



① ~~$F = K \delta L$~~ HOOKE'S LAW

② $E = \frac{\text{STRESS}}{\text{STRAIN}}$

$$E = \frac{F}{A} \div \frac{\delta L}{L}$$

$$\therefore E = \frac{FL}{A \delta L}$$

$$\therefore \delta L = \frac{FL}{AE}$$

③ FROM ①
HOOKE'S LAW

$$F = K \delta L$$

$$\therefore K = \frac{F}{\delta L}$$

$$= \frac{FAE}{FL}$$

$$\therefore K = \frac{AE}{L}$$

④ USING SIGN CONVENTION THAT FORCES & DISPLACEMENTS IN +ve x ARE POSITIVE, THEN

$$F_1 = K_1(u_1 - u_2) = K_1 u_1 - K_1 u_2 \quad (a)$$

$$F_2 = K_1(u_2 - u_1) = -K_1 u_1 + K_1 u_2 \quad (b)$$

WHERE $K = \frac{AE}{L}$

5) WE MAY WRITE (a) & (b) IN MATRIX FORM

(2)

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

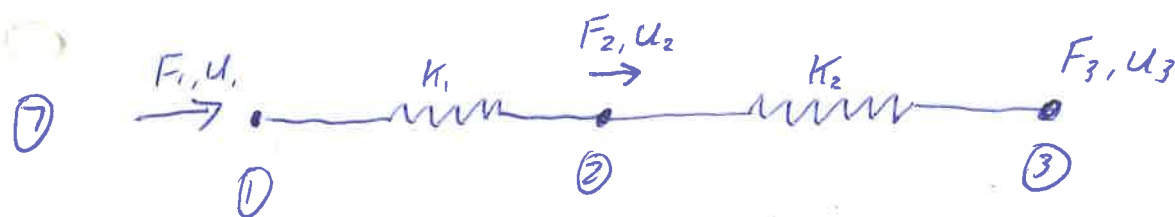
$$F_1 = K_1 u_1 - K_1 u_2$$

$$F_2 = -K_1 u_1 + K_1 u_2$$

6) IN SHORTHAND THIS MAY BE WRITTEN AS,

$$\{F\} = [K^e] \{u\}$$

$[K^e]$ IS THE STIFFNESS MATRIX AND AS CAN BE SEEN IT IS SYMMETRICAL *



* * LINEAR SOLVERS WORK OUT STIFFNESS MATRIX FIRST
- MULTIPLE SUBCASES MORE EFFICIENT
- 'NEGATIVE PIVOT RATIOS' - ZEROS IN THE STIFFNESS MATRIX - NOT CONSTRAINED PROPERLY!
*

SPRING 1
$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

SPRING 2
$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

EXPAND TO EQUIVALENT FORM:

SPRING 1
$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

SPRING 2
$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

ENTIRE
SYSTEM

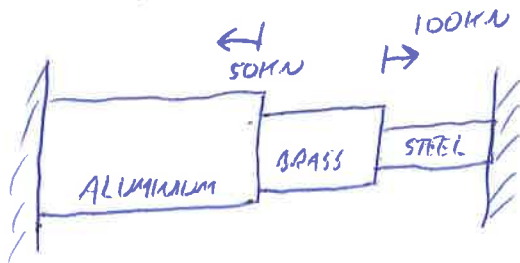
$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1 + K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \{F\} = [K^e] \{u\}$$

↑
STILL SYMMETRICAL

I FEM (3)

①



AL: $A = 400 \text{ mm}^2$
 $L = 280 \text{ mm}$
 $E = 70 \text{ GPa}$

$\rho = 2800 \text{ kg/m}^3$

BRASS: $A = 200 \text{ mm}^2$

$L = 100 \text{ mm}$

$E = 100 \text{ GPa}$

$\rho = 8250 \text{ kg/m}^3$

STEEL: $A = 70 \text{ mm}^2$

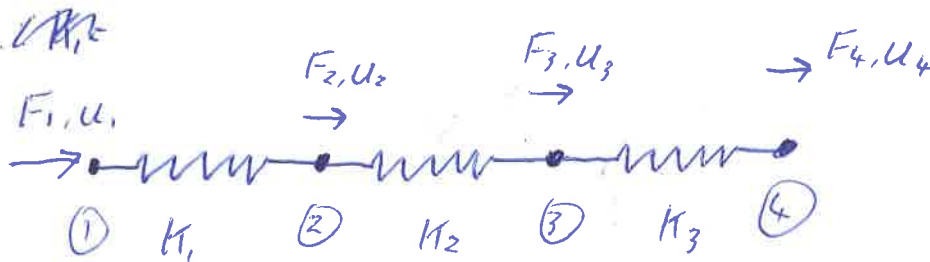
$L = 100 \text{ mm}$

$E = 200 \text{ GPa}$

$\rho = 7700 \text{ kg/m}^3$

~~WE ARE~~

$\rho = AL =$



$K = \frac{AE}{L} = \text{STIFFNESS OF EACH ELEMENT}$

NOTE WE ARE WORKING IN kN & mm!
 ↓

$K_1 = \frac{400 \times 10^{-6} \times 70 \times 10^9}{280 \times 10^{-3}} = 100 \times 10^6 \text{ N/m} = 100 \text{ kN/mm}$

$K_2 = \frac{200 \times 10^{-6} \times 100 \times 10^9}{100 \times 10^{-3}} = 200 \times 10^6 \text{ N/m} = 200 \text{ kN/mm}$

$K_3 = \frac{70 \times 10^{-6} \times 200 \times 10^9}{100 \times 10^{-3}} = 140 \times 10^6 \text{ N/m} = 140 \text{ kN/mm}$

