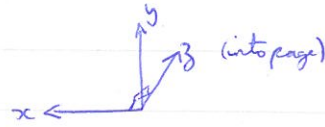


Plane (Transverse Electric and Magnetic) Waves

For sinusoidal time variations, frequency ω , a plane electromagnetic wave travelling in the +ve z -direction in a uniform medium, can have the electric field

$$E_x = E_0 e^{-jkz}, \quad E_0, k \text{ constant.}$$



This is a phasor*. The actual field is given by:-

*(complex)

$$\begin{aligned} \operatorname{Re} [E_x e^{j\omega t}] &= \operatorname{Re} [E_0 \exp \{j(\omega t - kz)\}] \\ &= |E_0| \cos (\omega t - kz + \phi) \end{aligned}$$

where $E_0 = |E_0| e^{j\phi}$ ← phase at reference point $z=0$
 $\uparrow$
 peak amplitude

Phase velocity from $\omega t - kz + \phi = \text{const}$

$$\text{as } \frac{dz}{dt} = \frac{\omega}{k}$$

; k is phase constant

In fact, wavelength λ is distance over which phase changes by 2π , and so $k = \frac{2\pi}{\lambda}$

Note: over some plane $z = \text{const}$, amplitude and phase are unchanging

Plane Wave



$$E_x = E_0 e^{-jkz}$$

can show that

$$H_y = \frac{E_0}{Z} e^{-jkz}$$

where Z and k are constant

$$Z = \sqrt{\frac{\mu}{\epsilon}} \quad k = \omega \sqrt{\mu\epsilon}$$

μ is permeability, ϵ is permittivity.

Note (a) $\frac{E_x}{H_y} = Z$ $\left\{ \begin{array}{l} \text{real; } E_x, H_y \text{ in phase} \\ \text{dimensions of } Z = \Omega \text{ Characteristic impedance} \\ \text{(plane-wave) (intrinsic) impedance of the medium.} \end{array} \right.$

for free space = 377Ω

$$Z = \frac{\text{transverse } E}{\text{transverse } H}$$

(b) $\underline{E}, \underline{H}, \underline{u}_z$ form a right-hand set (like $\underline{u}_x, \underline{u}_y, \underline{u}_z$)

(c) Form a second, independent solution

$$E_y = E_1 e^{-jkz}, \quad H_x = -\frac{E_1}{Z} e^{-jkz}$$



(d) Reflected wave e^{+jkz} is also possible,
and $\underline{E}$ (or $\underline{H}$) must reverse in sign

(e) Velocities of Propagation

phase vel^y $V_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$

group vel^y $V_g = \frac{d\omega}{dk} = \frac{1}{\sqrt{\mu\epsilon}}$

for $k = \omega \sqrt{\mu\epsilon}$
is for a lossless
medium, for a
plane wave there is
no dispersion

From Clarke + Brown, p. 239.

A.3 POWER FLOW IN A PLANE WAVE

If we return to the transmission line of Fig. A.1 we find that the power flow in the z -direction is $\frac{1}{2} \text{Re } VI^*$, provided that V and I are such that $|V|, |I|$ are peak values. Since the electric and magnetic fields are constant over the line cross-section, the power flux, that is, power flux per unit area, will be $\frac{1}{2} \text{Re } VI^* / (ab)$. Using equations (A.8) and (A.10), we find that the power flux is $\frac{1}{2} \text{Re } E_x H_y^*$ (W/m^2). The power flux is directed in the positive z -direction and the form of this result suggests the vector form

$$\underline{S} = \frac{1}{2} \text{Re } \underline{E} \times \underline{H}^* \quad (\text{A.15})$$

where $\underline{S}$, usually known as Poynting's vector, defines the direction of the power flux and its magnitude in W/m^2 . $\underline{E}, \underline{H}$ are vectors which represent the directions and magnitudes of the electric and magnetic fields. Note that $\underline{E}, \underline{H}$ are in general complex corresponding to voltage and current phasors. They therefore also contain phase information.

Equation (A.15) has only been derived for the special case of a plane electromagnetic wave. A fuller discussion starting from Maxwell's equations shows that $\underline{S}$ may be interpreted as a power flux for any form of electromagnetic wave with sinusoidal time dependence (see sections B.2.2 and B.2.5).

