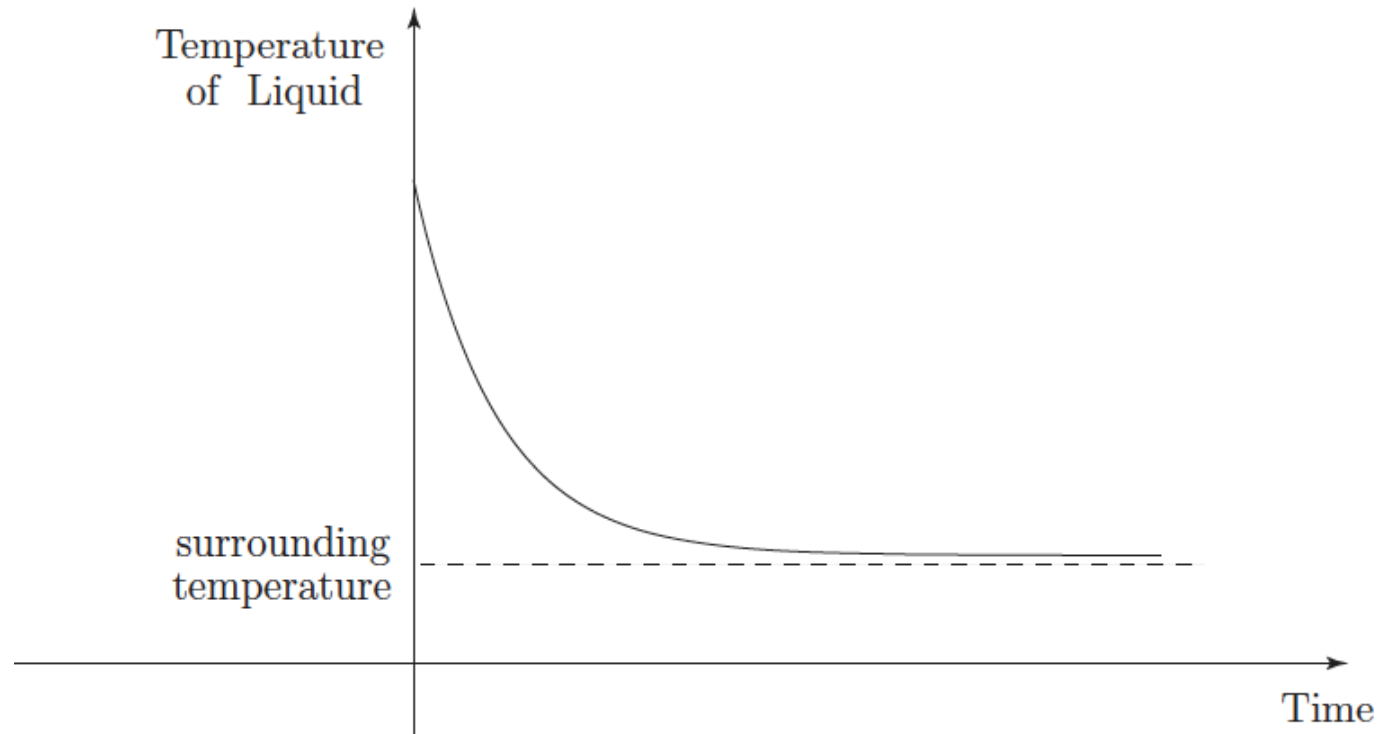


Maths

(Differential Equations)

Case study: Newton's law of cooling

When a hot liquid is placed in a cooler environment, experimental observation shows that its temperature decreases to approximately that of its surroundings. A typical graph of the temperature of the liquid plotted against time is shown in Figure



Example

Find the unique solution of the differential equation $\frac{dy}{dx} = 3x^2$ which satisfies the condition $y(1) = 4$.

$$\frac{d^2y}{dx^2} = 6x$$

Answer

You should obtain $y = x^3 + 3$ since, by a single integration we have $y = x^3 + C$, where C is an arbitrary constant. Now when $x = 1$, $y = 4$ so that $4 = 1 + C$. Hence $C = 3$ and the unique solution is $y = x^3 + 3$.