SPRING ONE

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$F = Kx$$

SPRING TWO

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

SPRING THREE

$$\begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\textcircled{1} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\textcircled{2} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 200 & -200 & 0 \\ 0 & -200 & 200 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\textcircled{3} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 140 & -140 \\ 0 & 0 & -140 & 140 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

STIFFNESS MATRIX

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & (100+200) & -200 & 0 \\ 0 & -200 & (200+140) & -140 \\ 0 & 0 & -140 & 140 \end{bmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}$$

$$F_1 = 100 U_1 - 100 U_2 \quad (1)$$

$$F_2 = -100 U_1 + 300 U_2 - 200 U_3 \quad (2)$$

$$F_3 = -200 U_2 + 340 U_3 - 140 U_4 \quad (3)$$

$$F_4 = -140 U_3 + 140 U_4 \quad (4)$$

BOUNDARY CONDITIONS

ONCE STIFFNESS MATRIX IS FOUND BOUNDARY CONDITIONS ARE APPLIED AS PER CASE)

$$U_1 = 0 \quad \therefore \text{FROM (1)} \quad F_1 = -100 U_2 \quad (4a)$$

$$U_4 = 0 \quad \therefore \text{FROM (4)} \quad F_4 = -140 U_3 \quad (4b)$$

$$F_2 = -50 \text{ kN} \quad \therefore \text{FROM (2)}$$

$$-50 = 0 + 300 U_2 - 200 U_3 \quad (5)$$

$$F_3 = 100 \text{ kN} \quad \therefore \text{FROM (3)}$$

$$100 = -200 U_2 + 340 U_3 + 0 \quad (6)$$

SOLVING (5) & (6) SIMULTANEOUSLY

FROM (5)

(x2)

$$-100 = 600 U_2 - 400 U_3 \quad (7)$$

FROM (6)

(x3)

$$300 = -600 U_2 + 1020 U_3 \quad (8)$$

I FEM (6)

(7)

AMANG
⑦⑧⑧

$$200 = 620 U_3$$

$$\therefore U_3 = \underline{\underline{0.323 \text{ mm}}}$$

From (5) $-50 = 300 U_2 - 200 U_3$

$$\therefore U_2 = \frac{-50 + 200 U_3}{300}$$

$$= \frac{-50 + 200 \times 0.323}{300}$$

$$= \underline{\underline{0.049 \text{ mm}}}$$

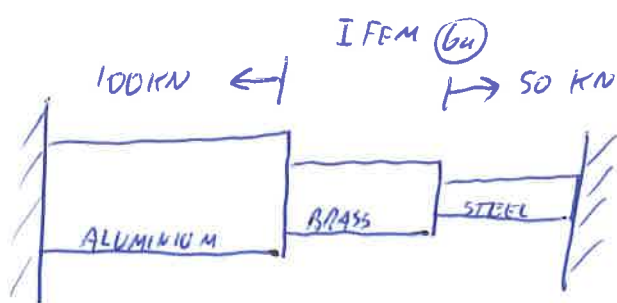
From (4a) $F_1 = -100 U_2$
 $= \underline{\underline{-4.9 \text{ kN}}}$

From (4b) $F_4 = -140 U_3$
 $= \underline{\underline{-45.22 \text{ kN}}}$

NEGATIVE
BECAUSE THEY
ACT TO THE
LEFT.

-50 kN

(2)



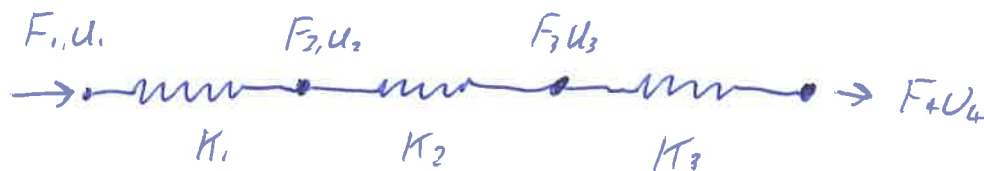
① WITH REVERSE FORCES

(8)

$$\begin{aligned} \text{AL} : A &= 400 \text{ mm}^2 \\ L &= 280 \text{ mm} \\ E &= 70 \text{ GN/m}^2 \end{aligned}$$

$$\begin{aligned} \text{BRASS} : A &= 200 \text{ mm}^2 \\ L &= 100 \text{ mm} \\ E &= 100 \text{ GN/m}^2 \end{aligned}$$

$$\begin{aligned} \text{STEEL} : A &= 70 \text{ mm}^2 \\ L &= 100 \text{ mm} \\ E &= 200 \text{ GN/m}^2 \end{aligned}$$



$$K = \frac{AE}{L}$$

~~$K_1 = 100 \text{ kN/mm}$~~
 ~~$K_2 = 200 \text{ kN/mm}$~~
 ~~$K_3 = 140 \text{ kN/mm}$~~

STIFFNESS MATRIX IS THE SAME!

