus (Z)** is often quoted

Second moment of area

Most beams can be considered of being made up of various rectangular sections. It is useful therefore to derive the second moment of area for a rectangular beam so that we can use this to calculate the second moment of area of a more complex beam.



By definition; $I = \int y^2 dA$

Therefore in the above example;

$$I = \int_{-\frac{D}{2}}^{\frac{D}{2}} y^2 B dy = B \int_{-\frac{D}{2}}^{\frac{D}{2}} y^2 dy$$

$$\therefore I = B \left[\frac{y^3}{3} \right]_{-\frac{D}{2}}^{\frac{D}{2}}$$

$$I_{NA} = \frac{BD^3}{12}$$

Notice that this is the second moment of area about the neutral axis because we integrated between $-D/2$ and $D/2$. If we are to integrate this between 0 and D we would get the second moment of area about the lower edge of the rectangle;

$$I = \int_{-0}^D y^2 B dy = B \int_{-0}^D y^2 dy$$

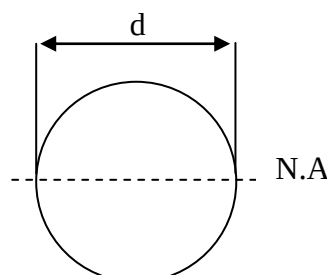
$$\therefore I = B \left[\frac{y^3}{3} \right]_0^D$$

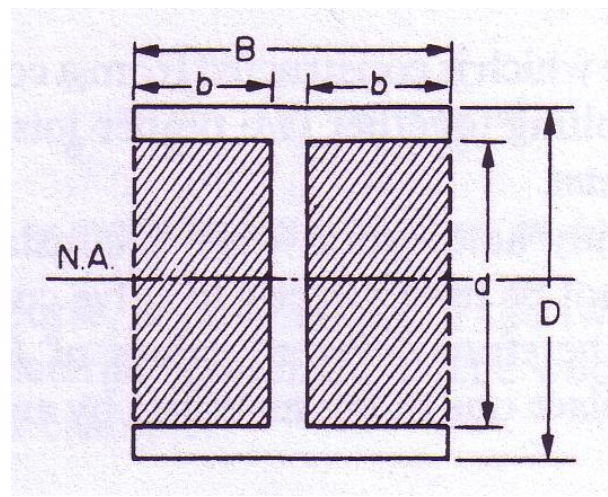
$$I = \frac{BD^3}{3}$$

These two equations prove very convenient when trying to calculate the second moment of area of a more complex beam as shown below;

In the same we can show that the second moment of area for a circle is;

$$I = \frac{\pi d^4}{64}$$

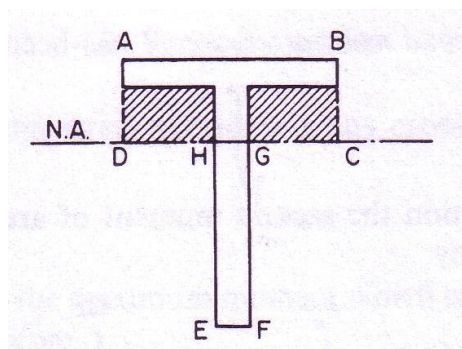




Consider the above 'I' beam. This can be considered as one solid beam ($B \times D$), minus two smaller beams shown by the shaded areas ($b \times d$). i.e.

$$I_{NA} = \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right)$$

This approach can be used with any symmetrical beam where the neutral axis runs through all of the rectangles that make up the solution to get a rapid solution. However, if the beam is unsymmetrical this is not possible, so it is more convenient to divide the section into rectangles with their edges on the neutral axis as shown below;



$$I_{NA} = I_{ABCD} - I_{\text{shaded areas}} + I_{EFGH}$$

Where each of these terms will be of the form $I = \frac{BD^3}{12}$

Alternatively, it is possible to determine each rectangle about its own neutral axis using $I_G = \frac{BD^3}{12}$, and then "shifting" them to the equivalent value about the N.A. by means of the **parallel axis theorem**.

$$I_{NA} = I_G + Ah^2$$

Where A is the area of the rectangle and h is the distance of its centroid G, from the neutral axis.

Tables of Sections

Structural steels (and other materials) are manufactured and supplied in a range of 'standard sections', and therefore the properties of these sections can also be made available for easy reference in any calculations associated with their use. Examples of these tables of sections are shown below.

