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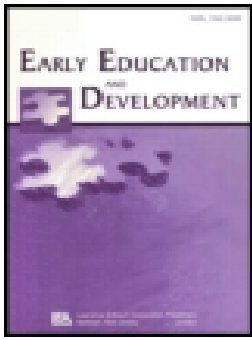


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Investigating U.S. Preschool Teachers' Math Teaching Knowledge in Counting and Numbers

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ABSTRACT

Research Findings: The aim of this study was to investigate U.S. preschool teachers' math teaching knowledge in a specific content domain: counting and numbers. One hundred in-service and pre-service teachers participated in the study; they completed a questionnaire that is composed of learning scenarios and scenario-based math teaching questions. A mixed qualitative and quantitative method was applied in analyzing the data. Results show that the majority of U.S. preschool teachers failed to identify the concept of count-out that is embedded in a math game, the one-to-one correspondence error was the most identified counting error comparing to the other errors such as the cardinality principle error, and the knowledge of content and teaching appears to be more satisfactory than specialized content knowledge or knowledge of content and students. *Practice or Policy:* The study serves to fill some gap in addressing a pressing need in the field that is to identify and measure the unique body of preschool teachers' math teaching knowledge. The study also justifies that preschool teachers need intensive training such as a coursework that wholly focuses on math teaching in order for their math teaching knowledge to be more refined and teaching to be more knowledge-informed.

Early math skills have great predictive power for later school success and socioeconomic status in adulthood (Burchinal et al., 2000; Duncan et al., 2007; NICHD Early Child Care Research Network, 2005; Peisner-Feinberg et al., 2001; Reynolds et al., 2002; Ritchie & Bates, 2013); early math intervention can also generate long-run impact for the underlying learning process (Watts et al., 2014). In the United States, increasingly more 4-year olds and in some states (e.g., NY, IL) 3-year olds are served in government-supported education programs, which presents both opportunities and challenges: how to optimize young children's exposure to math to empower them for future success? It is only natural that we turn to their teachers for some answers (Kagan et al., 2008).

Teachers' knowledge matters (e.g., Dickinson, 2011; Shulman, 1986, 1987). Research shows that pedagogical content knowledge (PCK) is a central piece of teachers' knowledge and directly associates with effective teaching and student achievement (Ball et al., 2008; Hill et al., 2004; Ma, 1999; McCray & Chen, 2012; Shulman, 1986, 1987). However, studies on teachers' math PCK have been mostly conducted with elementary or secondary education teachers (e.g., Ball et al., 2008; Hill et al., 2005, 2004; Ma, 1999; Rockoff et al., 2008; Rowan et al., 1997); there are only a few studies conducted with preschool teachers (Lee, 2017; McCray & Chen, 2012; Platas, 2014). McCray and Chen (2012) is a study that was conducted with U.S. preschool teachers and focused on their PCK: they took into account the distinctive educational context of early childhood education and designed play-based scenarios for PCK interview. A positive relation between preschool teachers' PCK and children's math learning outcomes was found in this study. One important limitation of the study, however, is that the

instrument measured preschool teachers' PCK in a holistic manner, which deviates from the rationale behind PCK that emphasizes content-specific teaching knowledge. For example, in order to respond to questions such as "Where do you see any math in this play?" or "What part of the children's play has math in it?" the participants only needed to identify the basic mathematics categories – number sense, pattern, measurement, shape, spatial relationships, and classification, etc.; they were not required to go any deeper within each content area. Deeply acknowledging the content-specific characteristic of PCK, the author undertook the study to investigate U.S. preschool teachers' teaching knowledge in a specific domain – counting and numbers.

The present study responds to the current trend of promoting early math learning for young children (Ginsburg Lee & Boyd, 2008; National Association for the Education of Young Children and National Council of Teachers of Mathematics [NAEYC & NCTM], 2010) and promoting early math teaching to be more systematic and research-based (e.g., Baroody, 2004; Clements & Sarama, 2004; Fuson, 2004; Sarama & Clements, 2009). Sarama and Clements' learning trajectories (Sarama & Clements, 2009) illustrated refined developmental sequences in the major content areas such as verbal and object counting, quantity comparing and ordering, geometric shape, measurement of length, etc. Ideally, preschool teachers should have correspondingly refined teaching knowledge. Previous studies (Li, Chi, DeBey, & Baroody, 2015; Li, Liu, DeBey, McFadden, & Pan, 2016) showed that U.S. preschool teachers' math teaching knowledge was not as refined. For example, when reporting learning goals, U.S. preschool teachers reported general and vague goals (Li et al., 2015), and their lessons were found to lack specific focuses (Li et al., 2016). Those results however need to be further validated because math teaching knowledge was not directly investigated (Li et al., 2015, 2016).

The focus of the present study is preschool teachers' math teaching knowledge; a reveal of such knowledge can help better prepare preschool teachers in teaching mathematics to young children.

Note that, preschool teachers in this paper refer to teachers who teach preschool, pre-kindergarten, or kindergarten; "the author" in this paper refers to the author of the present study.

Theoretical Background and Literature Review

According to Shulman (1986), teachers' knowledge contains three domains: content knowledge, pedagogical content knowledge (PCK), and curricular knowledge. PCK was identified as the central piece of teachers' knowledge that "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching." Ball et al. (2008) further developed Shulman's PCK concept (Shulman, 1986, 1987) and identified two discernable subdomains (1) *knowledge of content and students* (KCS) that "combines knowing about students and knowing about mathematics" (p. 401), and (2) *knowledge of content and teaching* (KCT) that "combines knowing about teaching and knowing about mathematics" (p. 401). Moreover, Ball et al. (2008) identified another subdomain of Shulman's content knowledge (Shulman, 1986), namely *specialized content knowledge* (SCK), which is a type of mathematical knowledge that is "not typically needed for purposes other than teaching" (p. 400), for example, knowledge of the different subtraction models ("change-take-away" and "comparison") or knowledge of the subskills that are involved in object counting (number sequence, one-to-one correspondence, keeping track). The three domains of knowledge—SCK, KCS, and KCT compose the construct of math teaching knowledge in this present study.

In the following section, the author reviewed relevant studies that informed the design of the present study.