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 340 & -140 \\ 0 & 0 & -140 & 140 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}$$

$$F_1 = 100U_1 - 100U_2 \quad (1)$$

$$F_2 = -100U_1 + 300U_2 - 200U_3 \quad (2)$$

$$F_3 = -200U_2 + 340U_3 - 140U_4 \quad (3)$$

$$F_4 = -140U_3 + 140U_4 \quad (4)$$

NEW BOUNDARY CONDITIONS:

↓ FEM (6b)

(9)

$$U_1 = 0 \quad \therefore \text{FROM (1)} \quad F_1 = -100U_2 \quad (4a)$$

$$U_4 = 0 \quad \therefore \text{FROM (4)} \quad F_4 = -140U_3 \quad (4b)$$

$$F_2 = -100 \text{ kN} \quad \therefore \text{FROM (2)} \quad -100 = 0 + 300U_2 - 200U_3 \quad (5)$$

$$F_3 = 50 \text{ kN} \quad \therefore \text{FROM (3)} \quad 50 = -200U_2 + 340U_3 + 0 \quad (6)$$

$$\text{FROM (5)} \quad -200 = 600U_2 - 400U_3 \quad (7)$$

$$\text{FROM (6)} \quad 150 = -600U_2 + 1020U_3 \quad (8)$$

$$\text{ADDING (7) \& (8)} \quad -50 = 620U_3$$
$$\therefore U_3 = \underline{\underline{-0.081 \text{ mm}}}$$

$$\text{FROM (5)} \quad -100 = 300U_2 - 200U_3$$
$$U_2 = \frac{-100 + 200U_3}{300}$$
$$= \frac{-100 + 200 \times -0.081}{300}$$

$$U_2 = \underline{\underline{-0.387 \text{ mm}}}$$

$$\text{FROM (4a)} \quad F_1 = -100 \times U_2$$
$$= \underline{\underline{38.7 \text{ kN}}}$$

$$(4b) \quad F_4 = -140U_3$$
$$= \underline{\underline{11.34 \text{ kN}}}$$

$$\underline{\underline{50 \text{ kN}}}$$

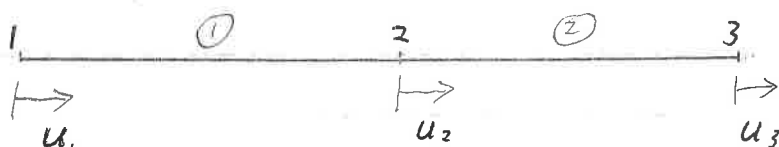
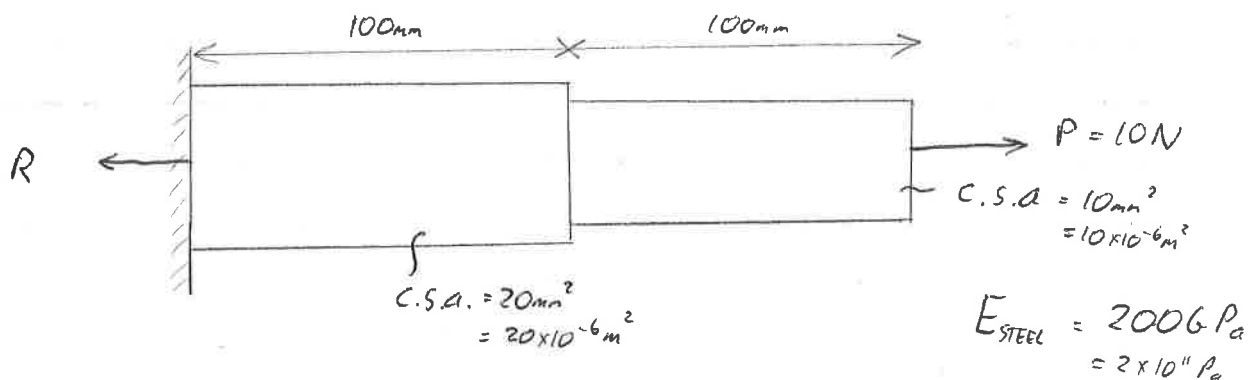
(3)

EXAMPLE

$$u_2 = ? \quad \epsilon_1 = ?$$

$$u_3 = ? \quad \sigma_1 = ?$$

(10)



For element 1 : $K^e = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \frac{(20 \times 10^{-6})(2 \times 10^{11})}{100 \times 10^{-3}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \times 10^7 \text{ N/m}$$

$$K^{e1} = \begin{matrix} u_1 & u_2 \\ u_2 & \end{matrix} \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \times 10^7 \text{ N/m}$$

For element 2 : $K^e = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \frac{(10 \times 10^{-6})(2 \times 10^{11})}{100 \times 10^{-3}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K^{e2} = \begin{matrix} u_2 & u_3 \\ u_3 & \end{matrix} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \times 10^7 \text{ N/m}$$

$$\therefore [K] = \begin{bmatrix} u_1 & u_2 & u_3 \\ 4 & -4 & 0 \\ -4 & 4+2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{matrix} u_1 \\ u_2 \times 10^7 \text{ N/m} \\ u_3 \end{matrix}$$

$$= \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \times 10^7 \text{ N/m}$$

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 10 \end{Bmatrix}$$

$$\{F\} = [K]\{U\}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 10 \end{Bmatrix} = \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \times 10^7 \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ 0 \\ 10 \end{Bmatrix} = 10^7 \begin{bmatrix} 4 & -4 & 0 \\ -4 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \text{APPLYING B.C.'s}$$