Recall from elementary circuit theory:

Power Flow in a Circuit

$$\text{Let } v = |V| \cos \omega t \quad (= \text{Re } V e^{j\omega t}; \quad V = |V| e^{j0})$$

$$i = |I| \cos (\omega t + \phi) \quad (= \text{Re } I e^{j\omega t}; \quad I = |I| e^{j\phi})$$

$$\text{Av. power} = \frac{1}{T} \int_T v i dt \quad (\text{integration over one cycle})$$

$$= \frac{1}{2} |V| |I| \cos \phi = \frac{1}{2} \text{Re } V I^*$$

Additional References

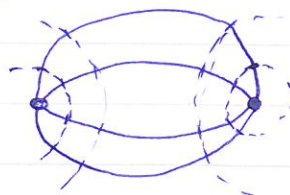
- F.E. Gardiol, "Introduction to Microwaves", Artech '84
 T.C. Edwards, "Foundations for Microstrip Circuit Design", Wiley, 1981.

(f) Power flow, averaged for sinusoidal fields,
 is $\underline{S} = \frac{1}{2} \operatorname{Re} \underline{E} \times \underline{H}^*$ (cf. $\operatorname{Re} \frac{1}{2} \underline{V} \underline{I}^*$)
 $= \frac{\underline{E}_0 \underline{E}_0^*}{2Z}$

$= \frac{|\underline{E}_0|^2}{2Z}$ for $\underline{E}_z = 0$.

(Inst power flow = $\underline{V} \underline{I}$,
 Av. = $\frac{1}{2} \operatorname{Re} \underline{V} \underline{I}^*$)

Arises in practice: locally in two-wire transmission lines



+ in ~~coax~~ coaxial lines



+ in microstrip

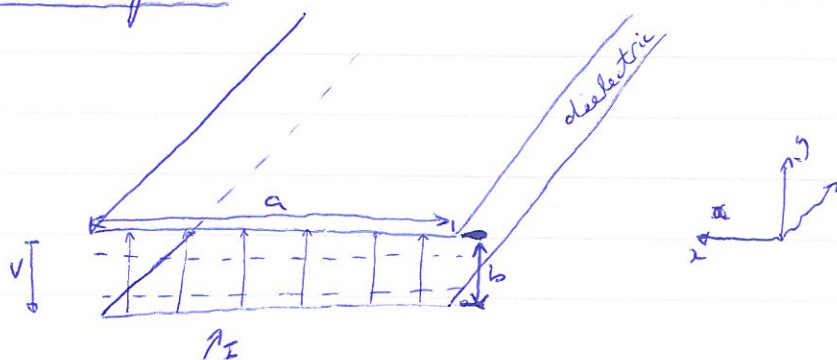


+ in far-field radiation

Arises in theory as plane-wave composition of more complicated fields.

Parallel-Plate TEM Waveguide

Two perfectly conducting plates, assume no fringing.



∴ have plane wave

$$E_y = E_1 e^{-jkz}$$

By definition

$$V = b E_y = b E_1 e^{-jkz}$$

By Ampere

$$I = a H_x = a \frac{E_1}{Z} e^{-jkz}$$

$$Z_c = \frac{V}{I} = \frac{b}{a} Z$$

$$\beta = k$$

$$k = \omega \sqrt{\mu \epsilon}$$

(Characteristic transmission line impedance = $Z_c (= \frac{V}{I} = \frac{b}{a} Z$ as above))

Mistaken
to be 1
for microstrip

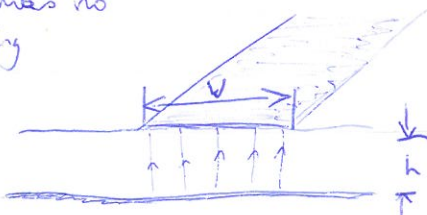
$$Z_c = \frac{V}{I} = \frac{b}{a} Z = \frac{b}{a} \frac{377}{\sqrt{\epsilon_r}}$$

$$\text{Phase constant } \beta = k = \frac{2\pi}{\lambda_g} \Rightarrow \lambda_g = \frac{\lambda_0}{\sqrt{\epsilon_r}}$$