Studies of Preschool Teachers' Math PCK and the Present Study

Two studies have been undertaken explicitly investigating preschool teachers' math PCK—McCray and Chen (2012) and Lee (2017). Specifically, McCray and Chen (2012) measured preschool teachers' PCK from two aspects: the ability to identify embedded math concepts in play scenarios and the ability to formulate questions for promoting those concepts. Related with this work, Lee (2017) defined PCK for

preschool teachers as consisting of three aspects: noticing children's mathematical thinking (that occurs during free play or other situations), interpreting the nature of mathematical situations, and enhancing children's mathematical thinking in those situations. Both studies referenced Ball et al. (2008) PCK framework, but both substantially re-conceptualized the framework. Specifically, one, in contrast to Ball et al. (2008) focus on explicit math tasks, McCray and Chen (2012) and Lee (2017) used implicit math tasks that take the form of children's free play. Two, while Ball et al. (2008) contended that PCK involves teachers examining a concept deeply and unpacking it, McCray and Chen (2012) and Lee (2017) contended that preschool school teachers' PCK involves noticing all the relevant math concepts in a particular situation. Three, children's development and common misconceptions with particular concepts were not explicitly addressed in those two studies as in Ball et al. (2008).

Different from McCray and Chen (2012) and Lee (2017), the author adopted a more direct application of Ball and et al. PCK (Ball et al., 2008) for a few important reasons. First, math lessons that take place in large or small groups are one of the major learning forms for young children (National Association for the Education of Young Children [NAEYC], 2009). Preschool teachers' PCK should include explicit math teaching context in addition to free play (e.g., Frye et al., 2013; NAEYC & NCTM, 2010). Second, teaching math to young children is somewhat of similar nature to teaching elementary grades—Ginsburg and Amit (2008) showed that teaching math to young children “entails all the processes involved in teaching mathematics to older children, including profound knowledge of the subject matter, pedagogical content knowledge, lecturing, introduction of symbolism, and connecting everyday experience to abstract ideas” (p. 274). Third, the Common Core movement (Council of Chief State School Officers, 2010) advocated that math teaching including that in early years should focus on fewer concepts but deeper learning. A more direct approach of applying Ball et al. (2008) in defining preschool teachers' math teaching knowledge resonates with this call and can potentially guide early math teaching to a more refined level.

Choosing the Specific Content for Measuring Math Teaching Knowledge

According to Ball et al. (2008), PCK is content specific. Therefore, the author's first task was to determine a specific content area. Early childhood math encompasses a range of contents including pre-number skills (e.g., matching, sorting, or sequencing), number and operations, geometry, measurement, and pattern and algebra, etc. Able to focus on only one content area the author chose number and operations, with two considerations: one, the Common Core standards (Council of Chief State School Officers, 2010) recommended number concepts as the primary focus for early years; two, there is plenty of research in this content area.

According to the research literature (e.g., Jordan et al., 2007; Purpura, 2010), early number concepts contain three major domains: *numbering*, which entails determining the total number of items in a collection or using a cardinal term to create a collection of a specified number; *numerical relations*, which involves knowledge of how two or more sets of numbers are related to each other and the association between the numbers on the mental number line; and *arithmetic reasoning*, which is a child's ability to understand changes in quantity and obtain new quantities from the change in the size of set. Correspondingly, the author designed an instrument with three sections: counting and numbers (corresponds to numbering [Purpura, 2010], but only focuses on counting and number recognition), numerical relation, and arithmetic reasoning. Note that, due to the limit on paper length, reported in this paper is only Scenario 1 *Counting and Numbers*. The other two scenario sets will be reported in another separate paper should it be published.

Children's Number Development and Defining Preschool Teachers' Math Teaching Knowledge

The next task was to define what makes up the SCK, KCS, and KCT in counting and numbers, which cannot be possibly done without referencing studies in children's number development (e.g., Baroody, 1992; Baroody & Coslick, 1998; Sarama & Clements, 2009; Gelman et al., 1978; Wynn, 1992).

Start with SCK

According to Ball et al. (2008), SCK "involves an uncanny kind of unpacking of mathematics that is not needed – or even desirable – in settings other than teaching" (p. 400). In early childhood math, take counting as an example, any typically developed adults can count, but some knowledge *about* counting is uniquely needed only by preschool teachers. Studies on children's number development (e.g., Gelman et al., 1978; Wynn, 1992) indicate that object counting involves three subskills: reciting correct number sequence, applying one-to-one correspondence, and keeping track. Apparently, not every adult is aware of those subskills, but knowledge of those subskills is needed by an effective preschool teacher to diagnose a child's counting difficulty. Another example, number recognition involves different forms: a point-to-*n* task requires that a child associate a number word/numeral with a quantity; a how-many task requires that the child not only make the word-quantity association but also speak out the correct word; a give-*n* task further requires that the child create the set (Wynn, 1992). Any adult may ask the how-many question to informally assess a child's number knowledge, but an effective preschool teacher needs the knowledge of the other two tasks to either use the point-to-*n* task when a child has difficulty in responding with words or to use the give-*n* task to move the child to a higher level.

Therefore, by the definition (Ball et al., 2008), SCK in counting and numbers should entail unpacked and therefore differentiated knowledge such as distinguishing oral counting, object counting, and counting out (set production), relational knowledge regarding how different concepts are sequentially related (see a list of SCK knowledge points in [Appendix A](#)). In the questionnaire for this present study, participants were asked to identify the key concept that is embedded in a math game as the measure of SCK. It is inferred that if a teacher can identify the targeted concept using specific terms instead of general terms, for example, "counting out" instead of "counting," the teacher has a differentiated knowledge, a knowledge that involves "unpacking mathematics" (Ball et al., 2008, p. 400).

Move on to KCS

When explaining *knowledge of content and students*, Ball et al. (2008) stated that central to the teaching tasks "is knowledge of common student concepts and misconceptions about particular mathematical content" (p. 401). Certainly, KCS should include knowledge of children's common errors and misconceptions with particular concepts, for example, the common error with toddlers that one number word is associated with multiple quantities. In the questionnaire, the participants were asked to identify a child's counting errors as the index for KCS. The author chose to use knowledge of children's common errors as the measure for two reasons: one, diagnosing students' errors has been widely used in PCK assessments (e.g., Ball et al., 2008; Hill et al., 2004; Ma, 1999), and two, children's common errors in number development are well documented by the research literature (e.g., Clements & Sarama, 2009; Wynn, 1992).

