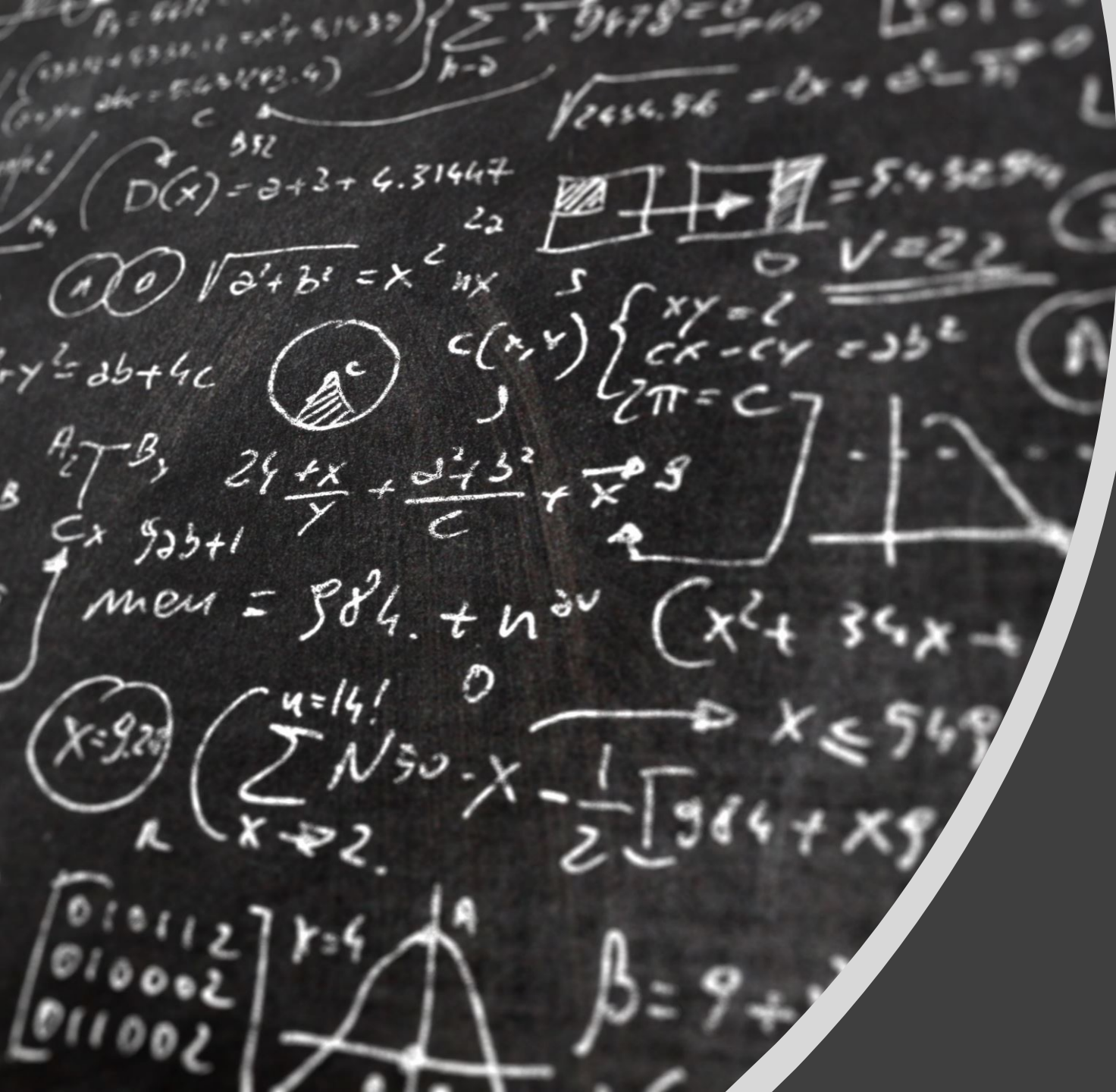




Maths

Lecture 7

Matrix Multiplication

- Multiplying row matrices and column matrices together
 - Let A be a 1X2 row matrix and B be a 2 X 1 column matrix:

$$A = \begin{bmatrix} a & b \end{bmatrix} \quad B = \begin{bmatrix} c \\ d \end{bmatrix}$$