$$y = x^3 + 2$$

$$\frac{dy}{dx} = 3x^2$$

$$\frac{d^2y}{dx^2} = 6x$$

$$\int (3x^2 + A) dx$$

$$\int 6x dx$$

$$y = x^3 + Ax^2 + B$$

$$= 3x^2 + A$$

$$Y(0)=2$$

$$Y(1)=3$$

$$y = x^3 + Ax^2 + B$$

$$2 = 0 + A \cdot 0 + B$$

$$Y(0)=2$$

$$Y(1)=5$$

$$2 = 0 + A \cdot 0 + B$$

$$5 = 1 + A \cdot 1 + B$$

$$A = 2$$

$$y = x^3 + 2x^2 + 2$$

Example

Solve the differential equation $\frac{d^2y}{dx^2} = 6x$ subject to the conditions

(a) $y(0) = 2$ and $y(1) = 3$

(b) $y(0) = 2$ and $y(1) = 5$

(c) $y(0) = 2$ and $\frac{dy}{dx} = 1$ at $x = 0$.

Solution

(a) Integrating the differential equation once produces $\frac{dy}{dx} = 3x^2 + A$. The general solution is found by integrating a second time to give $y = x^3 + Ax + B$, where A and B are arbitrary constants.

Imposing the conditions $y(0) = 2$ and $y(1) = 3$: at $x = 0$ we have $y = 2 = 0 + 0 + B = B$ so that $B = 2$, and at $x = 1$ we have $y = 3 = 1 + A + B = 1 + A + 2$. Therefore $A = 0$ and the solution is

$$y = x^3 + 2.$$

(b) Here the second condition is $y(1) = 5$ so at $x = 1$

$$y = 5 = 1 + A + 2 \quad \text{so that} \quad A = 2$$

and the solution in this case is

$$y = x^3 + 2x + 2.$$

(c) Here the second condition is

$$\frac{dy}{dx} = 1 \text{ at } x = 0 \quad \text{i.e.} \quad y'(0) = 1$$

then since $\frac{dy}{dx} = 3x^2 + A$, putting $x = 0$ we get:

$$\frac{dy}{dx} = 1 = 0 + A$$

so that $A = 1$ and the solution in this case is $y = x^3 + x + 2$.

Classifying differential equations

- An ordinary differential equation (ODE) is any relation between a function of a single variable and its derivatives. (All differential equations studied in this workbook are ordinary.)
- The order of a differential equation is the order of the highest derivative in the equation.
- A differential equation is linear if the dependent variable and its derivatives occur to the first power only and if there are no products involving the dependent variable or its derivatives.

Example

Classify the differential equations specifying the order and type (linear/non-linear)

$$(a) \frac{d^2 y}{dx^2} - \frac{dy}{dx} = x^2$$

$$(b) \frac{d^2 x}{dt^2} = \left(\frac{dx}{dt} \right)^3 + 3x$$

$$(c) \frac{dx}{dt} - x = t^2$$

$$(d) \frac{dy}{dt} + \cos y = 0$$

$$(e) \frac{dy}{dt} + y^2 = 4$$

Solution

- (a) Second order, linear.
- (b) Second order, non-linear (because of the cubic term).
- (c) First order, linear.
- (d) First order, non-linear (because of the $\cos y$ term).
- (e) First order, non-linear (because of the y^2 term).

Note that in (a) the independent variable is x whereas in the other cases it is t .

In (a), (d) and (e) the dependent variable is y and in (b) and (c) it is x .

Separating the variables in first order ODEs

In this Section we consider differential equations which can be written in the form:

$$\frac{dy}{dx} = f(x)g(y)$$

Note that the right-hand side is a product of a function of x , and a function of y .
Examples of such equations are

$$\frac{dy}{dx} = x^2 y^3, \quad \frac{dy}{dx} = y^2 \sin x \quad \text{and} \quad \frac{dy}{dx} = y \ln x$$

Example

Determine which of the following differential equations can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

If possible, rewrite each equation in this form.

$$(a) \frac{dy}{dx} = \frac{x^2}{y^2}, \quad (b) \frac{dy}{dx} = 4x^2 + 2y^2, \quad (c) y \frac{dy}{dx} + 3x = 7$$