Lastly for KCT

In the present study, KCT is measured from the angle of whether a teacher can propose appropriate tasks/strategies to help a child with specific errors. Teachers constantly encounter children's errors; effective teachers need to accurately diagnose the errors, come up with a curing prescription, and move students forward from their errors (Ball et al., 2008). The tasks/strategies that a teacher proposes for helping with a child's errors can reflect the teacher's knowledge across content, child, and teaching. In addition, Sarama and Clements (2009) contended that teachers should be able to identify a specific goal and design instructional tasks to suit children's developmental needs. In the questionnaire, participants were asked to choose/propose tasks/strategies that can help with a child correcting his

counting errors. This angle of measuring KCT is consistent with the logics of both theories (Ball et al., 2008; Sarama & Clements, 2009).

The Design of the Questionnaire for Math Teaching Knowledge

With the SCK, KCS, and KCT being defined in counting and numbers, the author moved on to the final task: how to measure them.

PCK research literature provided two important guidelines (Ball et al., 2008; Hill et al., 2004; Ma, 1999; McCray & Chen, 2012). One, Math teaching questions should be situated in math teaching scenarios. This guideline reflects Ball et al.'s (2008) practice-based approach to categorizing teachers' knowledge – what teachers need to know to carry out a specific math teaching task. Hill et al. (2004) illustrated examples of using teaching scenarios to embed questions that measure teachers' content knowledge and knowledge of students and content. Besides the theoretical justification, the use of teaching scenarios has its practical advantage that scenarios can situate participants in a teaching position and elicit their responses from the teaching perspective. Accordingly, all the math teaching questions in the present study are scenario-based. Two, for early childhood math, the scenarios should reflect the unique learning context for young children (McCray & Chen, 2014; Lee, 2017). Free play has been viewed as the typical learning context for young children (Bredekamp, 1987; Bredekamp & Copple, 1997) and used as the scenario for PCK interview (McCray & Chen, 2014; Lee, 2017). But the author thought that the teaching scenario needs to be explicitly math focused in order for math teaching knowledge to be gauged in a specific manner and at a deep level. The author used math games instead because math games target at specific concepts as well as present typical learning context for young children.

There is another important characteristic of the questionnaire in the present study: two sets of questions were designed following each teaching scenario. Set I presents short answer questions that participants need to answer with their own words; Set II presents multi-choice questions measuring the same concepts as in Set I but participants only need to choose the correct answer. The purpose of this design was to assure response reliability: consistency between participants' responses to the two sets of questions can reveal how reliable their responses are. Moreover, the design served another purpose: it helps reveal preschool teachers' math teaching vocabulary.

Research Questions

In contrast with using free play and gauging preschool teachers' math PCK in a holistic manner (Lee, 2017; McCray & Chen, 2012), the present study used explicit math tasks and investigated preschool teachers' math teaching knowledge in a specific content area – counting and numbers. Three research questions were outlined for the present study:

- (1) To what extent can preschool teachers identify specific math concepts that are embedded in a math game (the SCK question)?
- (2) To what extent can preschool teachers diagnose children's cardinality principle error, one-to-one correspondence error, and counting-out error (the KCS question)?
- (3) What do preschool teachers propose to help a child with his or her cardinality principle error, one-to-one correspondence error, and counting-out error (the KCT question)?
- (4) How do in-service and pre-service teachers differ in those three domains?

Method

Participants

About 150 participants were recruited, of whom 100 completed the study. The participants were recruited from graduate and undergraduate early childhood teacher preparation programs (at the author's department) as well as from preschools in campus vicinity. Among the participants, 29 were head teachers and 18 were teaching assistants, who altogether made up the in-service group; the other 53, who either were graduate students but had no formal teaching experience (e.g., paraprofessional or substitute teacher) or were undergraduates, made up the pre-service group. The participant pool represents a diverse population along ethnicity (28% White, 35% Black, 11% Hispanic, 17% Asian, & 9% other) and age (57% 21–30, 17% 31–40, 20% 41–50, & 4% 51+). There is no significant difference between the two groups along ethnicity ($\chi^2 = 10.094$, $df = 6$, $p > .05$) or age ($\chi^2 = 4.879$, $df = 7$, $p > .05$).

Procedure

The author handed out the questionnaire to the participants in their classes or their school sites, or the participants picked up the questionnaire from a designated mailbox that was outside the author's office. The participants completed the questionnaire either during the author's visit (64 participants) or completed it at other time and returned it to the mailbox within 3 weeks (36 participants). The participants completed the Scenario 1 set and then were instructed—by an instruction page in the questionnaire—to complete either the Scenario 2 set (for teachers of Pre-K or younger) or the Scenario 3 set (for teachers of Kindergarten or higher grades), or if they wished, completed both sets. Altogether, it took about 30 min for the participants to complete the questionnaire. The participants received no compensation for completing the questionnaire.

For all the participants who were graduate or undergraduate students, they either had not taken any math pedagogy course or were currently enrolled in a math pedagogy course. For the latter, participants completed the questionnaire before early math topics were covered (i.e., taking the questionnaire in the first 3 weeks of class before any specific concepts were taught).

Instrument

The questionnaire is composed of three scenarios. Each scenario depicts a math game that is played by a teacher and a group of children. Following each scenario are two sets of questions: Set I short answer questions that participants write responses; Set II multi-choice questions that participants choose the correct response/s. Below is an excerpt of Scenario 1:

Ms. Levy is playing a card game with two 4-year-olds, Ben and Alice, in their Pre-K classroom. Ben and Alice take turns to draw a card with a numeral of 1–10, and then put that number of plastic bears from a bin onto a plate.

Ben draws a card with the numeral 5. He grabbed 6 bears saying “five,” and then put the 6 bears onto his plate.

Ms. Levy asks Ben to count the bears on his plate.

Ben uses his index finger pointing to each bear and counts, “one, two, three, four, five, six, seven.” Ben says “six, seven” when he counts the last bear.

Ms. Levy says, “Ben, let's count again together.” Ms. Levy points to each bear and counts together with Ben, “One, two, three, four, and five, and then Six. **Six.** So, Ben, how many bears did you put on the plate?”

Ben says, “one, two, three, four, five, six.”

...