The product of these two matrices is written AB and is the 1 X 1 matrix defined by:

$$AB = \begin{bmatrix} a & b \end{bmatrix} \times \begin{bmatrix} c \\ d \end{bmatrix} = [ac + bd]$$

Note that corresponding elements are multiplied together and the results are then added together.
For example:

$$\begin{bmatrix} 2 & -3 \end{bmatrix} \times \begin{bmatrix} 6 \\ 5 \end{bmatrix} = [12 - 15] = [-3]$$

The only requirement is that the number of elements of the row matrix is the same as the number of elements of the column matrix.

$$C = [2 \quad -4 \quad 3 \quad 2] \quad B = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 5 \end{bmatrix}$$

$$CD = [2 \quad -4 \quad 3 \quad 2] \times \begin{bmatrix} 3 \\ 3 \\ -2 \\ 5 \end{bmatrix} = [6 - 12 - 6 + 10] = [-2]$$

Example 1 In each of the following cases, find the product AB .

(a) $A = [1 \ 2]$, $B = [4 \ 3]^T$. (b) $A = [1 \ 1 \ 1]$, $B = [2 \ 3 \ 4]^T$.

(c) $A = [1 \ -1 \ 2 \ 3]$, $B = [1 \ 1 \ -3 \ 2]^T$.

$$\text{(a)} \ AB = (1 \ 2) \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 1 \times 4 + 2 \times 3 = 4 + 6 = 10.$$

$$\text{(b)} \ AB = (1 \ 1 \ 1) \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 1 \times 2 + 1 \times 3 + 1 \times 4 = 2 + 3 + 4 = 9.$$

$$\begin{aligned} \text{(c)} \ AB &= (1 \ -1 \ 2 \ 3) \begin{pmatrix} 1 \\ 1 \\ -3 \\ 2 \end{pmatrix} = 1 \times 1 + 1 \times (-1) + 2 \times (-3) + 3 \times 2 \\ &= 1 + (-1) + (-6) + 6 = 0. \end{aligned}$$

Multiplying two 2x2 matrices

- If A and B are two matrices then the product AB is obtained by multiplying the rows of A with the columns of B in the manner that is described in the previous slides.
- This will only be possible if the number of elements in the rows of A is the same as the number of elements in the columns of B.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

$A \qquad B \qquad = \qquad C$

Example:

Find the product AB when $A = \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix}$.

$$AB = \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ -1 & 9 \end{pmatrix} = \begin{pmatrix} 2 \times 3 + 4 \times (-1) & 2 \times 6 + 4 \times 9 \\ 5 \times 3 + 3 \times (-1) & 5 \times 6 + 3 \times 9 \end{pmatrix} = \begin{pmatrix} 2 & 48 \\ 12 & 57 \end{pmatrix}$$

More general matrix multiplication

- For two matrices, A and B say, the product AB can only be found if the number of columns in the first matrix, A, is the same as the number of rows in the second, B.
- If A has size $p \times q$, that is, it has p rows and q columns, and B has size $r \times s$, that is, it has r rows and s columns, we can only multiply them together if $q = r$. When this is so, the result of multiplying them together, C say, is a $p \times s$ matrix.

$$\begin{array}{ccc} A & B & = & C \\ p \times q & r \times s & & p \times s \\ & \underbrace{\qquad\qquad\qquad}_{q=r} & & \end{array}$$

Example.

$$\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} * \\ * \end{pmatrix}$$

To obtain the first entry in the solution, ignore the second row of the first matrix.

$$\begin{pmatrix} 3 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \times 2 + 7 \times 9 \end{pmatrix} = \begin{pmatrix} 69 \end{pmatrix}$$

To obtain the second entry in the solution, ignore the first row of the first matrix.

$$\begin{pmatrix} 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \times 2 + 5 \times 9 \end{pmatrix} = \begin{pmatrix} 53 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 7 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \times 2 + 7 \times 9 \\ 4 \times 2 + 5 \times 9 \end{pmatrix} = \begin{pmatrix} 69 \\ 53 \end{pmatrix}$$

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the identity matrix or unit matrix of order 2, and is usually denoted by the symbol I . I plays the same role in matrix multiplication as the number 1 does in number multiplication.

just as $a \times 1 = 1 \times a = a$ for any number a , so $AI = IA = A$ for any matrix A .