Answer

$$(a) \frac{dy}{dx} = x^2 \left(\frac{1}{y^2} \right), \quad (b) \text{ cannot be written in the stated form,}$$

$$(c) \text{ Reformulating gives } \frac{dy}{dx} = (7 - 3x) \times \frac{1}{y} \text{ which is in the required form.}$$

First Order Ordinary Differential Equation

Prerequisites: You will need to know about trigonometry, differentiation, integration, complex numbers in order to make the most of this teach-yourself resource. We are looking at equations involving a function $y(x)$ and its first derivative:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

This method is used to solve ODEs in the form:

$$\frac{dy}{dx} = f(x)$$

These ODEs can be solved as follows:

$$\begin{aligned} dy &= f(x)dx \\ \text{so: } y &= \int f(x)dx \end{aligned}$$

Example:

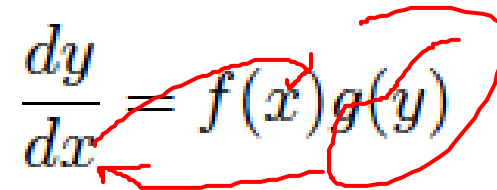
$$\frac{dy}{dx} = 3x^2 - 6x + 5$$

$$y = x^3 - 3x^2 + 5x + C \text{ with } C \text{ constant of integration}$$

Separating the variables in first order ODEs

Method of Separation of Variables

The solution of the equation

$$\frac{dy}{dx} = f(x)g(y)$$
A red circle is drawn around the function g(y) on the right side of the equation. A red arrow points from the circle to the denominator dx of the fraction dy/dx on the left side.

may be found from separating the variables and integrating:

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

Example

Use the method of separation of variables to solve the differential equation

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

Solution

The equation already has the form

$$\frac{dy}{dx} = f(x)g(y)$$

where

$$f(x) = 3x^2 \quad \text{and} \quad g(y) = 1/y.$$

Dividing both sides by $g(y)$ we find

$$y \frac{dy}{dx} = 3x^2$$

Integrating both sides with respect to x gives

$$\int y \frac{dy}{dx} dx = \int 3x^2 dx$$

that is

$$\int y dy = \int 3x^2 dx$$

Note that the left-hand side is an integral involving just y ; the right-hand side is an integral involving just x . After integrating both sides with respect to the stated variables we find

$$\frac{1}{2}y^2 = x^3 + c$$

We now have a relationship between y and x as required. Often it is sufficient to leave your answer in this form but you may also be required to obtain an explicit relation for y in terms of x . In this particular case

$$y^2 = 2x^3 + 2c$$

so that

$$y = \pm \sqrt{2x^3 + 2c}$$

Example

Find the general solution of the following equations:

$$(a) \frac{dy}{dx} = 3, \quad (b) \frac{dy}{dx} = \frac{6 \sin x}{y}$$

Answers

$$(a) y = 3x + C, \quad (b) \frac{1}{2}y^2 = C - 6 \cos x$$

$$\frac{dy}{dx} = \frac{6 \sin x}{y}$$

$$y \, dy = 6 \sin x \, dx$$

$$\int y \, dy = \int 6 \sin x \, dx$$

$$y^2/2 = -6 \cos x + C$$

$$y^2 = -12 \cos x + 2C$$

Example

Solve the equations

$$(a) \quad \frac{dy}{dx} = 5x^4 \qquad (b) \quad \frac{d}{dx}(x^3y) = 5x^4$$

Answer

$$(a) \quad y = x^5 + C \qquad (b) \quad x^3y = x^5 + C \text{ so that } y = x^2 + \frac{C}{x^3}.$$

Second Order Differential Equations

We now proceed to study those second order linear equations which have constant coefficients. The general form of such an equation is:

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

where a; b; c are constants. The homogeneous form this equation is the case when $f(x) = 0$:

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

Finding the complementary function

If $y_1(x)$ and $y_2(x)$ are any two (linearly independent) solutions of a linear, homogeneous second order differential equation then the general solution $y_{cf}(x)$, is

$$y_{cf}(x) = Ay_1(x) + By_2(x)$$

where A , B are constants.

We see that the second order linear ordinary differential equation has two arbitrary constants in its **general solution**. The functions $y_1(x)$ and $y_2(x)$ are **linearly independent** if one is not a multiple of the other.