$$(k_0 = \omega \sqrt{\mu_0 \epsilon_0})$$

Model for Microstrip

assumes no
fringing



really fringes
quite drastically

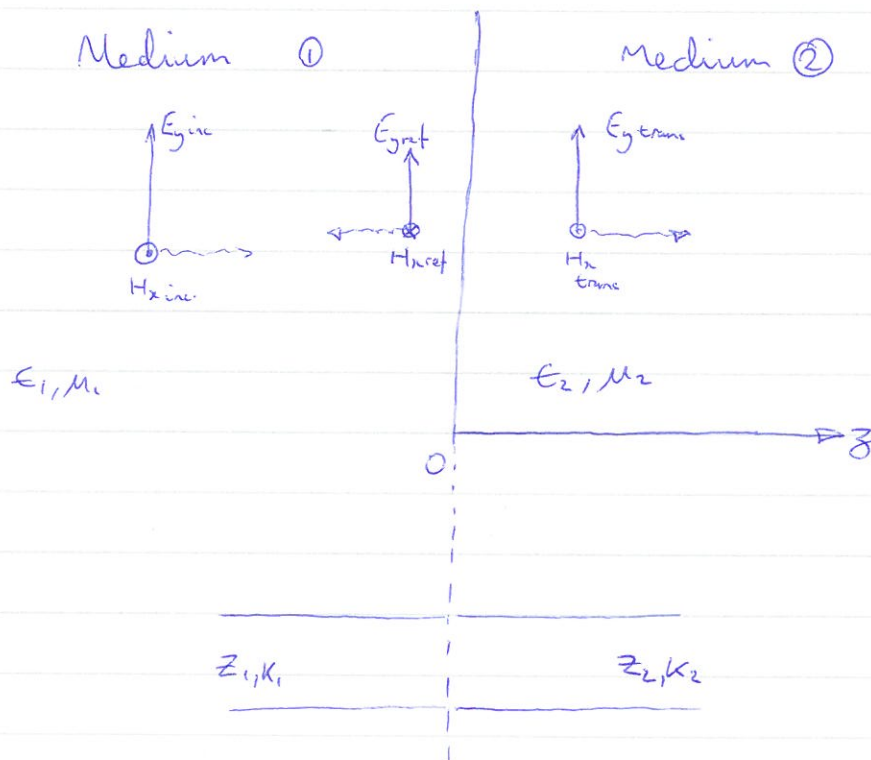


May need to use very complex analysis
for total accuracy

phase constant may vary by 10%. Characteristic impedance may vary by 50%.

Transmission Line Analogy

Plane wave incident normally on a plane boundary between two dielectrics:-



E and H are formally the same as V and I on a transmission line, with $Z_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$, $K_1 = \omega \sqrt{\mu_1 \epsilon_1}$, etc

At the boundary, tangential components of E and H must be the same on either side. Hence equivalent transmission lines can be joined directly.

Reflection Coefficient

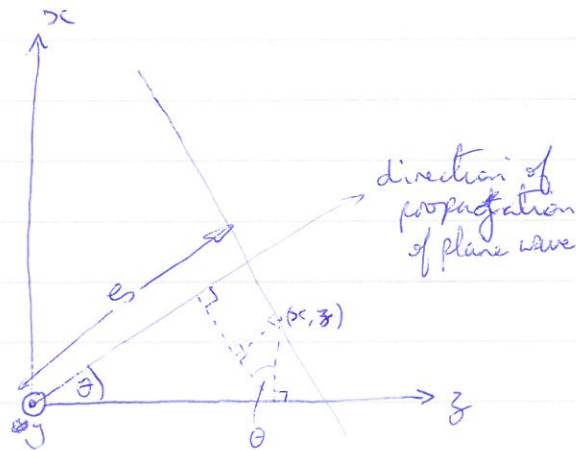
$$\rho = \frac{E_{yref}}{E_{yinc}} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

