

Maths

(Polynomial division, and Quadratic Equations)

Lecture 5

Polynomial division

- Simplify certain algebraic fraction through a process known as polynomial division.
- Example:

$$\frac{27x^3 + 9x^2 - 3x - 10}{3x - 2}$$
$$3x - 2 \overline{) 27x^3 + 9x^2 - 3x - 10}$$

The question we ask is ‘how many times does $3x$, NOT $3x - 2$, go into $27x^3$. The answer is $9x^2$ times.

$$9x^2$$
$$3x - 2 \overline{) 27x^3 + 9x^2 - 3x - 10}$$

Just as we did with the numbers we need to find the remainder, and so we multiply $9x^2$ by $3x-2$ and write the answer down under $27x^3 + 9x^2$. Thus we get:

$$\begin{array}{r} 9x^2 \\ 3x - 2 \overline{) 27x^3 + 9x^2 - 3x - 10} \\ \underline{27x^3 - 18x^2} \end{array}$$

- To find out what is left we now subtract, and get $27x^2$, and as with the numbers we now bring down the next term, the $-3x$, and write it alongside the $27x^2$ to give:

$$\begin{array}{r} 9x^2 \\ 3x - 2 \overline{) 27x^3 + 9x^2 - 3x - 10} \\ \underline{27x^3 - 18x^2} \\ 27x^2 - 3x \end{array}$$

- Again we want to know what is left over from the division, so we multiply $3x - 2$ by the $9x$ and write the answer down so we can subtract it from $27x^2 - 3x$, giving us $15x$:

$$\begin{array}{r}
 9x^2 + 9x \\
 3x - 2 \overline{) 27x^3 + 9x^2 - 3x - 10} \\
 \underline{27x^3 - 18x^2} \\
 27x^2 - 3x \\
 \underline{27x^2 - 18x} \\
 15x
 \end{array}$$

- The process is now repeated for the third time; bring down the next term, -10 , write it next to the $15x$, and ask ‘how many times does $3x$, not $3x - 2$, go into $15x$?’.

$$\begin{array}{r}
 9x^2 + 9x \\
 3x - 2 \overline{) 27x^3 + 9x^2 - 3x - 10} \\
 \underline{27x^3 - 18x^2} \\
 27x^2 - 3x \\
 \underline{27x^2 - 18x} \\
 15x - 10
 \end{array}$$

- $3x$ divides into $15x$ five times, and so we record this above the number place, above the 10. We multiply $3x - 2$ by 5 and write the answer down so we can subtract it from $15x - 10$ and see what the remainder is:

$$\begin{array}{r}
 9x^2 + 9x + 5 \\
 3x - 2 \overline{) 27x^3 + 9x^2 - 3x - 10} \\
 \underline{27x^3 - 18x^2} \\
 27x^2 - 3x \\
 \underline{27x^2 - 18x} \\
 15x - 10 \\
 \underline{15x - 10} \\
 0
 \end{array}$$

$$\frac{27x^3 + 9x^2 - 3x - 10}{3x - 2} = 9x^2 + 9x + 5$$

- Example:

$$\frac{x^4 + x^3 + 7x^2 - 6x + 8}{x^2 + 2x + 8} = x^2 - x + 1$$

The calculation is set out as follows:

$$\begin{array}{r}
 x^2 - x + 1 \\
 \hline
 x^2 + 2x + 8 \big) x^4 + x^3 + 7x^2 - 6x + 8 \\
 \underline{x^4 + 2x^3 + 8x^2} \\
 -x^3 - x^2 - 6x \\
 \underline{-x^3 - 2x^2 - 8x} \\
 x^2 + 2x + 8 \\
 \underline{x^2 + 2x + 8} \\
 0
 \end{array}$$

- Exercise:

$$\frac{x^2 + 5x + 6}{x + 2}$$

$$\begin{array}{r} x+3 \\ x+2 \overline{) x^2 + 5x + 6} \\ \underline{x^2 + 2x} \\ 3x + 6 \\ \underline{3x + 6} \\ 0 \end{array}$$

Quadratic Equations

- A quadratic equation is one which must contain a term involving x^2 , e.g. $3x^2$, $-5x^2$ or just x^2 on its own.
- It may also contain terms involving x , and constant e.g. $3x^2+x+5$.
- It cannot have terms involving higher powers of x than x^2 .
- In general a quadratic equation will take the form:

$$ax^2 + bx + c = 0$$

Where a , b , and c can be any number including zero.