This scenario is used to embed the SCK, KCS, and KCT questions. Specifically, participants' SCK is measured by their ability to identify the key concept that is embedded in the game—count out/set production: produces a set by a number word or written numeral. KCS is measured by participants'

ability to identify Ben's errors; specifically, Ben made three errors: count-out error – grabbed six instead of five items, one-to-one correspondence error – said “six, seven” for the last bear, and cardinality principle error – said “one, two, three, four, five, six” instead of “six” when responding Ms. Levy's how-many question, which means that he cannot use the last count word to tell the cardinal value of the whole set. Lastly, KCT is measured by participants' ability to propose tasks that help with Ben's errors. The full scenario can be obtained by contacting the author; selective scenario questions are presented in Appendix B.

Validity and Reliability

Content validity indicates the degree to which an instrument measures what it is intended to measure, and whether the instrument elicits accurate information (Cox, 1996). To establish validity for the questionnaire in the present study, several measures were undertaken during each writing phase. First, for initial writing, (a) key knowledge points were established (see Appendix A) based on the author's review of relevant literature such as Common Core state standards (Council of Chief State School Officers, 2010), math pedagogy textbooks (e.g., Van De Walle et al., 2017), and research literature (e.g., Sarama & Clements, 2009), (b) a matrix was created in order to ensure the distribution of questions across key knowledge points, teaching knowledge domains, and the Common Core State standards (see Table 1). Second, during the refinement phase, (a) the director and teachers from the college lab preschool evaluated its content validity and readability by reviewing the scenarios/questions, and commenting on how the questionnaire content pertains to classroom practice, (b) a pre-k teacher was hired as the research assistant to edit the questionnaire to further improve the readability of the teaching scenarios as well as the questions; and (c) three 5-point Likert questions were included at the end of the questionnaire to measure its general validity (see the questions and statistics in Appendix C). Third, the questionnaire was piloted with two classes and altogether 38 students (including both in- and pre-service teachers) in order to evaluate whether the question items were adequately sensitive to generate sufficient response variability.

Reliability refers to the consistency of a research study or measuring test. For the present study, the author used an important strategy to ensure the internal reliability of the questionnaire—the extent to which the measure is consistent within itself. Specifically, for each scenario, two sets of questions were used to measure the same set of concepts. The similar data patterns obtained from the two sets provided solid evidence for the internal reliability (see details in Results).

Data Analysis

Two research assistants (RAs) were hired for data entry and coding.

For the data obtained from Set I (the short answer questions), the author used a mixed methods approach: qualitative analysis was applied to identify response categories, and quantitative analysis

Table 1. Matrix of targeted concepts, math teaching knowledge domains, and the common core standards.

Question	Targeted concepts	Knowledge domain	Common core code
1-1-1 (Set I)	Counting-out	SCK	PK-CC 4
1-1-2	Cardinality error; one-to-one error; counting out	KCS	PK-CC 3a, 3b, 4
1-1-3	More-than; number composition (not reported)	KCS	PK-CC 5
1-1-4	One-to-one correspondence; cardinality; counting out	KCT	PK-CC 5
1-2-1 (Set II)	Counting-out	SCK	PK-CC 4
1-2-2	one-to-one error	KCS	PK-CC 3a
1-2-3	Cardinality error	KCS	PK-CC 3a
1-2-4	Counting-out	SCK	PK-CC 4
1-2-5	More than; number composition (not reported)	KCS	PK-CC 5
1-2-6	One-to-one correspondence; cardinality; counting out	KCT	PK-CC 5

SCK – Specialized Content Knowledge; KCS – Knowledge of Content and Student; KCT – Knowledge of Content and Teaching (Ball et al., 2008).

was applied to calculate frequencies of the response categories or compare proportions between the response categories. The procedure of qualitative analysis for each question varied (see details for each question below), but all questions entailed the RAs coding the data by categories and tallying the frequency for each category. The inter-rater reliability for the three research questions is 82%, 90%, 78%, respectively, and 83% on average. The discrepancies were resolved by the author reexamining the data and discussing them with the RAs. For the quantitative analysis, a z-test comparing one proportion (https://www.medcalc.org/calc/test_one_proportion.php) was used to compare two proportions from the same sampling (all participants)–the z-test was used to directly compare a proportion to a null hypothesis and another proportion to the same null hypothesis, and thus indirectly compare the two proportions. A z-score test (<https://www.socscistatistics.com/tests/ztest/default2.aspx>) was used to compare two proportions from two different samplings, in the present study, pre-service versus in-service participants.

For the data obtained from Set II (the multi-choice questions), quantitative analysis was used. Specifically, the same z-test and z-score test were used to compare proportions. Chi-square test was not used because the multi choices in the questionnaire are not mutually exclusive – participants could choose more than one options.

Results

Presented below are the analysis and results for each research question.

Can Preschool Teachers Identify the Embedded Math Concepts in a Specific Manner?

Question 1 (Q1) asks “For the card game in Scenario 1, what mathematical skills are involved? Please list up to 3 math skills.” This question is intended to investigate whether participants can identify the key concept that is embedded in the depicted math game – count out by a given numeral.

Set I (Short Answer): Analysis and Results

A review of the Q1 data indicates that the participants’ responses diverged substantially. Based on this review, the author expanded the data analysis from using pre-fixed categories examining whether and how the participants’ responses matched with the count-out concept to also including open search for emergent themes.

For the open search, response categories were identified from the data and defined based on occurring frequencies. Specifically, the author reviewed a subset of the data (15 cases) and identified five most frequent responses; each of the two RAs reviewed another subset of the data (15 cases) and also identified five most frequent responses. The author then finalized five response categories by comparing the top five frequent responses among the three coders. Lastly, by the five response categories, the RAs coded each participant’s response and tallied the frequency for each response category across the participants. Any responses that were not matching with the five categories were documented verbatim in the “other column.” The RAs’ inter-rater reliability for this question is 82%,

Table 2. The top five skills reported by the participants for set I “what math skills are involved?”.

Response categories	Frequency
Counting	65
One-to-one correspondence	54
Cardinality	33
Addition/subtraction	31
Number recognition	22

A participant may list 1–3 or more skills.

and the discrepancies were settled by the author reexamining the discrepant cases and discussing them with the RAs.