Transmission Coefficient

$$T = \frac{2Z_2}{Z_1 + Z_2} = \frac{E_{y \text{ trans.}}}{E_{y \text{ inc.}}}$$

Interference of Two Inclined Plane Waves

Take Electric field transverse to plane of propagation



But $\mathcal{S} = x \sin \theta + z \cos \theta$
and the electric field due to single inclined plane wave
at pt (x, z) is

$$E_y(x, z) = E_1 \exp\{-jk(x \sin \theta + z \cos \theta)\}$$

Remaining magnetic field component



$$H = \frac{E_y}{z}$$

$$H_x(x, z) = \frac{-E_1 \cos \theta}{z} \exp\{-jk(\mathcal{S})\}$$

$$H_{y0}(x, z) = \frac{E_1 \sin \theta}{z} \exp \{-jk(\beta)\}$$

Consider a second plane wave, similar to the first, but travelling at $-\theta$ to the z -axis. Then, substituting $-\theta$ for θ , the fields at (x, z) due to this plane wave are

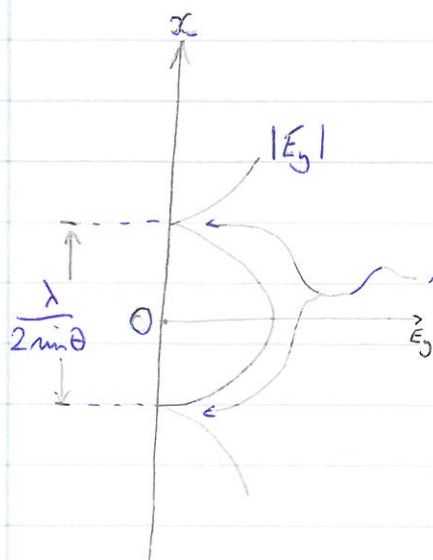
$$E_y(x, z) = E_1 \exp \{-jk(-x \sin \theta + z \cos \theta)\} \quad \text{etc}$$

The total electric field at (x, z) is

$$\begin{aligned} E_y(x, z) &= E_1 \left[e^{-jkx \sin \theta} + e^{+jkx \sin \theta} \right] e^{-jkz \cos \theta} \\ &= 2E_1 \cos(kx \sin \theta) e^{-jkz \cos \theta} \end{aligned}$$

sinusoidal variation
of amplitude across
wavefront

propagating in z -direction,
 $\beta = k \cos \theta$



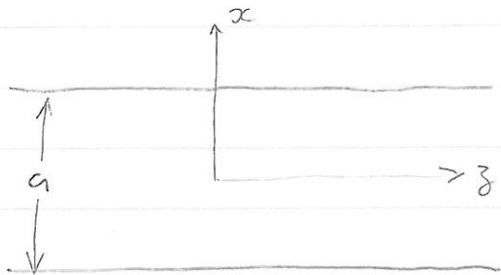
first zeroes occur at

$$kx \sin \theta = \pm \frac{\pi}{2} \quad ; \quad k = \frac{2\pi}{\lambda}$$

$$\text{ie at } x = \pm \frac{\lambda}{4 \sin \theta}$$

Parallel-Plate Waveguides (T.E. Mode)

Place two parallel perfectly conducting plates at the zeros,



then $a = \frac{\lambda}{2 \sin \theta}$

$$\text{or } \sin \theta = \frac{\lambda}{2a} ; \quad \cos \theta = \sqrt{1 - \frac{\lambda^2}{4a^2}}$$

Note: Fields need exist only between plates

: λ determined by frequency of source

: Angle θ determined by geometry, i.e., by a

$$k = \frac{2\pi}{\lambda} \\ = \omega \sqrt{\mu \epsilon'}$$

Cut Off

If $\lambda > 2a$, $\sin \theta > 1$, θ is not real.

When $\lambda = 2a = \lambda_c$, cut-off wavelength, which corresponds to the cut-off frequency $f_c = \frac{c}{2a}$.

c = Vel^y of
EM wave
in dielectric

When $f < f_c$, i.e., $\lambda > \lambda_c (= 2a)$

$$\cos \theta = \pm \sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}$$

$$= \pm j \sqrt{\left(\frac{\lambda}{2a}\right)^2 - 1}$$

pure imaginary

giving z dependence

$$e^{-j\beta z} = \exp\{-jKz\cos\theta\} = \exp\left\{-Kz\sqrt{\left(\frac{\lambda}{2a}\right)^2 - 1}\right\}$$

i.e., a negative exponential decay in the z -direction.
Wave is evanescent i.e., vanishing.