Solving quadratic equations by factorisation

- **Factorising an expression:**

$$x^2 + 5x + 6 = 0$$

We would like to write the expression in the form $(\quad + \quad)(\quad + \quad)$

1. First note that we can achieve the x^2 term by placing an x in each bracket:

$$(x + \quad)(x + \quad)$$

The next place to look is the constant term in $x^2 + 5x + 6$, we need to find the numbers in the brackets that when we multiply them the result is 6, and we add them the result is 5. It is clear that they are 2, and 3.

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

- In general when we factorise the expression $ax^2 + bx + c$ the result is :

$$ax^2 + bx + c = (x+d)(x+e)$$

Where $c=d \times e$, and $b = d + e$

- The difference of two squares which can be factorised as:

$$x^2 - c^2 = (x + c)(x - c) = 0$$

- Example

- Suppose we wish to solve $3x^2 = 27$.

$$3x^2 - 27 = 3(x^2 - 9) = 3(x-3)(x+3) = 0$$

If two quantities are multiplied together and the result is zero then either or both of the quantities must be zero.

$$x-3=0 \text{ or } x+3=0$$

$$x=3 \text{ or } x=-3$$

- Example:

Suppose we wish to solve $x^2 - 7x + 12 = 0$

- We factorise the quadratic by looking for two numbers which multiply together to give 12, and add to give -7. which are -3, and -4.

$$x^2 - 7x + 12 = (x - 3)(x - 4) = 0$$

$$x-3=0 \text{ or } x-4=0$$

$$X=3 \text{ or } x=4$$

- Exercise

- Suppose we wish to solve $x^2 - 3x + 2 = 0$.

- $x^2 - 3x + 2 = (x+(-1))(x+(-2)) = (x-1)(x-2) = 0$

- $x-1=0 \Rightarrow x=1$, or $x-2=0 \Rightarrow x=2$

Solving quadratic equations using a formula

- Consider the quadratic equation $x^2 - 3x - 2 = 0$
 - We are looking for two numbers which multiply to give -2 and add together to give -3 .
 - It is hard to find these two number. So this equation will not factorise.

$$ax^2 + bx + c = 0.$$

There is a formula for solving this: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (-2)}}{2 \times 1} = \frac{3 \pm \sqrt{17}}{2}$$

- Example

$$2x^2 + x - 6 = 0$$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4 \times 2 \times (-6)}}{2 \times 2} = \frac{-1 \pm \sqrt{49}}{4}$$

$$x = \frac{-1+7}{4} = \frac{6}{4} = \frac{3}{2}, \text{ or } x = \frac{-8}{4} = -2$$

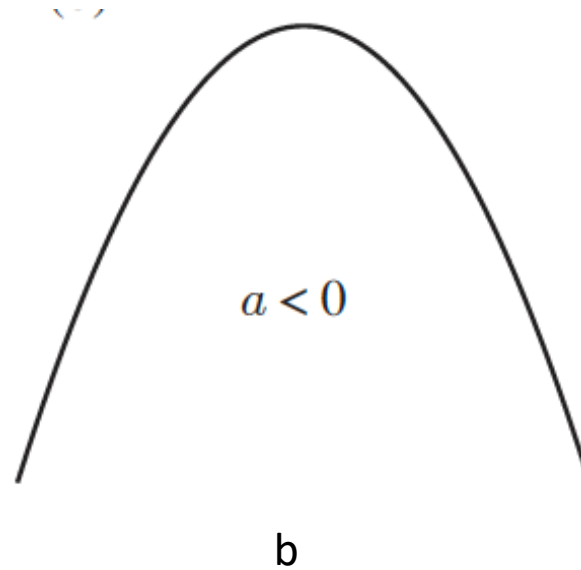
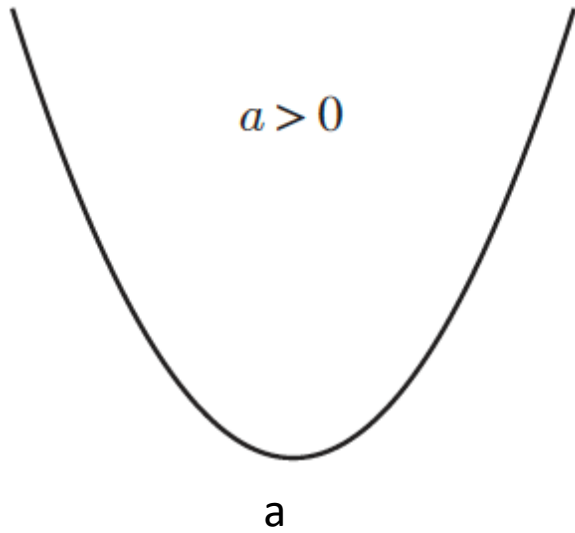
- Exercise
- Suppose we wish to solve: $3x^2 = 5x - 1 \Rightarrow 3x^2 - 5x + 1 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{5 \pm \sqrt{25 - 12}}{6} = \frac{5 \pm \sqrt{13}}{6} \Rightarrow x = \frac{5 + \sqrt{13}}{6}, \text{ or } x = \frac{5 - \sqrt{13}}{6}$$