Multiplying two 3x3 matrices

- The definition of the product $C = AB$ where A and B are two 3×3 matrices is as follows:

$$C = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} ar + bu + cx & as + bv + cy & at + bw + cz \\ dr + eu + fx & ds + ev + fy & dt + ew + fz \\ gr + hu + ix & gs + hv + iy & gt + hw + iz \end{bmatrix}$$

The construction of the matrix on the right-hand side is straightforward if we follow the simple rule from Key Point 4 that the element in the i^{th} row and j^{th} column of C is obtained by multiplying the i^{th} row of A with the j^{th} column of B.

For example, to obtain the element in row 2, column 3 of C we take row 2 of A: [d, e, f] and multiply it with column 3 of B in the usual way to produce [dt + ew + fz].

Example:

• Calculate $AB = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

$$\begin{bmatrix} 1 \times 2 + 2 \times 1 + (-1) \times 0 & 1 \times (-1) + 2 \times (-2) + (-1) \times 3 & 1 \times 3 + 2 \times 1 + (-1) \times (-2) \\ 3 \times 2 + 4 \times 1 + 0 \times 0 & 3 \times (-1) + 4 \times (-2) + 0 \times 3 & 3 \times 3 + 4 \times 1 + 0 \times (-2) \\ 1 \times 2 + 5 \times 1 + (-2) \times 0 & 1 \times (-1) + 5 \times (-2) + (-2) \times 3 & 1 \times 3 + 5 \times 1 + (-2) \times (-2) \end{bmatrix}$$

i.e. $AB = \begin{bmatrix} 4 & -8 & 7 \\ 10 & -11 & 13 \\ 7 & -17 & 12 \end{bmatrix}$

The 3 x 3 unit matrix is: $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$AI = IA = A$$

Multiplying non-square matrices together

- So far, we have just looked at multiplying 2 x 2 matrices and 3 x 3 matrices. However, products between non-square matrices may be possible.

Example

Find the product AB if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 6 & 1 \\ 4 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 6 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 22 & 13 \\ 38 & 25 \end{bmatrix}$$

Exercise:

Obtain the product AB if $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 1 \\ 6 & 1 & 0 \end{bmatrix}$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 1 \\ 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} -10 & 2 & 1 \\ -14 & 5 & 2 \end{bmatrix} \end{aligned}$$

Exercise:

$$\text{Given } A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

State which of the products AB , BA , AC , CA , BC , CB , $(AB)C$, $A(CB)$ is defined and state the size ($n \times m$) of the product when defined.

$$\begin{array}{cc} A & B \\ 2 \times 3 & 2 \times 2 \end{array} \quad \text{not possible}$$

$$\begin{array}{cc} B & A \\ 2 \times 2 & 2 \times 3 \end{array} \quad \text{possible; result } 2 \times 3$$

$$\begin{array}{cc} A & C \\ 2 \times 3 & 3 \times 2 \end{array} \quad \text{possible; result } 2 \times 2$$

$$\begin{array}{cc} C & A \\ 3 \times 2 & 2 \times 3 \end{array} \quad \text{possible; result } 3 \times 3$$

$$\begin{array}{cc} B & C \\ 2 \times 2 & 3 \times 2 \end{array} \quad \text{not possible}$$

$$\begin{array}{cc} C & B \\ 3 \times 2 & 2 \times 2 \end{array} \quad \text{possible; result } 3 \times 2$$

$$(AB)C \quad \text{not possible, } AB \text{ not defined.}$$

$$\begin{array}{cc} A & (CB) \\ 2 \times 3 & 3 \times 2 \end{array} \quad \text{possible; result } 2 \times 2$$

Exercise:

3. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ -1 & 2 \\ 5 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, find AB and BC .

4. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, verify $A(BC) = (AB)C$.