Note: This wave does not propagate; decays in the order of wavelengths in the z -direction

Modes

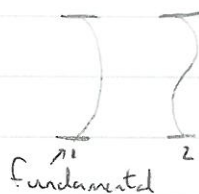
A field pattern with the minimum variation across the guide is called the fundamental or dominant mode.

For T.E. mode, this is one half sinusoid.

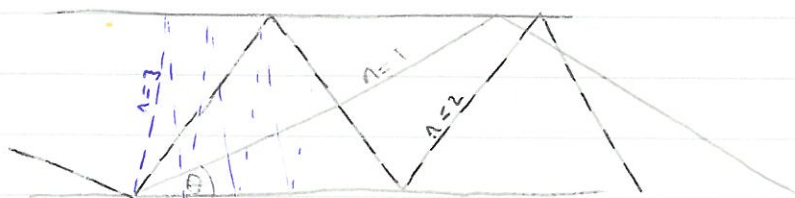
But boundary conditions can be satisfied for integral multiples of half sinusoids.

$$\text{Thus } \frac{n\lambda}{2\sin\theta} = a$$

for $n = 1, 2, 3, 4, \dots$
 higher order modes



$$\text{i.e. } \sin\theta = \frac{n\lambda}{2a} \quad \text{and} \quad \cos\theta = \sqrt{1 - \left(\frac{n\lambda}{2a}\right)^2}$$



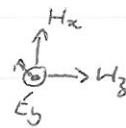
Mode is cut-off when
 $n > \frac{2a}{\lambda}$

Note: Only fundamental mode propagates if $a < \lambda < 2a$
 $\underbrace{a < \lambda}_{n=2 \text{ mode cut off}} < \underbrace{\lambda < 2a}_{n=1 \text{ mode not cut off}}$

Magnetic field components

$$H_x(x, z) = -\frac{2E_1}{Z} \cos \theta \cos(kx \sin \theta) e^{-jkz \cos \theta}$$

$$H_z(x, z) = -j \frac{2E_1}{Z} \sin \theta \sin(kx \sin \theta) e^{-jkz \cos \theta}$$



Wave impedance for TE mode

$$Z_{TE} = \frac{E_{trans}}{H_{trans}} = \frac{E_y}{-H_x} = \frac{Z}{\cos \theta} = \frac{Z}{\sqrt{1 - \frac{\lambda^2}{4a^2}}}$$

for
fundamen-
tal mode.
(etc.)

Mode is cut off when n is such that
 $n > \frac{2a}{\lambda}$

Note only fundamental mode propagates if λ lies between a and $2a$.
 $a < \lambda < 2a$.

$n=2$ mode
cut off.

Fundamental ($n=1$)
is not cut off.

Magnetic field components.

$$H_x(x, z) = -\frac{2E_1}{Z} \cos \theta \cos(k_x \sin \theta) e^{-jk_z z}$$

$$H_z(x, z) = -j \frac{2E_1}{Z} \sin \theta \sin(k_x \sin \theta) e^{-jk_z z}$$

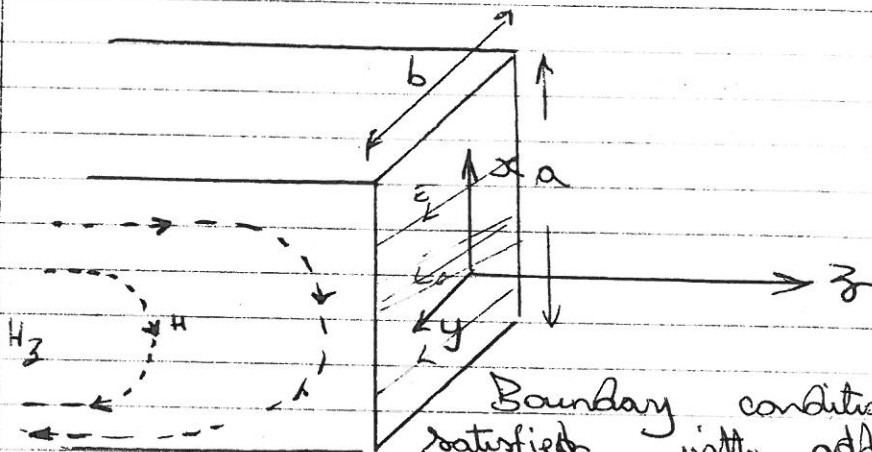
Wave impedance for TE mode

$$Z_{TE} = \frac{E_{\text{Transverse}}}{H_{\text{Transverse}}} = \frac{E_y}{H_x} = \frac{Z}{\cos \theta} = \frac{Z}{\sqrt{1 - \frac{1^2}{4a^2}}}$$

for the fundamental mode.