The open-search analysis shows that participants reported as many as 24 math skills. The top five most reported ones in order are “counting,” “one-to-one correspondence,” “cardinality,” “addition/subtraction,” and “number recognition” (see frequency for each category in Table 2). “Counting out” or “set production” was not among the top five and reported by nine participants (9% of the participants) only, which is not significantly higher than a null hypothesis of 10% by the z -test comparing one proportion ($z = 0.33$, $p > .05$). In comparison, 22% of the participants reported “number recognition,” which is significantly higher than the null hypothesis of 10% ($z = 4.0$, $p < .01$).

Regarding the analysis by the pre-fixed categories, four categories were used: *explicit level* involves participants literally using the term “counting out,” “set production,” or “producing a small number of items;” *implicit level* involves participants stating that the game involved identifying a number and producing the corresponding amount but without using those explicit terms as in the explicit level; *generic level* involves participants reporting counting or sub-counting skills but with no indication of how the counting in this game (counting out) is different from counting objects to tell how many; *unfocused* involves participants reporting irrelevant skills (e.g., addition/subtraction) or general skills (e.g., problem-solving or algebraic thinking). The author and one RA conducted this analysis; the inter-rater reliability is 80%. Discrepancies were resolved by the author reexamining the cases and discussing them with the RA.

The analysis with the pre-fixed categories was conducted only with the 47 in-service participants, who presumably should have a higher level of teaching knowledge than the pre-service group. Examples and frequencies for each of the four levels are presented in Table 3. We can see that 43% of the participants (including the explicit level and implicit level) identified the count-out concept that is embedded in the game, but 57% of the participants could not recognize that the depicted game involved counting-out objects to make sets.

Set II (Multi-choice): Analysis and Results

For the corresponding multi-choice question, “Which math concepts/skills are required in order to play the card game depicted in Scenario 1?” four options (and the definitions) are provided (see more details in Appendix B): A. cardinality principle, B. number recognition (telling how-many items are in a set), C. set production (producing a set by a given number), and D. number composition. The correct response should be C set production. As can be seen from Figure 1, C (by C only or together with other options) was the second least choice, chosen by 46% of the participants, which is not significantly

Table 3. The four pre-fixed categories for in-service participants: frequency and examples.

Levels	Explicit level	Implicit level	Generic level	Unfocused
Frequency	3 (6%)	17 (37%)	10 (21%)	17 (36%)
Examples	#17: Number recognition; producing small number of items; counting using 1:1 correspondence #78: Cardinality principle, number recognition, set production #89: Set matching; set production	#6: Counting and cardinality, number recognition, making connections between numbers and quantities #9: 1:1 correspondence; counting up by ones; concrete number modeling #73: rote counting; 1:1 correspondence; representation of a number	#11: counting; cardinality; 1:1 correspondence #57: oral counting; object counting; cardinality principle #66: 1:1 correspondence; cardinality principle, counting forward #90: counting skills from 1–10; 1:1 correspondence	#15: sequence of numbers; counting & number recognition, addition/subtraction/concept of more or less than #38: addition; subtraction; counting forward #93: critical thinking; procedural fluency; memorization

Specialized Content Knowledge in Counting and Numbers (multi-choice section)

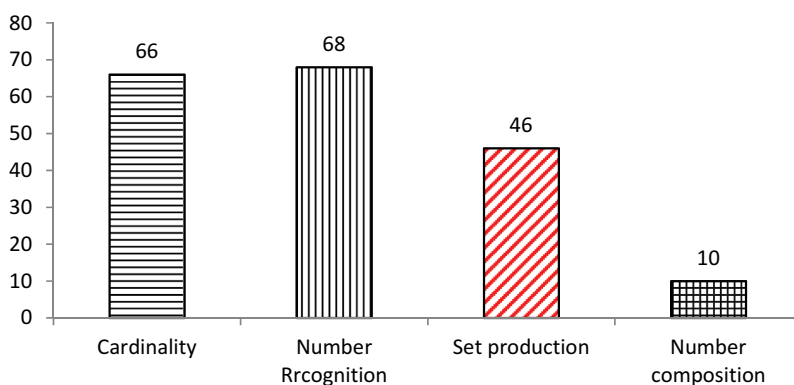


Figure 1. Frequencies of participants' response choices for the question of what math skills are involved in the math game. Counting out/set production is the targeted concept. Frequency of each category is how-many times each category was chosen by itself or together with other categories. Note that, a participant may choose more than one categories.

different than a null hypothesis of 50% by the z -test comparing one proportion: $z = 0.8$, $p > .05$. In comparison, the most chosen answer was B number recognition (by B only or together with other options), by 68% of the participants, which is significantly higher than the null hypothesis of 50% by the same z -test: $z = 3.6$, $p < .001$. The z -score test shows no significant difference between pre-service and in-service participants in choosing *set production* as the answer (53% vs. 40%), $z = 1.24$, $p > .05$; the same test shows no significant difference between the two groups in choosing *number recognition*, $z = 0.88$, $p > .05$.

The results of both Set I and Set II show that about 40% of the participants could identify the key math concept in a specific manner, that is they could identify that count-out/set production is involved in the depicted math game; for the rest of the participants, they could not recognize that the game is a count-out game that is different from counting objects to tell how many.

Can Preschool Teachers Diagnose a Child's Counting Errors?

In Set I, Question 2 asks "Ben had some difficulties when playing this card game. Please list up to three errors that you noticed." Note that, Ben made three counting errors in the scenario – one-to-one correspondence error, cardinality principle error, and counting-out error.

Set I (Short Answer): Analysis and Results

The three errors were used as the pre-fixed categories to analyze the data. The author also adopted the participants' wordings in defining the categories: 1) one-to-one correspondence error: no reporting "one-to-one correspondence," or "say two numbers for the same bear;" 2) cardinality error: no reporting of "cardinality," or no "using the last number to say the value;" and 3) keeping track error: "no stop putting out bears," or "keep counting." The two RAs conducted the analysis, and the inter-rater reliability is 90%.

The analysis shows that only two participants—one in-service participant, one pre-service participant—identified all the three errors. Twenty-five participants—5 [11%] in-service participants, 17 [32%] pre-service participants – did not identify any of the three errors; the proportions between these two groups are significantly different by the z -score test comparing two proportions: $z = 2.58$, $p < .05$.

Regarding the one-to-one correspondence error, 61% of the participants identified the error (29 [61%] in-service participants; 32 [60%] pre-service participants): 33 of them explicitly used the term

Table 4. The list of other terms participants used describing Ben's counting errors.