Answers

3. $AB = \begin{bmatrix} 15 & 26 \\ -6 & -12 \\ 12 & 24 \end{bmatrix}$, $BC = \begin{bmatrix} 8 & 10 \\ 0 & 3 \\ 16 & 17 \end{bmatrix}$ 4. $A(BC) = (AB)C = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$

Matrix algebra

$$A(B + C) = \bar{A}B + AC$$

$$AB \neq BA \text{ in general}$$

$$A(BC) = (AB)C$$

$$AI = IA = A$$

$$A\underline{0} = \underline{0}A = \underline{0}$$

AB may not be possible

$$AB = \underline{0} \text{ does not imply } A = \underline{0} \text{ or } B = \underline{0}$$

Determinants

The determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Note the change from square brackets to vertical lines, and is defined to be the number $ad - bc$. That is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We can use the notation $\det(A)$ or $|A|$ to denote the determinant of A .

Example

Find the determinants of the matrices

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, & B &= \begin{bmatrix} 4 & -1 \\ -2 & -3 \end{bmatrix}, & C &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & D &= \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \\ E &= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, & F &= \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, & G &= \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}. \end{aligned}$$

Answer

$$|A| = 1 \times 4 - 2 \times 3 = -2$$

$$|B| = 4 \times (-3) - (-1) \times (-2) = -12 - 2 = -14$$

$$|C| = 0 \qquad |D| = 3$$

$$|E| = 8$$

$$|F| = 3$$

$$|G| = -4 + 4 = 0$$

Laplace expansion along the top row

This is a technique which can be used to evaluate determinants of any order. In principle, this method can use any row or any column as its starting point. We quote one example: using the top row.

$$\text{Det} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}.$$

First we introduce the idea of a minor. Each element in this array of numbers has an associated **minor** formed by removing the column and row in which the element lies and taking the determinant of the remainder.

For example consider element $a_{23} = 3$. We strike out the second row and the third column:

$$\begin{array}{ccc|c} 4 & 1 & 1 & \\ \hline 1 & 2 & 3 & \\ 3 & 1 & 2 & \end{array} \quad \text{to leave} \quad \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 4 - 3 = 1.$$

For the element $a_{31} = 3$ we strike out the third row and first column:

$$\begin{array}{ccc|c} 4 & 1 & 1 & \\ 1 & 2 & 3 & \\ \hline 3 & 1 & 2 & \end{array} \quad \text{to leave} \quad \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1.$$

Next we introduce the idea of a **cofactor**. This is a minor with a sign attached. The appropriate sign comes from the pattern of signs appropriate to a 3 x 3 array:

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

Each element has a cofactor associated with it. The cofactor of element a_{11} is denoted by A_{11} , that of a_{23} by A_{23} and so on.

Cofactors are important as it can be shown that the value of the determinant of a 3 x 3 matrix can be found from the formula

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

Example

In the case of $\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$ we have $a_{11} = 4$, $a_{12} = 1$, $a_{13} = 1$,

$$A_{11} = + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1$$

$$A_{12} = - \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -(2 - 9) = 7$$

$$A_{13} = + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5$$

Hence $\Delta = 4 \times 1 + 1 \times 7 + 1 \times -5 = 6$.

Alternatively, choosing to expand along the second row:

$$\begin{aligned} \Delta &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ &= 1 \left(- \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right) + 2 \left(\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} \right) + 3 \left(- \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} \right) = 6 \end{aligned}$$

Example

Use expansion along the first row to find $\Delta = \begin{vmatrix} 1 & -1 & 3 \\ 0 & 2 & 6 \\ -2 & 1 & 5 \end{vmatrix}$

Answer

$$a_{11} = 1, \quad a_{12} = -1, \quad a_{13} = 3$$

$$A_{11} = + \begin{vmatrix} 2 & 6 \\ 1 & 5 \end{vmatrix} = 10 - 6 = 4$$

$$A_{12} = - \begin{vmatrix} 0 & 6 \\ -2 & 5 \end{vmatrix} = -(0 + 12) = -12$$

$$A_{13} = + \begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} = 2 + 2 = 4.$$

Hence $\Delta = 1 \times 4 + (-1) \times (-12) + 3 \times 4 = 4 + 12 + 12 = 28$, as before.