8/10/84.

RECTANGULAR WAVE GUIDE.



Boundary conditions are satisfied with addition of two further conducting parallel planes giving an enclosed waveguide.

Fields only penetrate to skin depth. Hence isolation.

Design with $b < a$ so that structure only supports y directed electric field. Then for fundamental mode

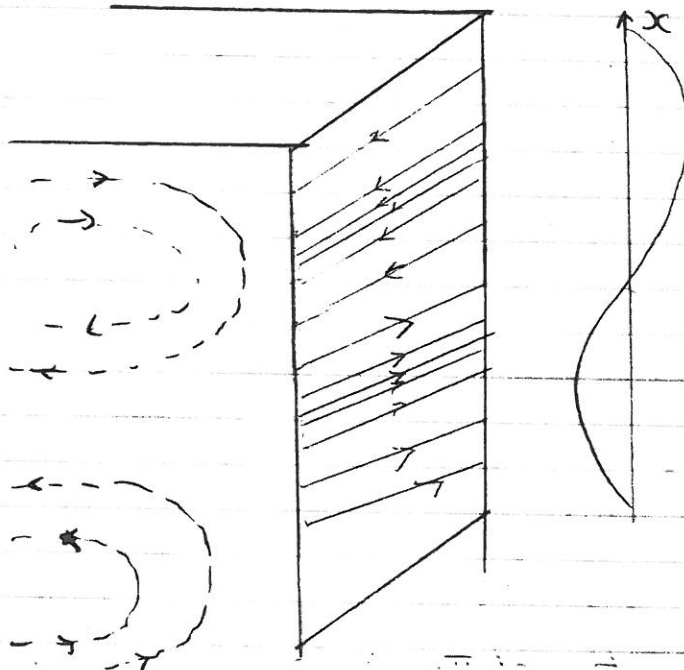
$$a < \lambda < 2a$$

Fundamental only propagate.

So that orthogonal fundamental mode is cut off.

MODE DESIGNATION:

TE_{10} ← 3rd half sinusoidal variation in y direction
 Electric field wholly transverse
 Higher order modes
 One half sinusoidal variation of E field in x direction for example TE_{20} .



In general TE_{mn} modes exist.
 Also TM_{mn}

But continue with TE_{10}
 Propagation constant

$$\beta = k \cos \theta = k \sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2} = k \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

where $\lambda_c = 2a$ and $k = \omega \sqrt{\mu \epsilon}$

Phase velocity of TE_{10} mode

$$V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu \epsilon} \sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2}}$$

$$V_p = \frac{c}{\sqrt{1 - \frac{\lambda^2}{\lambda_c^2}}} > c$$

$$V_g = \frac{d\omega}{d\beta} = c \sqrt{1 - \frac{\lambda^2}{\lambda_c^2}} < c$$

Guide Wavelength λ_g given by

$$\beta = \frac{2\pi}{\lambda_g} = \frac{2\pi}{\lambda} \sqrt{1 - \frac{\lambda^2}{\lambda_c^2}}$$

$$\rightarrow \frac{1}{\lambda_g^2} = \frac{1}{\lambda^2} - \frac{1}{\lambda_c^2} \quad ; \quad \lambda_c = 2a$$

$$\lambda = \frac{c}{f}$$

Characteristic impedance

$$Z_{TE} = \frac{E_y}{-H_x} = \frac{Z}{\sqrt{1 - \frac{\lambda^2}{\lambda_c^2}}}$$

Remember transmission analogy.

$$\begin{matrix} E_y & \longleftrightarrow & V \\ H_x & \longleftrightarrow & I \end{matrix}$$