Error	Terms/phrases describing the error
One-to-one correspondence error	"could not keep track" "counting too fast" "sequence counting"
Cardinality principle error	"started counting from beginning the 2nd time" "identify how many [error]" "identify quantity [error]" "recall [error]" "identifying numbers [error]"
Counting-out error	"2nd time put too little bears" "ended with smaller number" "place 4 instead of 5" "incorrect total" "words to quantity [error]"
Other	"guessing" "subitizing" "add/subtract" "part of whole" "counting/value of 5" "representation" "needs to pay attention" "miscounting"

"one-to-one correspondence," and 28 of them identified the error but could not name it—they used other terms instead describing the error (see Table 4).

For the cardinality principle error, 27% of the participants (13 [28%] in-service participants; 14 [26%] pre-service participants) identified the error: 12 of them explicitly used the term "cardinality," and 15 identified the error but could not name it (Table 4).

Lastly, for the counting-out error, only 9% of the participants (five [11%] in-service participants; four [7%] pre-service participants) identified the error – one explicitly used the term "set production," and the other eight participants identified the error but used other terms describing the error (Table 4).

Participants' responses that were considered as incorrect were also examined. One interesting finding is that about 10 participants attributed subjective reasons to Ben's errors, for example, attention reason such as "not understanding he only needed 5 bears," "not paying attention," lack of patience such as "did not take his time to count," or memory reason such as "difficulty remembering the last number he count."

Set II (Multi-choice): Analysis and Results

In Set II, the three errors are asked in three separate questions (see Appendix B), and for each question, four options (and the definitions) are provided.

Regarding the one-to-one correspondence error, the four options are: A. counting sequence, B. one-to-one correspondence (cannot assign one and only one number to an object), C. keeping track (cannot make sure that each item is counted only once), and D. cardinality principle. The correct response is B. Result shows that B (as a single option) was chosen the most and was chosen by 59% of the participants, and an additional 22% chose B with another choice (e.g., keeping track error); 10% of the participants chose C. The z-score test shows no significant difference between the pre-service and in-service groups in choosing B as the single correct answer (55% vs. 64%): $z = 0.71, p > .05$.

Regarding the cardinality principle error, the four options are: A. one-to-one correspondence, B. keeping track, C. cardinality, and D. no difficulty is observed. The correct response is C. The result shows that 52% of the participants chose C as the single answer, which is significantly higher than a null hypothesis of 40% by the z-test comparing one proportion ($z = 2.45, p < .05$). The second most choice was "no difficulty is observed" and was reported by 20% of the participants, which is significantly less than the null hypothesis of 40% by the same z-test ($z = 4.08, p < .01$). No significant difference was found between in-service and pre-service participants in choosing cardinality as the single answer by the same z-score test (43% vs. 61%), $z = 1.83, p > .05$.

Regarding the counting-out error, the four options are: A. numeral reading, B. number recognition, C. set production, and D. set matching. The correct response is C. Result shows that 50% of the participants chose C as the single answer, which is significantly more than a null hypothesis of 30% by the z -test comparing one proportion ($z = 4.36, p < .01$). The second most choice is *number recognition* chosen by 31% of the participants, which is not significantly more than the null hypothesis of 30% by the same z -test ($z = 0.22, p > .05$). Again, no significant difference was found between in-service and pre-service participants by the same z -score test (47% vs. 53%), $z = .60, p > .05$.

In sum, results from Set I and II are generally consistent. Specifically, the one-to-one correspondence error was more identified than the cardinality principle error even though significant difference is only found in Set I (short answer)–61% vs. 27%: 61% is significantly higher than a null hypothesis of 25% by the z -test ($z = 8.3, p < .01$), while 27% is not significantly higher than the null hypothesis of 25% ($z = 0.46, p > .05$). The cardinality principle error, in turn, was significantly more identified than the counting-out error even though the difference is also only found in Set I–27% vs. 9%: 27% is significantly higher than a null hypothesis of 10% by the z -test ($z = 5.6, p < .01$), while 9% is not significantly higher than the null hypothesis of 10% ($z = 0.33, 5.6, p > .05$).

How Do Preschool Teachers Attempt to Help with the Child's Errors?

In Set I, Question 3 asks “How would you help Ben in order for him to play the Card game more successfully?” Note that, it is not specified in Question 3 which errors to help because identifying Ben's errors was what the participants needed to answer for Question 2.

Set I (Short Answer): Analysis and Results

For this question, a preliminary review showed that the participants' responses were substantially divergent; a lot of information would be lost if grouping participants' responses only into a few categories. Therefore, the analysis was approached by listing as many response categories as needed, and merging the categories only if the responses are highly similar (e.g., “lower the numbers to 1–4” and “start off with lower numbers”). Each RA made a list of response categories: RA1 ended up with 28 categories and RA2 with 27. The author compared their categories, combined some of them, and ended up with 16 categories as listed in Table 5. The two RAs then coded each participant's response by the 16 categories independently. The RAs' inter-rater reliability is 78%; discrepancies were resolved by the author examining the cases and discussing the discrepancies with the RAs.

Table 5. Participants' proposed teaching strategies for helping Ben.

Targeted error (added by the researcher)	Strategies	# of participants
One-to-one correspondence error	Touch and count	5 (3;2) ^a
	Techer model/count with Ben	12 (6;6)
	Practicing 1:1 (e.g., using ten-frame)	8 (5;3)
	Using objects with number numbers writing on them	4 (3;1)
Counting-out error	Practice counting with fingers/manipulatives	5 (2;3)
	Take out one at a time and count aloud	8(2;6)
	Give one spot to each bear	2(2; 0)
	Matching bears with dots/fingers/other objects first	9(5;4)
	Comparing target quantity with other quantity	2 (2;0)
Cardinality principle error	Understand number representation, subitize	6 (1;5)
	Model number and quantity	7 (4;3)
	Understand cardinality principle	5 (1;4)
Overall	Lower the numbers for the game	17 (8;9)
	Use cards with both numeral and quantity	5 (0; 5)
	Know more or less	3 (2;1)
	Count forward	2 (1;1)
Other		11
Total		122

A participant may propose more than 1 strategy. The participants did not specify which error/s their strategies targeted at.

^a(3;2) “3” refers to the number of in-service participants, and “2” refers to the number of pre-service participants.