$$\textcircled{1} \quad -4u_2 \times 10^7 = R_1$$

$$\textcircled{2} \quad (6u_2 - 2u_3) \times 10^7 = 0$$

$$6u_2 = 2u_3$$

$$u_2 = \frac{1}{3}u_3$$

$$\textcircled{3} \quad (-2u_2 + 2u_3) \times 10^7 = 10$$

$$\text{From } \textcircled{3} \quad -2u_2 \times 10^7 + 2u_3 \times 10^7 = 10$$

$$-\frac{2}{3}u_3 + 2u_3 = 10 \times 10^7$$

$$\frac{10}{3}u_3 = 10 \times 10^7$$

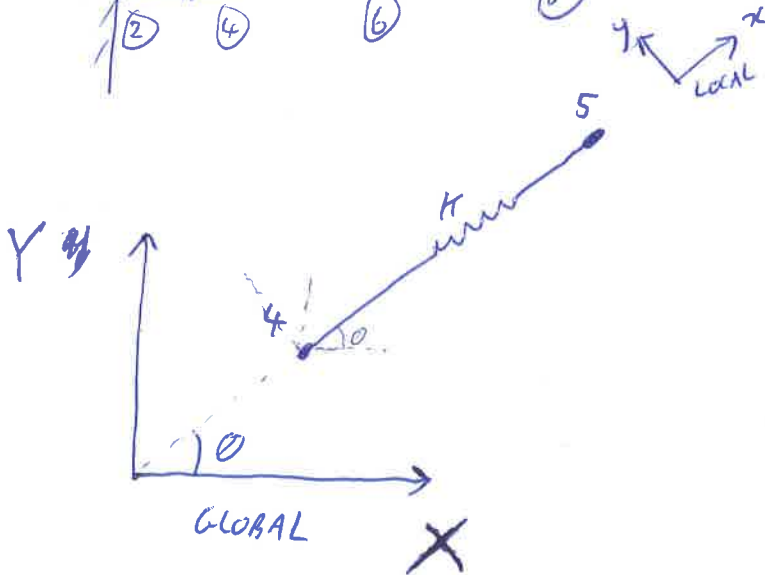
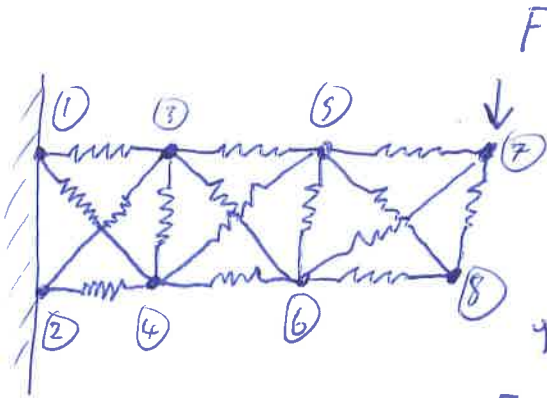
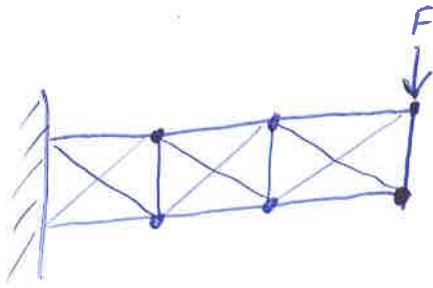
$$u_3 = 7.5 \times 10^{-7} \text{ m}$$

$$u_2 = 2.5 \times 10^{-7} \text{ m}$$

$$R = -10 \text{ N}$$

$$E_0 = \frac{-u_1 + u_2}{L} = \frac{-0 + 2.5 \times 10^{-7}}{100 \times 10^{-3}} = 2.5 \times 10^{-6}$$

$$\sigma_1 = E E_0 = 2.5 \times 10^{-6} \times 2 \times 10^{11} = 500 \times 10^3 \text{ N/m}^2$$



FOR A SIMPLE SPRING IN GLOBAL COORDINATES LOADED IN ON X:

$$F = U$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & +K_1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$$

IN A PIN JOINTED STRUCTURE F_y FORCES WILL BE ZERO

(K_{ij}) : STIFFNESS MATRIX FOR ELEMENT IN LOCAL COORDINATES

$$\begin{Bmatrix} F_{x4} \\ F_{y4} \\ F_{x5} \\ F_{y5} \end{Bmatrix} = \begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta x_4 \\ \delta y_4 \\ \delta x_5 \\ \delta y_5 \end{Bmatrix}$$

← LOCAL COORDINATE SYSTEM

TO CONVERT TO GLOBAL AXIS

$$F_{x4} = F_{x4} \cos \theta + F_{y4} \sin \theta$$

(SEE ←)

$$F_{y4} = -F_{x4} \sin \theta + F_{y4} \cos \theta$$

$$F_{x5} = F_{x5} \cos \theta + F_{y5} \sin \theta$$

$$F_{y5} = -F_{x5} \sin \theta + F_{y5} \cos \theta$$

BRING $S = \sin \theta$

$C = \cos \theta$

$$\begin{matrix} \text{LOCAL} \\ \begin{pmatrix} F_{x4} \\ F_{y4} \\ F_{x5} \\ F_{y5} \end{pmatrix} \end{matrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{matrix} \text{GLOBAL} \\ \begin{pmatrix} F_{x4} \\ F_{y4} \\ F_{x5} \\ F_{y5} \end{pmatrix} \end{matrix} =$$

OR

$$\begin{matrix} \text{LOCAL} \\ \{F\}_L \end{matrix} = [T] \begin{matrix} \text{GLOBAL} \\ \{F\}_G \end{matrix} \quad (9)$$

$[T]$ IS THE TRANSFORMATION MATRIX

IT CAN SIMILARLY BE SHOWN THAT

$$\{\delta\}_L = [T]^T \{\delta\}_G \quad (10)$$

$$\{u\}_L = [T]^T \{u\}_G \quad \delta = u$$

OR

$$\{\delta\}_L = [T]^T \{\delta\}_G$$

WE HAVE SEEN PREVIOUSLY

I FEM (9)