Exercise:

1. Use Laplace expansion along the 1st row to determine

$$\begin{vmatrix} 3 & 1 & -4 \\ 6 & 9 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

Answers

$$1. 3 \begin{vmatrix} 9 & -2 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 6 & -2 \\ -1 & 1 \end{vmatrix} - 4 \begin{vmatrix} 6 & 9 \\ -1 & 2 \end{vmatrix} = 3(9 + 4) - 1(6 - 2) - 4(12 + 9) = -49$$

Properties of determinants

- If two rows (or two columns) of a determinant are interchanged then the value of the determinant is multiplied by (-1) .

For example $\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 3 = 5$ but (interchanging columns) $\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = 3 - 8 = -5$ and
(interchanging rows) $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5$.

Properties of determinants

- The determinant of a matrix A and the determinant of its transpose A^T are equal.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2$$

- If two rows (or two columns) of a matrix A are equal, or they are multiple of another row (or column), then it has zero determinant.

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} &= 1 \times \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} + 2 \times \left(- \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \right) + 3 \times \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \\ &= -3 + 2 \times (6) + 3 \times (-3) = 0. \end{aligned}$$

Properties of determinants

- The determinant of a lower triangular matrix, an upper triangular matrix or a diagonal matrix is the product of the elements on the leading diagonal.

As an example, it is easily confirmed that each of the following determinants has the same value $1 \times 4 \times 6 = 24$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{vmatrix}$$

The Inverse of a Matrix

In number arithmetic every number $a \neq 0$ has a reciprocal b written as a^{-1} or $\frac{1}{a}$ such that $ba = ab = 1$.

Some, but not all, square matrices have inverses. If a square matrix A has an inverse, A^{-1} , then

$$AA^{-1} = A^{-1}A = I.$$

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \text{ is } B = \begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Inverse of a Matrix

- We make three important remarks:
 - Non-square matrices do not have inverses.
 - The inverse of A is usually written A^{-1} .
 - Not all square matrices have inverses.
 - The matrix A therefore has no inverse and is said to be a singular matrix.

The inverse of a 2x2 matrix

Assume we have a matrix A

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Note that $ad - bc = |A|$, the determinant of the matrix A. If $ad - bc \neq 0$ then A has a (unique) inverse.
- To find the inverse of a 2x2 matrix A we interchange the diagonal elements, change the sign of the other two elements, and then divide by the determinant of A.

Example

Which of the following matrices has an inverse?

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Answer

$$|A| = 1 \times 3 - 0 \times 2 = 3; \quad |B| = 1 + 1 = 2; \quad |C| = 2 - 2 = 0; \quad |D| = 1 - 0 = 1.$$

Therefore, A , B and D each has an inverse. C does not because it has a zero determinant.

Example

Find the inverses of the matrices A, B and D in the previous Task.

Answer

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}, \quad B^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = D$$

Exercise

1. Find the inverses of the following matrices

$$(a) \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad (b) \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Answers

$$1. (a) -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \quad (c) \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

The inverse of a 3x3 matrix Gauss elimination method

This method involves operating on the rows of a matrix in order to reduce it to a unit matrix. The row operations we shall use are:

- (i) interchanging two rows
- (ii) multiplying a row by a constant factor
- (iii) adding a multiple of one row to another.

Note that in (ii) and (iii) the multiple could be negative or fractional, or both.

The same operation **MUST** be done on the unit matrix

Example

Suppose we wish to find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix}$

1- We first place A and I adjacent to each other.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Phase 1

We now proceed by changing A to be form an upper triangular matrix

1- First we subtract row 1 from row 2 and twice row 1 from row 3.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R2 - R1 \\ R3 - 2R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

1- Now we subtract row 2 from row 3.

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad R3 - R2 \Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Phase 2

This consists of continuing the row operations to reduce the elements above the leading diagonal to zero.

1-We subtract 3 times row 3 from row 1

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad R1 - 3R3 \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

2- Finally we subtract 3 times row 2 from row 1.

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad R1 - 3R2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Then we have $A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

(This can be verified by showing that $AA^{-1} = I$ or $A^{-1}A = I$.)