Fields in Rect. TE₀ Waveguide

Putting $\sin \theta = \frac{1}{2a}$ and $\beta = k \cos \theta = k \sqrt{1 - \left(\frac{\lambda}{2a}\right)^2}$

$$E_0 = 2E_1$$

$$\begin{aligned} E_y(x, y, z) &= E_0 \cos\left(\frac{\pi x}{a}\right) e^{-j\beta z} \\ H_x(x, y, z) &= -\frac{E_0}{Z_{TE}} \cos\left(\frac{\pi x}{a}\right) e^{-j\beta z} \\ H_z(x, y, z) &= -j \frac{E_0}{Z} \frac{\lambda}{2a} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \end{aligned} \quad \left. \begin{array}{l} \text{No} \\ y \text{ variation} \\ \text{of fields} \\ \text{within} \\ a \times b. \end{array} \right\}$$

Power Flow

Average Poynting Vector $\underline{S} = \frac{1}{2} \text{Re}(\underline{E} \times \underline{H}^*)$

gives $P = \frac{1}{2} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} (-E_y \cdot H_x^*) dx dy$

total power flowing down guide.

$$= \frac{|E_0|^2}{2 Z_{TE}} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \cos^2\left(\frac{\pi x}{a}\right) dx dy = \frac{|E_0|^2 ab}{4 Z_{TE}}$$

Losses - Attenuation

Finite conductivity of materials gives ohmic losses. (No loss in perfect insulator ($\sigma=0$) and perfect conductor ($\sigma=\infty$)).

Propagation in the z-direction

$$\exp(-j\beta z) \exp[-\alpha z] = e^{-\gamma z}$$

where $\gamma = \alpha + j\beta$ is the propagation constant and is complex.

β phase constant rad/m
 α attenuation constant nepers/m

Principle

Allowance can be made for finite σ by taking the ideal solution ($\sigma=0$) and replacing ϵ by the effective (complex) permittivity

$$\epsilon_{\text{eff}} = \epsilon + \frac{\sigma}{j\omega}$$

$$= \epsilon \left(1 + \frac{\sigma}{j\omega\epsilon} \right) \quad \text{ratio of conduction current to displacement current}$$

$$= \epsilon_0 (\epsilon' - j\epsilon'')$$

relative permittivity is complex

and $\frac{\epsilon''}{\epsilon'}$ is known as the loss tangent *

Example of a Plane Wave :

a) in an imperfect dielectric (σ is small)

* gives a measure of the 'quality' of the dielectric

lossless solution: e^{-jkz} ; $k = \omega \sqrt{\mu \epsilon}$

replace: k by $\omega \sqrt{\mu \epsilon_{\text{eff}}}$

and: identify γ with $j k$

$$\gamma = j \omega \sqrt{\mu \epsilon \left(1 + \frac{\sigma}{j \omega \epsilon}\right)} \approx j \omega \sqrt{\mu \epsilon} \left[1 + \frac{\sigma}{j 2 \omega \epsilon}\right] \quad \text{binomial expansion}$$
$$= \alpha + j \beta$$

then

$$\beta \approx k \quad \text{and} \quad \alpha \approx \frac{k \sigma}{2 \omega \epsilon}$$

(b) in a good conductor (σ large)

$$\epsilon_{\text{eff}} = \epsilon \left(1 + \frac{\sigma}{j \omega \epsilon}\right) \approx \frac{\sigma}{j \omega}$$

$$\gamma = j \omega \sqrt{\mu \epsilon_{\text{eff}}} \approx j \omega \sqrt{\frac{\mu \sigma}{j \omega}} = \sqrt{j \omega \mu \sigma}$$

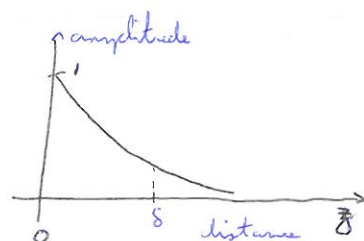
$$\text{but } \sqrt{j} = \frac{1+j}{\sqrt{2}}, \quad \text{hence}$$

$$\gamma = \alpha + j \beta = (1+j) \sqrt{\pi f \mu \sigma}$$

$$\omega = 2\pi f$$

Wave attenuates as: $e^{-\alpha z} = \exp\{-\sqrt{\pi f \mu \sigma} z\}$

$$\text{this} = \frac{1}{e} \quad \text{when} \quad z = \frac{1}{\sqrt{\pi f \mu \sigma}} = \delta$$