(14)

$$\{F\} = |K^e|_L \{u\}$$

← LOCAL FORCES ON SPRING

FROM (9) $\{F\}_L = |T| \{F\}_G$

$$|K^e|_L \{\delta\}_L = |T| \{F\}_G$$

$$\delta = u$$

FROM (10)

$$|K^e|_L |T| \{\delta\}_G = |T| \{F\}_G$$

PRE-MULTIPLY EACH SIDE BY $|T|^{-1}$ WE GET (& SWAP SIDES!)

$$\{F\}_G = |T|^{-1} |K^e|_L |T| \{\delta\}_G \quad (|T|^{-1} |T| = I)$$

OR

$$\{F\}_G = |K^e|_G \{\delta\}_G \quad (F = kx)$$

WHERE $|K^e|_G$ IS THE STIFFNESS MATRIX FOR THE ELEMENT IN GLOBAL CO-ORDINATES AND

$$|K^e|_G = |T|^{-1} |K^e|_L |T|$$

BECAUSE

$|T|$

IS

ORTHOGONAL

$$|T|^T = |T|^{-1}$$

(SEE OVER FOR PROOF)

$$|K^e|_G = |T|^T |K^e|_L |T|$$

THIS USES THE GLOBAL STIFFNESS MATRIX TO CALCULATE THE GLOBAL FORCES AND DISPLACEMENTS FOR EACH ELEMENT

$$|T| = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \quad \text{I FEM (10)}$$

$$c = \cos \theta$$

$$s = \sin \theta$$

(15)

$$|T|^T = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix}$$

(INFO ONLY)

$$|T||T|^T = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix}$$

$$= \begin{bmatrix} c^2 + s^2 + 0 + 0 & -cs + cs + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 \\ -cs + cs + 0 + 0 & s^2 + c^2 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 \\ 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 + c^2 + s^2 & 0 + 0 - cs + cs \\ 0 + 0 + 0 + 0 & 0 + 0 + 0 + 0 & 0 + 0 - cs + cs & 0 + 0 + s^2 + c^2 \end{bmatrix}$$

$$(c^2 \theta + s^2 \theta = 1)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

= IDENTITY MATRIX

~~ALWAYS~~ ~~TRANSFORMATION~~

$$|T||T|^T = [I] \quad \leftarrow \text{BECAUSE IT IS ORTHOGONAL}$$

$$|T|^{-1}|T| = [I] \quad \leftarrow \text{ALWAYS}$$

(SEE OPPOSITE FOR PROOF)

$$|T|^T = |T|^{-1}$$

$$|K^e|_G = |T|^T |K^e|_L |T|$$

I FEM (11)

(16)

$$|T|^T = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix}$$

$$|T| = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

TO WORK OUT
THE GLOBAL STIFFNESS
MATRIX OF THE ELEMENT
SO THAT GLOBAL FORCES
AND DISPLACEMENTS CAN
BE FOUND!

$$|K^e|_L = \begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left(\frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\therefore |K^e|_G = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} K & 0 & -K & 0 \\ 0 & 0 & 0 & 0 \\ -K & 0 & K & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

$$= \begin{bmatrix} cK+0+0+0 & 0+0+0+0 & -cK+0+0+0 & 0+0+0+0 \\ Ks+0+0+0 & 0+0+0+0 & -Ks+0+0+0 & 0+0+0+0 \\ 0+0-cK+0 & 0+0+0+0 & 0+0+cK+0 & 0+0+0+0 \\ 0+0+Ks+0 & 0+0+0+0 & 0+0+Ks+0 & 0+0+0+0 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

$$= \begin{bmatrix} cK & 0 & -cK & 0 \\ Ks & 0 & -Ks & 0 \\ -cK & 0 & cK & 0 \\ -Ks & 0 & Ks & 0 \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

$$= \begin{bmatrix} C^2K + 0 + 0 + 0 & CSK + 0 + 0 + 0 & 0 + 0 - C^2K + 0 & 0 + 0 - CSK + 0 \\ CKS + 0 + 0 + 0 & KS^2 + 0 + 0 + 0 & 0 + 0 - KS^2 + 0 & 0 + 0 - KS^2 + 0 \\ -C^2K + 0 + 0 + 0 & -CKS + 0 + 0 + 0 & 0 + 0 + C^2K + 0 & 0 + 0 + CKS + 0 \\ -KCS + 0 + 0 + 0 & -KS^2 + 0 + 0 + 0 & 0 + 0 + KCS + 0 & 0 + 0 + KS^2 + 0 \end{bmatrix}$$

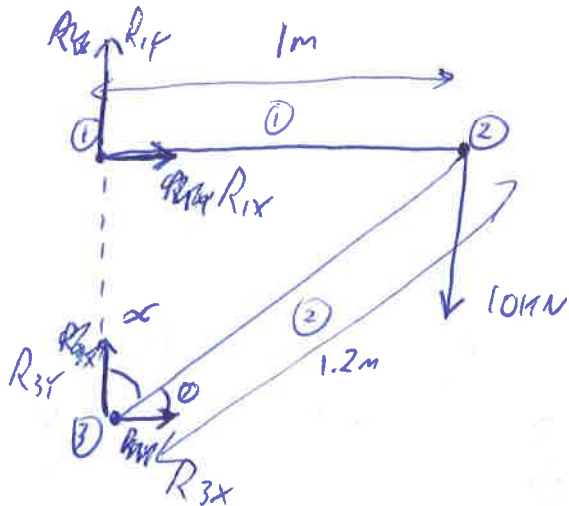
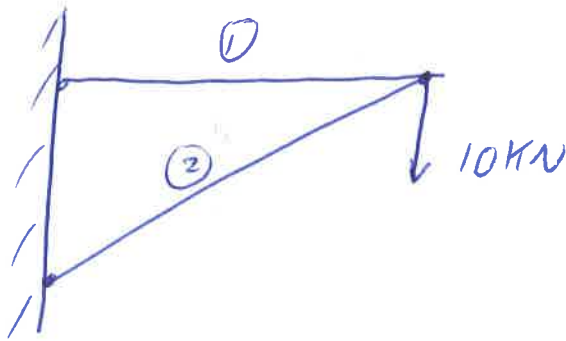
$$= \begin{bmatrix} KC^2 & KCS & -KC^2 & -KCS \\ KCS & KS^2 & -KCS & -KS^2 \\ -KC^2 & -KCS & +KC^2 & KCS \\ -KCS & -KS^2 & KCS & KS^2 \end{bmatrix}$$