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} \times \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

First interchange rows 1 and 2, then carry out the operation (row 3) + 1/2(row 1): $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix}$

$$\begin{array}{ccc} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & R1 \leftrightarrow R2 \Rightarrow \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & R3 + \frac{1}{2}R1 \Rightarrow \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & \frac{7}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & \frac{7}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \quad R3 - \frac{7}{2}R2 \quad \Rightarrow \quad \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix} \quad \begin{array}{l} R1 - \frac{1}{3}R3 \\ R2 + \frac{1}{3}R3 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} +\frac{7}{6} & +\frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{7}{6} & \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix} \quad R1 - 3R2 \quad \Rightarrow \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{10}{6} & \frac{2}{6} & -\frac{4}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{10}{6} & \frac{2}{6} & -\frac{4}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{7}{2} & \frac{1}{2} & 1 \end{bmatrix} \quad \begin{array}{l} R1 \div 2 \\ R3 \div (-3) \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{7}{6} & -\frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

$$\text{Hence } A^{-1} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{2}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{7}{6} & -\frac{1}{6} & -\frac{1}{3} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$$

The inverse of a 3x3 matrix determinant method

This method which employs determinants, is of importance from a theoretical perspective. The numerical computations involved are too heavy for matrices of higher order than 3 x 3 and in such cases the Gauss elimination approach is preferred.

Given a square matrix A:

- Find $|A|$. If $|A| = 0$ then A^{-1} does not exist. If $|A| \neq 0$ we can proceed to find the inverse matrix, as follows.
- Replace each element of A by its cofactor.
- Transpose the result to form the adjoint matrix, denoted by $\text{adj}(A)$
- Then calculate $A^{-1} = \frac{1}{|A|} \text{adj}(A)$.

Example

Find the inverse of $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix}$ This will require five stages.

(a) First find $|A|$: $|A| = 0 \times 5 + 1 \times (-1) + 1 \times 7 = 6$

(b) Now replace each element of A by its minor:

$$\begin{bmatrix} \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 1 & 7 \\ -1 & 1 & 1 \\ -4 & -2 & -2 \end{bmatrix}$$

(c) Now attach the signs from the array

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

$$\begin{bmatrix} 5 & -1 & 7 \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{bmatrix}$$

(d) Then transpose the result to obtain the adjoint matrix:

$$\text{Transposing, } \text{adj}(A) = \begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$$

(e) Finally obtain A^{-1} :

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{6} \begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$$

Exercise

Use the determinant method and also the Gauss elimination method to find the inverse of the following matrices

$$(a) A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 4 & 1 & 2 \end{bmatrix}$$

Answer

$$(a) A^{-1} = -\frac{1}{2} \begin{bmatrix} 0 & -2 & 1 \\ -2 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix}^T = -\frac{1}{2} \begin{bmatrix} 0 & -2 & 0 \\ -2 & 4 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

Matrix Solution of Equations

- Cramer's rule.

$$ax + by = e$$

$$cx + dy = f$$

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta} \quad \text{where} \quad \Delta_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

Example

Use Cramer's rule to solve the simultaneous equations

$$\begin{aligned}2x + y &= 7 \\ 3x - 4y &= 5\end{aligned}$$

Answer

Calculating $\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11$. Since $\Delta \neq 0$ we can proceed with Cramer's solution.

$$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11 \quad x = \frac{1}{\Delta} \begin{vmatrix} 7 & 1 \\ 5 & -4 \end{vmatrix}, \quad y = \frac{1}{\Delta} \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix}$$

$$\text{i.e. } x = \frac{(-28 - 5)}{(-11)}, \quad y = \frac{(10 - 21)}{(-11)} \quad \text{implying: } x = \frac{-33}{-11} = 3, \quad y = \frac{-11}{-11} = 1.$$

Exercise

Use Cramer's rule to solve the equations

$$\begin{aligned}2x - 3y &= 6 \\4x - 6y &= 12\end{aligned}$$

Cramer's rule to solve three equations in three unknowns

The unique solution to the system of equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

is

$$x_1 = \frac{\Delta_{x_1}}{\Delta}, \quad x_2 = \frac{\Delta_{x_2}}{\Delta}, \quad x_3 = \frac{\Delta_{x_3}}{\Delta}$$

in which

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and

$$\Delta_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \quad \Delta_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \quad \Delta_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Example