As Table 5 shows, the participants proposed a variety of teaching strategies. For the one-to-one correspondence error, the most proposed strategy is “teacher model/count with Ben.” For the counting-out error, the most proposed strategies are: “take out one at a time while count aloud,” and “matching bears with dots/fingers/other object first.” For the cardinality principle error, there was no most proposed strategy. One notable strategy is that a teacher should help Ben understand the number and its associated quantity, and another strategy is that the teacher should help Ben understand the cardinality principle. Lastly, for the game per se, the most proposed strategy is that the teacher should use smaller numbers (e.g., 1–5). In general, most of the proposed strategies are practical and helpful.

Note that, some participants proposed strategies that are not explicitly helpful, for example, trying different number combinations of five or six, using number line, or knowing more or less.

Set II (Multi-choice): Analysis and Results

In Set II, six options are provided for the multi-choice question: A. Counting forward, B. Using small sets to reinforce correct enumeration, C. Understanding more than/less than, D. Knowing that a number is composed of small numbers, E. Understanding of the cardinality principle, F. Using flash cards to recognize number patterns. The choice of both B and E is considered the best answer. A is not considered correct because Ben did not demonstrate difficulty in reciting the counting sequence; C or D is not directly relevant for counting, counting out, or cardinality, and therefore is not considered correct; F is considered relevant but not the best choice.

The analysis shows that the participants made a variety of choices (Figure 2), which mirrors their divergent responses in Set I. Specifically, 29% chose both B and E, which is the correct answer and the most chosen one. Even though some choices such as AE can be considered acceptable, the other choices such as AC, AF, C, CD, D, or DF (altogether 8%) are not relevant, for example, understanding more than/less than (C) would not help Ben with any of his errors neither would know that a number is composed of small numbers (D). The z-score test shows no significant difference between the pre-service and in-service groups in choosing both B and E as the correct answer (28% vs. 30%); $z = 0.16$, $p > .05$.

Results from Set I and Set II are mostly consistent in that participants' responses were substantially divergent, and that only a small portion of participants proposed strategies that are not explicitly helpful.

Knowledge of Content and Teaching in Counting and Numbers (multi-choice section)

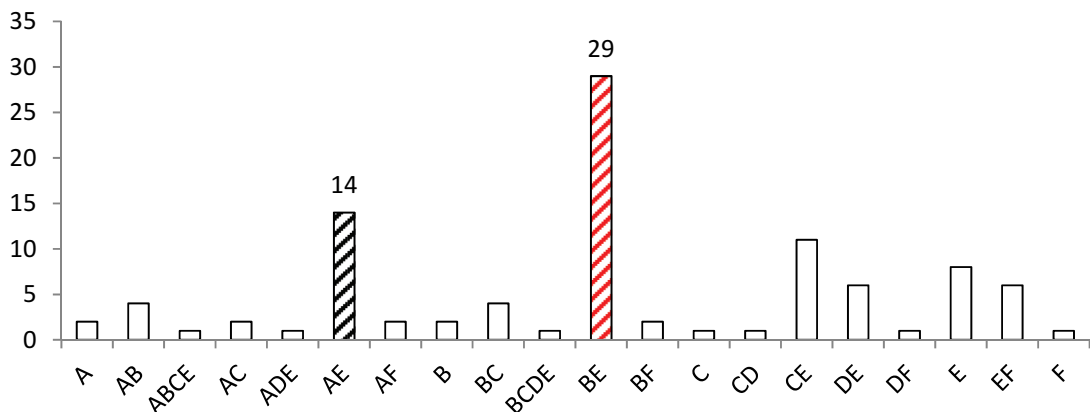


Figure 2. Frequencies of participants' response choices for the question of what areas should the teacher focus on to help Ben. Note. (a) counting forward; (b) using small sets; (c) understanding more than/less than; (d) number composition; (e) understanding cardinality; (f) recognize number patterns. B and E together make the best response.

How Do In-service and Pre-service Teachers Differ in Those Three Domains of Math Teaching Knowledge?

At both the expressive level (Set I, short answer section) and the receptive level (Set II, the multi-choice section), the in-service participants appeared to perform better than the pre-service participants – almost all the data are higher in favor of the in-service participants, but only one difference is significant—more in-service participants were able to identify all the three counting errors than the pre-service participants (Q2).

Discussion

The participants tended to either over-generalize the depicted math game as a counting game or narrowly focused on a counting subskill such as one-to-one correspondence; they appeared to lack the knowledge to recognize the essential nature of the game – counting out/producing a set by a given numeral. It is not technically wrong to view the counting-out game as a counting game or as involving one-to-one correspondence or cardinality, but there *is* a difference between counting out and object counting – the former entails producing a set, but the latter entails enumerating the set only. Is it reasonable to require that preschool teachers have the knowledge to discern this difference? The answer might be yes. SCK states that teachers should develop a unique body of subject knowledge that goes beyond what other adults typically know (Ball et al., 2008). Take the depicted game as an example, any adult probably can tell that it is a counting game, but preschool teachers should go beyond that recognition and discern that it is a counting-out game that entails something harder than counting objects. Lack of this knowledge may result in teachers' insensitivity to children's errors. As evidenced in the study, the counting-out error was identified only by 9% of the participants.

Another informative finding is that the cardinality principle error was much less identified than the one-to-one correspondence error. The percentage of participants who correctly identified the cardinality principle error was 27%; when choices were provided, the percentage increased to 52%, but there were still 20% of the participants reporting “no difficulty is observed.” The reason may be that, compared with the one-to-one correspondence error that bears obvious physical violation, cardinality principle error demands trained sensitivity—for many adults the error does not indicate anything but that the child *chose* to count again.

The third interesting finding is that, compared with responses in SCK and KCS, participants' responses in KCT seem to be more satisfactory. For example, even though only nine participants identified the counting-out error (KCS), 21 participants proposed strategies that are helpful for correcting the counting-out error (KCT).

This third finding seems to make sense if we think about the sources of learning for preschool teachers: they can informally learn KCT – memories of how they were taught or observations of how other teachers taught; in comparison, SCK (e.g., the knowledge of cardinality principle) or KCS (e.g., the knowledge of child's number development) can be learned almost solely from formal learning – courses that are offered by the teacher preparation or professional development programs. This finding, however, does not indicate that KCT is independent of SCK or KCS. Without systematic and refined knowledge in SCK and KCS, teaching will less likely to hit the target. As evidenced in the present study, only 29% of the participants made the correct choice for helping Ben with his errors. Therefore, this third finding instead indicates that there is a great need to elevate preschool teachers' KCT to be more knowledge informed.

