

Differentiating from First Principles -

1. (a) Given that $y = 2x^2 - 5x + 3$, find $\frac{dy}{dx}$ from first principles. [5]

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} & \left[\begin{array}{l} f(x) = 2x^2 - 5x + 3 \\ f(x+h) = 2(x+h)^2 - 5(x+h) + 3 \end{array} \right] \\
 &= \lim_{h \rightarrow 0} \frac{2(x+h)^2 - 5(x+h) + 3 - [2x^2 - 5x + 3]}{h} & \checkmark \text{ Replace all 'x' by 'x+h'} \\
 &= \lim_{h \rightarrow 0} \frac{2(x^2 + 2xh + h^2) - 5x - 5h + 3 - 2x^2 + 5x - 3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{5x} - 5h + \cancel{3} - \cancel{2x^2} + \cancel{5x} - \cancel{3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 - 5h}{h} \\
 &= \lim_{h \rightarrow 0} 4x + 2h - 5 \\
 &\text{As } h \rightarrow 0 \quad 2h \rightarrow 0 \quad \therefore \underline{\underline{\frac{dy}{dx} = 4x - 5}}
 \end{aligned}$$

- (b) Given that $y = \frac{a}{x} + 2x^{\frac{3}{2}}$ and $\frac{dy}{dx} = 7$ when $x = 4$, find the value of the constant a . [4]

$$\begin{aligned}
 \frac{dy}{dx} &= -ax^{-2} + 3x^{\frac{1}{2}} & \left[\begin{array}{l} y = \frac{a}{x} + 2x^{\frac{3}{2}} \\ = ax^{-1} + 2x^{\frac{3}{2}} \end{array} \right] \\
 \frac{dy}{dx} \Big|_{x=4} &= -a(4)^{-2} + 3(4)^{\frac{1}{2}} \\
 &= -\frac{a}{4^2} + 3\sqrt{4} \\
 &= -\frac{a}{16} + (3 \times 2) \\
 &= -\frac{a}{16} + 6
 \end{aligned}$$

$$\frac{dy}{dx} = 7 \Rightarrow -\frac{a}{16} + 6 = 7$$

$$-a + 96 = 112$$

$$a = 112 - 96 = \underline{\underline{-16}}$$

2. (a) Given that $y = x^2 - 3x + 4$, show from first principles that

[5]

$$\frac{dy}{dx} = 2x - 3$$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 3(x+h) + 4 - [x^2 - 3x + 4]}{h}$$

$f(x) = x^2 - 3x + 4$
 $f(x+h) = (x+h)^2 - 3(x+h) + 4$
 Replace all 'x' by 'x+h'

$$= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2 - 3x - 3h + 4 - x^2 + 3x - 4)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h}$$

$$= \lim_{h \rightarrow 0} 2x + h - 3$$

As $h \rightarrow 0$, $h \rightarrow 0$ $\therefore \frac{dy}{dx} = 2x - 3$

- (b) Differentiate $y = \frac{2}{x^2} + 7\sqrt{x}$ with respect to x .

[2]

$$y = 2x^{-2} + 7x^{1/2}$$

$$\frac{dy}{dx} = (-2)(2)x^{-2-1} + (7)\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}$$

$$= -4x^{-3} + \frac{7}{2}x^{-\frac{1}{2}}$$

$$= -\frac{4}{x^3} + \frac{7}{2\sqrt{x}}$$

← You can leave your answer like this

3. Given that $y = x^2 - 7x + 2$, find $\frac{dy}{dx}$ from first principles.

[5]

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \left[\begin{array}{l} f(x) = x^2 - 7x + 2 \\ f(x+h) = (x+h)^2 - 7(x+h) + 2 \end{array} \right]$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 7(x+h) + 2 - [x^2 - 7x + 2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + \cancel{h^2} - \cancel{7x} - 7h + \cancel{2} - \cancel{x^2} + \cancel{7x} - \cancel{2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2x\cancel{h} + \cancel{h^2} - 7\cancel{h}}{\cancel{h}}$$

$$= \lim_{h \rightarrow 0} 2x + h - 7$$

As $h \rightarrow 0$, $h \rightarrow 0$

$$\therefore \frac{dy}{dx} = \underline{\underline{2x - 7}}$$

4. (a) Differentiate $y = x^2 - 6x + 2$ from first principles.