Example

Verify that $y_1 = e^{4x}$ and $y_2 = e^{2x}$ both satisfy the constant coefficient linear homogeneous equation:

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

Solution

When $y_1 = e^{4x}$, differentiation yields:

$$\frac{dy_1}{dx} = 4e^{4x} \quad \text{and} \quad \frac{d^2y_1}{dx^2} = 16e^{4x}$$

Substitution into the left-hand side of the ODE gives $16e^{4x} - 6(4e^{4x}) + 8e^{4x}$, which equals 0, so that $y_1 = e^{4x}$ is indeed a solution.

Similarly if $y_2 = e^{2x}$, then

$$\frac{dy_2}{dx} = 2e^{2x} \quad \text{and} \quad \frac{d^2y_2}{dx^2} = 4e^{2x}.$$

Example

Find values of k so that $y = e^{kx}$ is a solution of:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

Solution

As suggested we try a solution of the form $y = e^{kx}$. Differentiating we find

$$\frac{dy}{dx} = ke^{kx} \quad \text{and} \quad \frac{d^2y}{dx^2} = k^2e^{kx}.$$

Substitution into the given equation yields:

$$k^2e^{kx} - ke^{kx} - 6e^{kx} = 0 \quad \text{that is} \quad (k^2 - k - 6)e^{kx} = 0$$

The only way this equation can be satisfied for all values of x is if

$$k^2 - k - 6 = 0$$

that is, $(k - 3)(k + 2) = 0$ so that $k = 3$ or $k = -2$. That is to say, if $y = e^{kx}$ is to be a solution of the differential equation, k must be either 3 or -2 . We therefore have found two solutions:

$$y_1(x) = e^{3x} \quad \text{and} \quad y_2(x) = e^{-2x}$$

These are linearly independent and therefore the general solution is

$$y_{\text{cf}}(x) = Ae^{3x} + Be^{-2x}$$

The equation $k^2 - k - 6 = 0$ for determining k is called the **auxiliary equation**.

Auxiliary equation

The **auxiliary equation** of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is $ak^2 + bk + c = 0$ where $y = e^{kx}$

If the auxiliary equation has real, distinct roots k_1 and k_2 , the **complementary function** will be:

$$y_{cf}(x) = Ae^{k_1x} + Be^{k_2x}$$

Example

Write down, but do not solve, the auxiliary equations of the following:

$$(a) \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0, \quad (b) \quad 2\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = 0$$

$$(c) \quad 4\frac{d^2y}{dx^2} + 7y = 0, \quad (d) \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

Answer

$$(a) \quad k^2 + k + 1 = 0 \quad (b) \quad 2k^2 + 7k - 3 = 0 \quad (c) \quad 4k^2 + 7 = 0 \quad (d) \quad k^2 + k = 0$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$$k^2 + k = 0$$

$$K(k+1)=0$$

$$K=0, \text{ or } k=-1$$

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= 1 \end{aligned}$$

$$y = Ae^{-x} + B$$

Example

Find the general solution of:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 0$$

Solution

By letting $y = e^{kx}$, so that $\frac{dy}{dx} = ke^{kx}$ and $\frac{d^2y}{dx^2} = k^2e^{kx}$

the auxiliary equation is found to be: $k^2 + 3k - 10 = 0$ and so $(k - 2)(k + 5) = 0$

so that $k = 2$ and $k = -5$. Thus there exist two solutions: $y_1 = e^{2x}$ and $y_2 = e^{-5x}$.

We can write the general solution as: $y = Ae^{2x} + Be^{-5x}$

Exercise

Obtain the general solutions, that is, the complementary functions, of the following equations:

$$(a) \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$$(b) \frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 6y = 0$$

$$(c) \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

$$(d) \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$$

$$(e) \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Answers

1. (a) $y = Ae^x + Be^{2x}$
- (b) $y = Ae^{-x} + Be^{-6x}$
- (c) $x = Ae^{-2t} + Be^{-3t}$
- (d) $y = Ae^{-t} + Bte^{-t}$
- (e) $y = Ae^{2x} + Bxe^{2x}$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$$k^2 - 3k + 2 = 0$$

$$(K-1)(K-2)=0$$

$$K=1 \text{ or } K=2$$

$$y_1 = e^x$$

$$y_2 = e^{2x}$$

$$Y = A e^x + B e^{2x}$$