Lastly, the study reveals an important finding that U.S. preschool teachers lack consistent and precise math teaching vocabulary. One piece of evidence comes from the participants' description of Ben's errors. As Table 4 shows, many participants lacked the right terms describing his error/s even though they identified the error/s (e.g., “words to quantity [error]” for counting-out error). Another piece of evidence comes from the participants' responses to Q1 (what math skills are involved?). Note that, Q1 does not ask for opinion nor does it invite open interpretation—it is a straightforward task analysis question, but the participants reported as many as 24 different skills. We, however, must

acknowledge that, for the present study, 53% of the participants were pre-service teachers, and many of them had not taken any math pedagogy course; even for those in-service teachers, it is not clear whether they had taken any math pedagogy course before. Therefore, this finding should not be surprising but does indicate an education need for early childhood teachers.

Significance and Implications

This study serves to address a pressing need in the field: briefly speaking, the need to identify and measure the unique body of preschool teachers' math teaching knowledge. Across the U.S., the requirement of math teaching knowledge for preschool teachers varies widely from state to state. For example, according to the author's online search, among the 10 largest states by population (<https://www.moving.com/tips/the-10-largest-states-by-population/>), seven states (e.g., NY, CA) require that ECE teacher candidates pass a license test that has a math section; among the seven states that have the math section, some states (e.g., PA) have only math pedagogy for the section, while the other states (GA, NY) have both math and math pedagogy. This wide variation reflects the lack of consensus regarding what constitutes the math teaching knowledge for preschool teachers and whether that knowledge matters.

The present study fills some gaps in addressing that need. Specifically, the author defined the unique body of preschool teachers' math knowledge in counting and numbers by the structure of SCK, KCS, and KCT, and measured such knowledge by scenario-based questions. Important findings regarding preschool teachers' math teaching knowledge are obtained, for example, the majority of preschool teachers were not familiar with the count-out concept that is a differentiated concept from object counting, only a few preschool teachers identified all the three common counting errors, and preschool teachers' vocabulary in counting and number was substantially divergent.

Findings of the present study provide important reference for policymakers, for example, the study shows that some basic teaching knowledge such as identifying the cardinality principle error is lacking in many of the participants, which justifies the need to include those concepts in teacher licensure test. In addition, the study justifies that preschool teachers need intensive training in math teaching: their SCK or KCS needs to be differentiated, and KCT needs to be more knowledge-informed. A coursework that wholly focuses on math teaching rather than a combined course on math, science, and technology is instrumentally needed.

To further the research in this line, studies that focus on improving the present instrument, measuring its validity and reliability, or designing new instruments in other content areas such as geometry are greatly needed. With a reliable instrument for preschool teachers' math teaching knowledge, we can investigate the factors that affect this knowledge as well as the factors that this knowledge affects, and we can also measure the growth of preschool teachers' math teaching knowledge.

Limitation

The present study reveals important findings regarding preschool teachers' math teaching knowledge, but limitations also apply. First, even though the sample size is adequately large to obtain quantitative data, it is not enormous to draw conclusions with inferential statistics. Second, most in-service teachers in the present study were not certified and many were preschool (3-year olds) teachers, and therefore, may not represent the general ECE teacher workforces. However, because we took careful measures to ensure the validity and reliability of the instrument, the study should have adequate generalizability for the teacher population that resembles this study that includes in-service teachers who seek certification.

Disclosure Statement

No potential conflict of interest was reported by the author(s).

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Appendices

Appendix A. Key math teaching knowledge points in counting and numbers

	Specialized Content Knowledge	Knowledge of Content & Student	Knowledge of Content & Teaching
Oral	counting	a. Oral counting involves various forms: counting forward, counting backward, counting on from random number, and skip counting; b. Counting-on is more challenging than counting forward	a. Identify the levels of a child's oral counting: consistently count up to n ; counting backwards from n ; counting on from n ; skip counting by 2, 5, or 10; b. Identify a child's oral counting error/s (e.g., skip a counting word)
a. To promote oral	counting, 1) lead a child to count consistently to a higher number, or 2) engage the child in more advanced counting forms such as counting backward, counting on from a random number, or 3) skip counting by 2, 5, or 10		
Object	counting	c. Object counting entails three skills: recite number sequence, one-to-one correspondence, and keeping track; d. Cardinality principle: the last number word in counting represents the value of the set	c. Identify a child's object counting error/s: one-to-one correspondence error, keeping track error, or a combination of multiple errors; d. Identify signs of lack of understanding the cardinality principle (e.g., a child counts a set again to respond to "how many")
b. Help a child overcome	enumeration errors: using small sets, counting slowly, marking what was counted, or identifying the other people's counting error; c. Help a child observe/understand cardinality principle: using small sets, emphasize the last number word; d. Advance a child's enumeration by using large sets, discussing counting strategies, or using irregular arrangements, etc.		
Number recognition	e. Number recognition involves various forms: recognize a set with a number word or numeral (e.g., point to the card with 3 cars); Label a set with a number word or numeral (e.g., how many cars do you see?); Produce a set by a number word or numeral (e.g., please give me 3 cars);	e. Know that a child may use the same number word labeling different sets (e.g., "five" to a set of 5, 6, or 8 items); f. Know that producing a set is more difficult than labeling the set	d. Use different tasks to promote number recognition: point to n task, how many task, or give me n task

Appendix A is based on Baroody et al. (2006), Clements and Sarama (2014), and Sarama and Clements (2009).

Appendix B. Selected questions from scenario 1

Set I: Short Answer Questions from Scenario 1 Ms. Levy and the Card Game

1. For the card game depicted in Scenario 1, what mathematical skills are required/involved? Please list 1–3 math skills.
2. Ben had some difficulties when playing this card game. Please list 1–3 difficulties/errors that you noticed.
3. How would you help Ben in order for him to play the Card game more successfully?

Set II: Multi-choice Questions for Scenario 1 Ms. Levy and the Card Game

Please circle the choice/choices that best applies/apply.

1. Which math concepts/skills are required in order to play the card game depicted in Scenario 1?