Solving quadratic equations by using graphs

- If the coefficient of x^2 in the quadratic expression $ax^2 + bx + c$ is positive then a graph of $y = ax^2 + bx + c$ will take the form in figure a, if it is negative the graph will look like (b):



graph of $y = ax^2 + bx + c$ for the case in which a is positive.

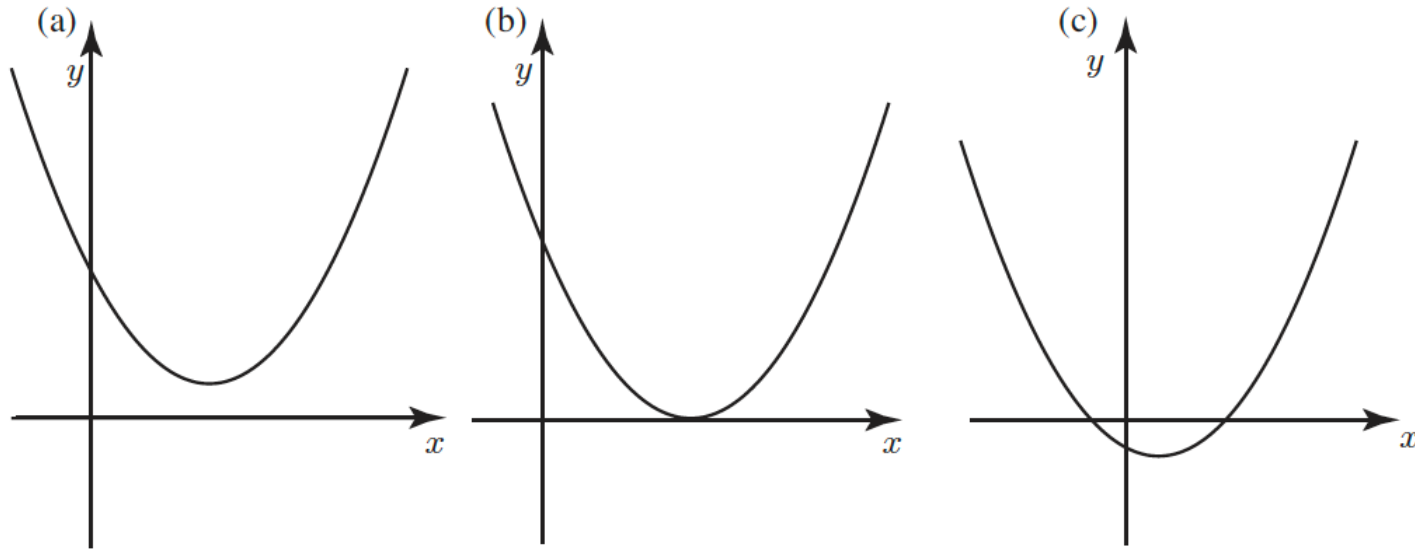


Figure 2. Graphs of $y = ax^2 + bx + c$ when a is positive

- (a) The graph never cuts or touches the horizontal axis and so this corresponds.
- (b) The graph just touches the horizontal axis corresponding to the case in which the quadratic equation has two equal roots.
- (c) The graph cuts the horizontal axis twice, corresponding to the case in which the quadratic equation has two different roots.