[5]

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^2 - 6x + 2$$

$$f(x+h) = (x+h)^2 - 6(x+h) + 2$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 6(x+h) + 2 - [x^2 - 6x + 2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{6x} - 6h + \cancel{2} - \cancel{x^2} + \cancel{6x} - \cancel{2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2x\cancel{h} + h^2 - 6\cancel{h}}{\cancel{h}}$$

$$= \lim_{h \rightarrow 0} 2x + h - 6$$

As $h \rightarrow 0$, $h \rightarrow 0$

$$\therefore \frac{dy}{dx} = 2x - 6$$

- (b) Differentiate $\frac{3}{x^2} + x^{\frac{5}{2}}$ with respect to x .

[2]

$$y = 3x^{-2} + x^{\frac{5}{2}}$$

$$\frac{dy}{dx} = (-2)(3)x^{-2-1} + \frac{5}{2}x^{\frac{5}{2}-1}$$

$$= -6x^{-3} + \frac{5}{2}x^{\frac{3}{2}}$$

$$= -\frac{6}{x^3} + \frac{5}{2}x^{\frac{3}{2}}$$

5. (a) Given that $y = x^2 + 5x - 2$, find $\frac{dy}{dx}$ from first principles.

[5]

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + 5(x+h) - 2 - [x^2 + 5x - 2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{5x} + 5h - \cancel{2} - \cancel{x^2} - \cancel{5x} + \cancel{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2x\cancel{h} + h^2 + 5\cancel{h}}{h} \\
 &= \lim_{h \rightarrow 0} 2x + h + 5
 \end{aligned}$$

As $h \rightarrow 0$, $h \rightarrow 0 \quad \therefore \frac{dy}{dx} = \underline{\underline{2x + 5}}$

- (b) Differentiate $\frac{3}{x} - 2x^{\frac{5}{2}}$ with respect to x .

[4]

$$\begin{aligned}
 y &= 3x^{-1} - 2x^{\frac{5}{2}} \\
 \frac{dy}{dx} &= (-1)(3)x^{-1-1} - \left(\frac{5}{2}\right)(2)x^{\frac{5}{2}-1} \\
 &= -3x^{-2} - 5x^{\frac{3}{2}} \\
 &= \underline{\underline{-\frac{3}{x^2} - 5x^{\frac{3}{2}}}}
 \end{aligned}$$

← You can leave your answer like this

6. (a) Given that $y = 2x^2 + x + 3$, find $\frac{dy}{dx}$ from first principles. [5]

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(x+h)^2 + (x+h) + 3 - [2x^2 + x + 3]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(x^2 + 2xh + h^2) + x + h + 3 - 2x^2 - x - 3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{x} + h + \cancel{3} - \cancel{2x^2} - \cancel{x} - \cancel{3}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 + h}{h} \quad \left[\frac{h}{h} = 1 \right] \\
 &= \lim_{h \rightarrow 0} 4x + 2h + 1
 \end{aligned}$$

As $h \rightarrow 0$, $2h \rightarrow 0 \quad \therefore \frac{dy}{dx} = 4x + 1$

- (b) Given that [5]

$$y = \sqrt{x} + \frac{k}{x}$$

- and that $\frac{dy}{dx} = 2$ when $x = 4$, find the value of the constant k . [4]

$$\begin{aligned}
 y &= x^{\frac{1}{2}} + kx^{-1} \\
 \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} - kx^{-2} \\
 \frac{dy}{dx} \Big|_{x=4} &= \frac{1}{2}(4)^{-\frac{1}{2}} - k(4)^{-2} \\
 &= \frac{1}{2\sqrt{4}} - \frac{k}{4^2} \\
 &= \frac{1}{4} - \frac{k}{16}
 \end{aligned}$$

$$\frac{dy}{dx} = 2 \Rightarrow \frac{1}{4} - \frac{k}{16} = 2$$

$$(\times 16) \quad 4 - k = 32$$

$$k = 4 - 32 = -28$$