Size $D \times B$ (mm)	Mass per metre (kg)	Thickness		I_{xx} (cm ⁴)	I_{yy} (cm ⁴)	Z_{xx} (cm ³)	Z_{yy} (cm ³)
		Web t (mm)	Flange T (mm)				
910.3 x 304.1	222	15.9	23.9	375 924	11 223	8259	738.1
840.7 x 292.4	194	14.7	21.7	279 450	9069	6648	620.4
762.0 x 266.7	173	14.3	21.6	205 177	6846	5385	513.4
677.9 x 253.0	125	11.7	16.2	118 003	4379	3481	346.1
617.5 x 307.0	113	14.1	23.6	151 631	11 412	4911	743.3
533.1 x 209.3	92	10.2	15.6	55 353	2392	2076	228.6
467.4 x 192.8	98	11.4	19.6	45 717	2343	1956	243.0
457.2 x 151.9	67	9.1	15.0	28 577	878	1250	115.5
402.3 x 142.4	46	6.9	11.2	15 647	539	777.8	75.7
364.0 x 173.2	67	9.1	15.7	19 522	1362	1073	157.3
352.8 x 126.0	39	6.5	10.7	10 087	357	571.8	56.6
310.9 x 166.8	54	7.7	13.7	11 710	1061	753.3	127.3
312.7 x 102.4	33	6.6	10.8	6487	193	415.0	37.8
254.0 x 101.6	22	5.8	6.8	2867	120	225.7	23.6
203.2 x 133.4	25	5.8	7.8	2356	310	231.9	46.4

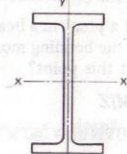
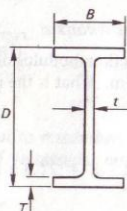


Fig. 2.31 Universal beam

Size $B \times A$ (mm)	Mass per metre (kg)	Thickness		I_{xx} (cm ⁴)	Z_{xx} (cm ³)	Extreme fibre distance	
		Web t (mm)	Flange T (mm)			y_f (cm)	y_t (cm)
304.1 x 455.2	112	15.9	23.9	29 036	869.8	12.1	33.42
292.4 x 420.4	97	14.7	21.7	21 381	690.9	11.1	30.94
266.7 x 381.0	87	14.3	21.6	15 495	551.3	9.99	28.11
253.0 x 339.0	63	11.7	16.2	8997	359.5	8.87	25.03
307.0 x 308.7	90	14.1	23.6	8949	369.5	6.65	24.22
209.3 x 266.6	46	10.2	15.6	3905	194.4	6.56	20.10
192.8 x 233.7	49	11.4	19.6	2978	167.2	5.55	17.82
151.9 x 228.6	34	9.1	15.0	2128	126.1	5.99	16.87
142.4 x 201.2	23	6.9	11.2	1130	75.0	5.05	15.07
173.2 x 182.0	34	9.1	15.7	1158	81.6	4.01	14.19
126.0 x 176.4	20	6.5	10.7	719.6	54.4	4.41	13.23
166.8 x 155.4	27	7.7	13.7	636.4	51.5	3.19	12.35
102.4 x 156.3	17	6.6	10.8	486.7	42.4	4.14	11.49
101.6 x 127.0	11	5.8	6.8	227.2	24.7	3.49	9.21
133.4 x 101.6	13	5.8	7.8	133.7	16.6	2.12	8.04

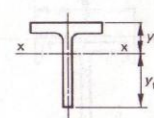
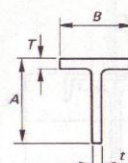


Fig. 2.32 Structural tee cut from universal beam

Size $D \times B$ (mm)	Mass per metre (kg)	Extreme fibre distance		I_{xx} (cm ⁴)	Z_{xx} (cm ³)
		y_1 (cm)	y_2 (cm)		
939 x 432	355	39.20	54.68	644 504	11 797
853 x 432	260	32.92	52.37	384 673	7345
774 x 432	239	29.16	48.26	288 718	5982
700 x 432	218	25.58	44.40	215 582	4856
630 x 432	245	24.11	38.86	206 436	5313
627 x 381	195	23.60	39.14	155 882	3982
622 x 381	180	22.74	39.49	140 490	3558
618 x 305	155	23.53	38.22	120 871	3163
555 x 381	177	20.27	35.23	108 879	3091
550 x 305	151	20.84	34.13	92 644	2715
478 x 305	140	17.77	29.98	64 431	2149
472 x 254	117	18.67	28.50	54 593	1916
465 x 229	93	17.57	28.91	39 916	1381
423 x 305	116	14.67	27.62	40 572	1469
418 x 254	95	15.52	26.23	34 059	1298

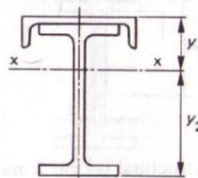
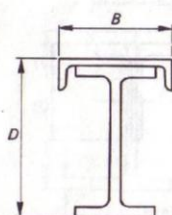


Fig. 2.33 Gantry girder

Examples

1) An I-section girder, 200mm wide by 300mm deep, with flange and web of thickness 20mm is used as a simply supported beam over a span of 7m. The girder carries a distributed load of 5kN/m and a concentrated load of 20kN at mid-span.