Polynomial functions

- A polynomial of degree n is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

- where the a 's are real numbers (sometimes called the coefficients of the polynomial).

For example:

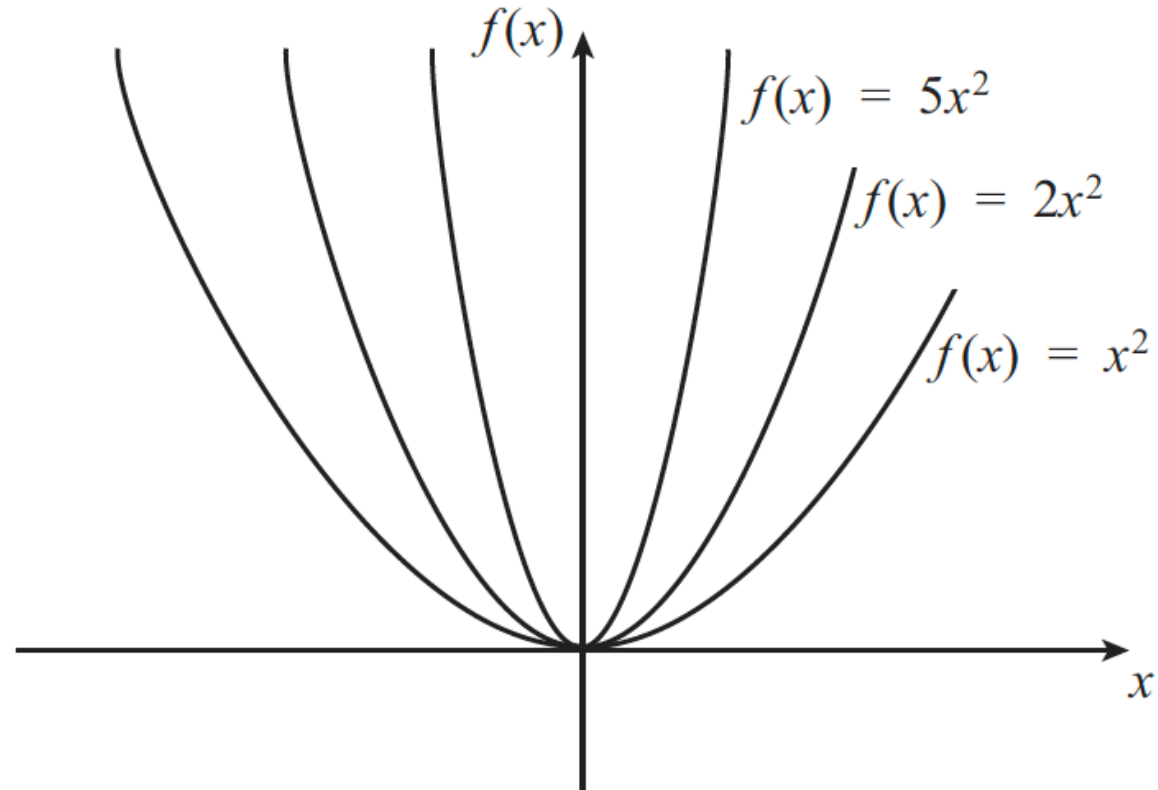
$$f(x) = x^7 - 4x^5 + 1 \Rightarrow \text{polynomial of degree 7}$$

- What is the degree of this polynomial:

$$f(x) = 4x^2 - 2x - 4$$

Graphs of polynomial functions

- Examples of quadratic functions: $f(x)=ax^2$



Roots of polynomial functions

- The number a is a root of the polynomial function $f(x)$ if $f(a) = 0$, and this occurs when $(x-a)$ is a factor of $f(x)$.

$$f(x) = (x - 2)^2(x + 1).$$

We can also see that the roots of the function are $x = 2$ and $x = -1$. The root $x = 2$ has even multiplicity and so the curve just touches the x -axis here, whilst $x = -1$ has odd multiplicity and so here the curve crosses the x -axis.