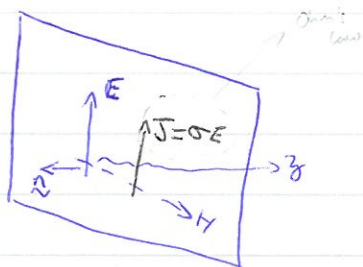
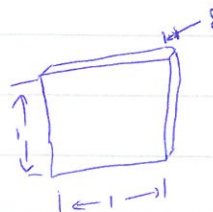


Wave has reduced in amplitude to e^{-1} in the "skin depth", $\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$

Wave impedance (plane wave) in (metal) a good conductor

$$Z_m = \sqrt{\frac{\mu}{\epsilon_{eff}}} \approx \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j) \sqrt{\frac{\pi f \mu \sigma}{\sigma}} \\ = (1+j) \frac{1}{\sigma \delta} = R_m + jX_m$$

Note that $R_m = \frac{1}{\sigma \delta}$ is the d.c. resistance of a metre square of metal of thickness δ .

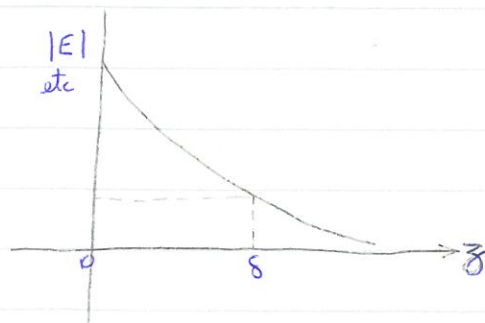


$\underline{E}$, $\underline{J}$ and $\underline{H}$ all propagate as

$$\exp\left\{-\frac{1+j}{\delta} z\right\}$$

Wave "skin depth"

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$



As $\sigma \rightarrow \infty$, (ie the conductor becomes perfect)
 $\delta \rightarrow 0$

Field in the metal $\rightarrow 0$, and the current is confined to the surface, ie $\underline{J}_s$ where

$$\underline{J}_s = \int_0^\infty \underline{J} dz$$

If tangential magnetic field just outside conductor is $\underline{H}_t$, then Ampere's law gives

$$\underline{J}_s = \Omega \underline{H}_t$$

$$\underline{J}_s = \Omega \underline{H}_t$$

Loss in conductor, given by Poynting as

ie $\frac{1}{2} \text{Re} [\underline{H}_t \cdot \underline{H}_t^* \Omega]$

$$W_L = \frac{1}{2} \text{Re} [|\underline{H}_t|^2 Z_m]$$

$$= \frac{1}{2} |\underline{H}_t|^2 R_m$$

$$= \frac{1}{2} |\underline{J}_s|^2 R_m$$

$$\text{where } R_m = \frac{1}{\sigma \delta}$$

(maybe written as R_s
 to accentuate
 is surface.)

Ω per square
 or Ω/Ω

Attenuation in a TE₁₀ Waveguide

Due to the dielectric (assume perfectly conducting walls)

$$\text{Lossless case: } e^{-j\beta z} ; \beta^2 = k^2 - \frac{\pi^2}{a^2} ; k = \omega \sqrt{\mu \epsilon}$$

$$\text{replace } \epsilon \text{ by } \epsilon_{\text{eff}} = \epsilon \left(1 + \frac{\sigma}{j\omega \epsilon} \right)$$

make equivalence of γ with $j\beta$

$$\beta' \approx \beta, \quad \alpha \approx \frac{k^2}{\beta} \frac{\sigma}{2\omega f}$$

$$\gamma = \alpha + j\beta'$$

Due to metal walls (assume perfect dielectric filling)

Use the lossless relation in the "perturbation formula"

$$\alpha = -\frac{1}{2P} \frac{dP}{dz}$$

$$\text{from } P = P_0 e^{-2\alpha z}$$

$$\frac{dP}{dz} = -2\alpha P_0 e^{-2\alpha z} = -2\alpha P$$

$-\frac{dP}{dz}$ is the rate of loss
of power with distance.

$$\therefore \alpha = -\frac{1}{2P} \frac{dP}{dz}$$

hence



Calculate the loss
in a strip Δz from
 $\frac{1}{2} |H_t|^2 R_m$