$$= K \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

WHERE $K = \frac{AE}{L}$

$$\text{i.e. } |K^e|_G = K \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

THE GLOBAL
STIFFNESS MATRIX



$$x^2 + 1^2 = 1.2^2$$

$$\therefore x = 0.663 \text{ m}$$

$$\sin \alpha = \frac{0}{1.2} \quad \therefore \alpha = \sin^{-1} \frac{1}{1.2}$$

$$\alpha = 56.44^\circ$$

$$\therefore \theta = 33.56^\circ$$

ELEMENT (1)

$$[K]_e = \frac{AE}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & +c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$\theta = 0$$

$$\therefore \cos \theta = 1$$

$$\sin \theta = 0$$

$$= \frac{40 \times 10^6 \times 200 \times 10^9}{1} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} U_{1x} \\ U_{1y} \\ U_{2x} \\ U_{2y} \end{matrix} \begin{bmatrix} 8 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^6 \text{ N/m}$$

ELEMENT (2)

1 FEM (14)

(19)

$$\theta = 33.56$$

$$\cos \theta = 0.833$$

$$\sin \theta = 0.553$$

$$\cos \theta \sin \theta = 0.461$$

$$\cos^2 \theta = 0.694$$

$$\sin^2 \theta = 0.306$$

$$[K_z]_G = \frac{40 \times 10^6 \times 200 \times 10^9}{1.2} \begin{bmatrix} 0.694 & 0.461 & -0.694 & -0.461 \\ 0.461 & 0.306 & -0.461 & -0.306 \\ -0.694 & -0.461 & 0.694 & 0.461 \\ -0.461 & -0.306 & 0.461 & 0.306 \end{bmatrix}$$

$$= \begin{matrix} & \begin{matrix} U_{2x} & U_{2y} & U_{2x} & U_{2y} \end{matrix} \\ \begin{matrix} U_{2x} \\ U_{2y} \\ U_{3x} \\ U_{3y} \end{matrix} & \begin{bmatrix} 4.627 & 3.073 & -4.627 & -3.073 \\ 3.073 & 2.040 & -3.073 & -2.040 \\ -4.627 & -3.073 & 4.627 & 3.073 \\ -3.073 & -2.040 & 3.073 & 2.040 \end{bmatrix} \end{matrix} \times 10^6 \text{ N/m}$$

$$[K^e]_L = \begin{matrix} & \begin{matrix} U_{1x} & U_{1y} & U_{2x} & U_{2y} & U_{3x} & U_{3y} \end{matrix} \\ \begin{matrix} U_{1x} \\ U_{1y} \\ U_{2x} \\ U_{2y} \\ U_{3x} \\ U_{3y} \end{matrix} & \begin{bmatrix} 8 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 0 & 8+4.627 & 3.073 & -4.627 & -3.073 \\ 0 & 0 & 3.073 & 2.040 & -3.073 & -2.040 \\ 0 & 0 & -4.627 & -3.073 & 4.627 & 3.073 \\ 0 & 0 & -3.073 & -2.040 & 3.073 & 2.040 \end{bmatrix} \end{matrix} \times 10^6 \text{ N/m}$$

$$\{F\} = [K] \{U\}$$

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ R_{3x} \\ R_{3y} \end{Bmatrix} = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} K \begin{Bmatrix} U_{1x} \\ U_{1y} \\ U_{2x} \\ U_{2y} \\ U_{3x} \\ U_{3y} \end{Bmatrix}$$

APPLYING BOUNDARY CONDITIONS.

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ 0 \\ -10,000 \\ R_{3x} \\ R_{3y} \end{Bmatrix} = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} K \begin{Bmatrix} 0 \\ 0 \\ U_{2x} \\ U_{2y} \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ 0 \\ -10,000 \\ R_{3x} \\ R_{3y} \end{Bmatrix} = \begin{bmatrix} 8 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 0 & 12.623 & 3.073 & -4.627 & -3.073 \\ 0 & 0 & 3.073 & 2.040 & -3.073 & -2.040 \\ 0 & 0 & -4.627 & 3.073 & 4.627 & 3.073 \\ 0 & 0 & -3.073 & 2.040 & 3.073 & 2.040 \end{bmatrix} \times 10^6 \begin{Bmatrix} 0 \\ 0 \\ U_{2x} \\ U_{2y} \\ 0 \\ 0 \end{Bmatrix}$$

$$R_{1x} = -8 \times 10^6 U_{2x} \quad (1)$$

$$R_{1y} = 0 \quad (2)$$

$$0 = (12.623 U_{2x} + 3.073 U_{2y}) \times 10^6 \quad (3)$$

$$-10000 = (3.073 U_{2x} + 2.040 U_{2y}) \times 10^6 \quad (4)$$

$$R_{3x} = (-4.627 U_{2x} - 3.073 U_{2y}) \times 10^6 \quad (5)$$

$$R_{3y} = (-3.073 U_{2x} - 2.040 U_{2y}) \times 10^6 \quad (6)$$

From (3) $U_{2x} = \frac{-3.073 U_{2y}}{12.623} \therefore U_{2x} = -0.243 U_{2y}$