Use Cramer's rule to solve the system

$$x_1 - 2x_2 + x_3 = 3$$

$$2x_1 + x_2 - x_3 = 5$$

$$3x_1 - x_2 + 2x_3 = 12.$$

First check that $\det \neq 0$:

Answer

$$\Delta = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix}.$$

Expanding along the top row,

$$\begin{aligned} \Delta &= 1 \times \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ &= 1 \times (2 - 1) + 2 \times (4 + 3) + 1 \times (-2 - 3) \\ &= 1 + 14 - 5 = 10 \end{aligned}$$

Now find the value of x_1 . First write down the expression for x_1 in terms of determinants:

$$x_1 = \begin{vmatrix} 3 & -2 & 1 \\ 5 & 1 & -1 \\ 12 & -1 & 2 \end{vmatrix} \div \Delta$$

$$\begin{aligned} & 3 \times \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} - (-2) \times \begin{vmatrix} 5 & -1 \\ 12 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 5 & 1 \\ 12 & -1 \end{vmatrix} \\ &= 3 \times 1 + 2 \times 22 + 1 \times (-17) \\ &= 30 \end{aligned}$$

$$\text{Hence } x_1 = \frac{1}{10} \times 30 = 3$$

In a similar way find the values of x_2 and x_3 :

$$\begin{aligned}x_2 &= \frac{1}{10} \begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & -1 \\ 3 & 12 & 2 \end{vmatrix} \\&= \frac{1}{10} \left\{ 1 \times \begin{vmatrix} 5 & -1 \\ 12 & 2 \end{vmatrix} - 3 \times \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 3 & 12 \end{vmatrix} \right\} \\&= \frac{1}{10} \{ 22 - 3 \times 7 + 9 \} = 1 \\x_3 &= \frac{1}{10} \left\{ 1 \times \begin{vmatrix} 1 & 5 \\ -1 & 12 \end{vmatrix} - (-2) \times \begin{vmatrix} 2 & 5 \\ 3 & 12 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \right\} \\&= \frac{1}{10} \{ 17 + 2 \times 9 + 3 \times (-5) \} = 2\end{aligned}$$

Solution by Inverse Matrix Method

Consider the system

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 - 2x_2 &= -1. \end{aligned}$$

In matrix form this becomes

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \quad \text{which is of the form } AX = B.$$

If A^{-1} exists then the solution is $X = A^{-1}B$

$$|A| = 2 \times (-2) - 1 \times 3 = -7 \quad \Rightarrow \quad A^{-1} = \frac{1}{(-7)} \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

since $|A| \neq 0$ then A^{-1} exists.

$$X = A^{-1}B = \frac{1}{7} \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \text{ Hence } x_1 = 1, x_2 = 1.$$

Example

Use the inverse matrix method to solve

$$2x_1 + 3x_2 = 3$$

$$5x_1 + 4x_2 = 11$$

$$AX = B \text{ is } \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$|A| = 2 \times 4 - 3 \times 5 = -7 \text{ and } A^{-1} = -\frac{1}{7} \begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix}$$

Using $X = A^{-1}B$:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 4 \times 3 - 3 \times 11 \\ -5 \times 3 + 2 \times 11 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -21 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

So $x_1 = 3$, $x_2 = -1$

Solving three equations in three unknowns

It is much more tedious to use the inverse matrix to solve a system of three equations

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 3 \\2x_1 + x_2 - x_3 &= 5 \\3x_1 - x_2 + 2x_3 &= 12\end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix} =$$

$$|A| = 10.$$

$$\begin{bmatrix} \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 7 & -5 \\ -3 & -1 & 5 \\ 1 & -3 & 5 \end{bmatrix}$$

The array of signs is $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ so that we obtain $\begin{bmatrix} 1 & -7 & -5 \\ 3 & -1 & -5 \\ 1 & 3 & 5 \end{bmatrix}$.

Transposing gives $\begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$. Finally, $A^{-1} = \frac{1}{10} \begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix}$

$$X = \frac{1}{10} \begin{bmatrix} 1 & 3 & 1 \\ -7 & -1 & 3 \\ -5 & -5 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 12 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 30 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ Then } x_1 = 3, \ x_2 = 1, \ x_3 = 2.$$