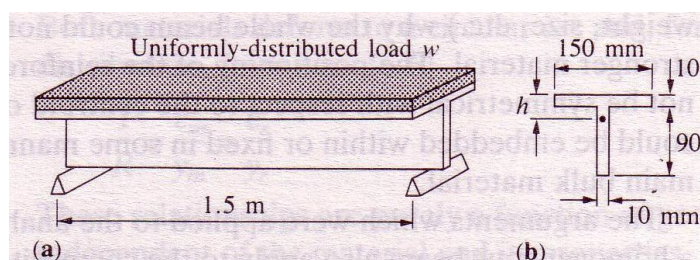
- Calculate the second moment of area using conventional methods
- Calculate the second moment of area using the parallel axis theorem
- Comment on the two methods used in parts (a) and (b)
- Calculate the maximum stress that the beam will see.
- Comment on where the maximum stress will be and if it will be compressive or tensile.

[$1.86 \times 10^{-4} \text{m}^4$, 52.9MPa]

2) A uniform T-section beam is 100mm wide and 150mm deep with a flange thickness of 25mm and a web thickness of 12mm. If the limiting bending stresses for the material for the beam are 80MPa in compression and 160MPa in tension, find the maximum u.d.l. that the beam can carry over a simply supported span of 5m.

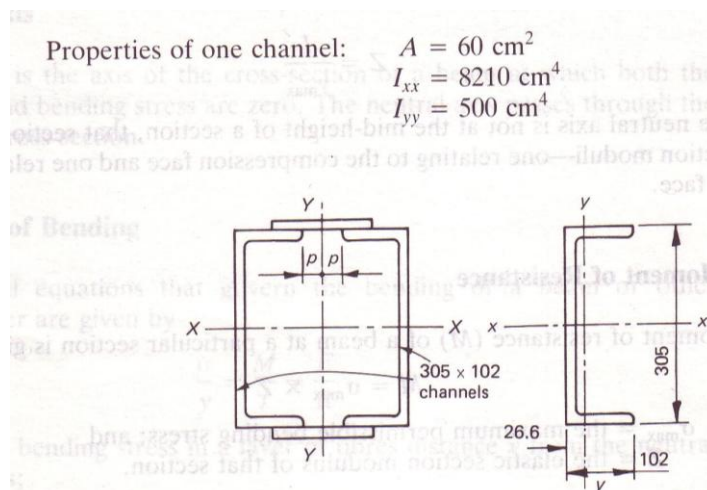
[3.4kN/m]

3) A tee-section bar has dimensions shown below. If the bar is used as a simply supported beam of span 1.5m, calculate the uniformly-distributed load which can be applied if the maximum tensile stress is not to exceed 100MN/m^2 .



[9.5kN/m]

4) A column section is built up by connecting together two $305 \text{ mm} \times 102 \text{ mm}$ channels, using battens (see Figure 5.1). If the column is to have equal second moments of area about its two axes of symmetry, calculate the spacing ($2p$) required between the two channels. (Note that the battens do not contribute to the second moments of area of the section.)



[75.92mm]

Limitations of the simple bending theory

- 1) *Stress is proportional to the distance from the neutral axis*
 - a) **Correct**, for homogeneous beams within the elastic range
 - b) **Incorrect**, outside of the elastic range or for non-homogeneous or composite beams.
- 2) *Strain is proportional to the distance from the neutral axis*
 - a) **Correct**, for initially straight beams or, for engineering purposes, beams with $R/d > 10$ (where d is the total depth of the beam)
 - b) **Incorrect**, for combined bending and axial load systems such as eccentric loading.
- 3) *Plane cross-sections remain plane*
 - a) **Correct**,
 - i. For cross sections at a reasonable distance from points of loading and stress concentration (usually taken as one depth of the beam)
 - ii. when the change of cross section with length is gradual.
 - iii. in the absence of end-condition spurious effects.
 - b) **Incorrect**,
 - iv. For points of local loading
 - v. At positions of stress concentrations (holes, fillets etc)
 - vi. In regions of rapid change of cross section.
- 4) *The axis of the applied bending moment is coincident with the neutral axis*
 - a) **Correct**, when the axis of bending is a principal axis (ie an axis of symmetry)
 - b) **Incorrect**, for 'unsymmetrical bending cases', when the axis of the applied bending moment is not a principal axis.
- 5) *Lateral contraction or expansion is not prevented*
 - a) **Correct**, when the beam is narrow (i.e. width is same order as depth)
 - b) **Incorrect**, for wide beams (when width is many times greater than the depth).