From (4) $\frac{-10000}{1 \times 10^6} = 3.073 U_{2x} + 2.040 U_{2y}$

$$-0.01 = 3.073 (-0.243 U_{2y}) + 2.040 U_{2y}$$

$$\therefore U_{2y} = \underline{\underline{-7.73 \times 10^{-3} \text{ m}}}$$

$$\begin{aligned} \therefore U_{2x} &= -0.243 \times -7.73 \times 10^{-3} \\ &= \underline{\underline{1.88 \times 10^{-3} \text{ m}}} \end{aligned}$$

FORCE IN ①

1 FEM (17)

$$E = \frac{\sigma}{\epsilon}$$

$$\therefore = \frac{F}{AE}$$

$$\therefore F = E A \epsilon$$

$$= E A \frac{\Delta L}{L}$$

$$= \frac{200 \times 10^9 \times 40 \times 10^{-6}}{1} \times 1.88 \times 10^{-3}$$

$$= \underline{\underline{15.04 \text{ kN}}} \quad \text{TENSILE FORCE IN ①}$$

FORCE IN ②

TO CONVERT FROM THE DISPLACEMENTS FROM GLOBAL
TO LOCAL

$$\{u\}_L = [T] \{u\}_G$$

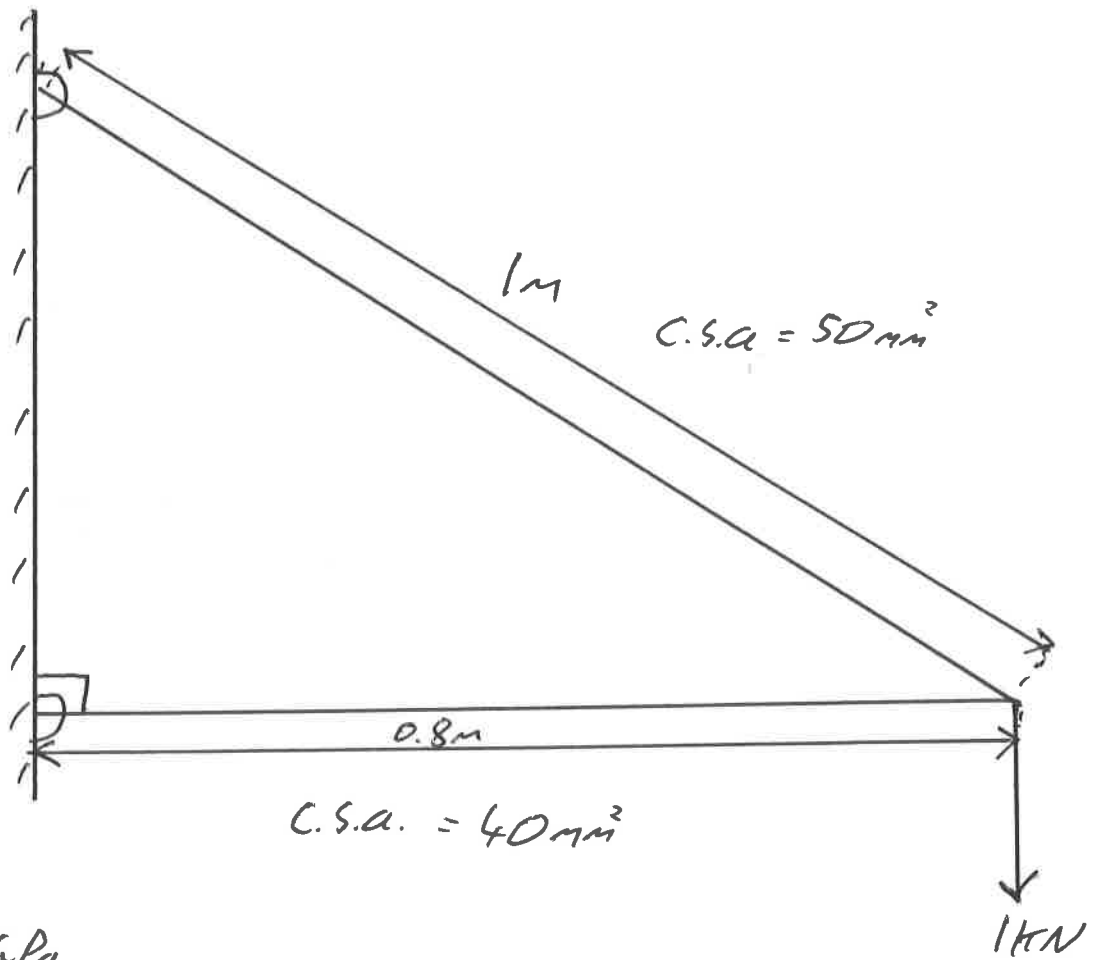
$$\begin{Bmatrix} u_{1x} \\ u_{2x} \end{Bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} u_{2x} \\ u_{2y} \end{Bmatrix}$$

$$= \begin{bmatrix} 0.833 & 0.553 \\ -0.553 & 0.833 \end{bmatrix} \begin{Bmatrix} 1.88 \times 10^{-3} \\ -7.73 \times 10^{-3} \end{Bmatrix}$$

$$= \begin{bmatrix} 1.566 \times 10^{-3} - 4.275 \times 10^{-3} \\ -1.040 \times 10^{-3} - 6.439 \times 10^{-3} \end{bmatrix}$$

$$= \begin{bmatrix} -2.71 \times 10^{-3} \\ -8.479 \times 10^{-3} \end{bmatrix}$$

$$F = \frac{200 \times 10^9 \times 40 \times 10^{-6}}{1.2} \times -2.71 \times 10^{-3} = \underline{\underline{-18.07 \text{ kN}}}$$

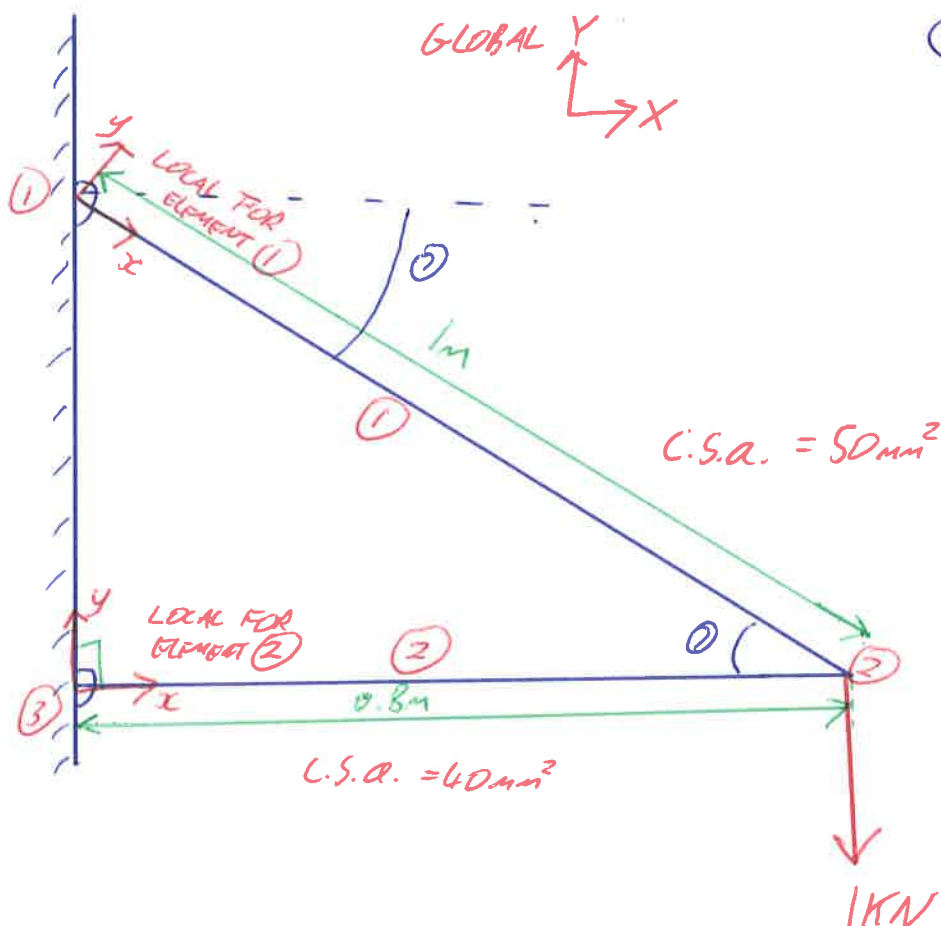


$$E_{\text{steel}} = 200\text{GPa}$$

FIND THE STRESS & STRAIN
IN THE DIAGONAL MEMBER
USING THE FINITE ELEMENT
METHOD

(5)

(24)



$$E_{\text{STEEL}} = 200 \text{ GPa}$$

FIND THE STRESS & STRAIN IN THE DIAGONAL MEMBER USING THE FINITE ELEMENT METHOD.

- FIRST FIND THE OFFSET BETWEEN THE LOCAL & THE GLOBAL COORDINATE SYSTEMS:

~~$$\sin \theta = \frac{A}{H}$$~~

$$\cos \theta = \frac{A}{H}$$

~~$$\theta = \cos^{-1} \frac{A}{H}$$~~

$$\theta = \cos^{-1} \frac{0.8}{1}$$

$$\theta = \underline{\underline{36.87^\circ}}$$

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$$\sin \theta = -0.6$$

$$= \begin{bmatrix} 0.64 & -0.48 & -0.64 & -0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \times 10^7 \text{ N/m}^2$$

FIND THE STIFFNESS MATRIX FOR THE 2ND ELEMENT
IN GLOBAL CO-ORDINATES

26

$$\theta = 0$$

$$\cos \theta = 1$$

$$\sin \theta = 0$$

$$|K^2|_G = \frac{40 \times 10^{-6} \times 2 \times 10^9}{0.8} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ N/m}^2$$

$$= 1 \times 10^7 \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \text{ N/m}^2$$

$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^7 \text{ N/m}^2$$

• FIND THE STIFFNESS MATRIX FOR THE ENTIRE STRUCTURE

(26)

(27)

$$|K|_G = \begin{bmatrix} 0.64 & -0.48 & -0.64 & -0.48 & 0 & 0 \\ -0.48 & 0.36 & 0.48 & -0.36 & 0 & 0 \\ -0.64 & 0.48 & 0.64+1 & -0.48+0 & -1 & 0 \\ 0.48 & -0.36 & -0.48+0 & 0.36+0 & +0 & +0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 10^7 \text{ N/m}^2$$

$$= \begin{bmatrix} 0.64 & -0.48 & -0.64 & -0.48 & 0 & 0 \\ -0.48 & 0.36 & 0.48 & -0.36 & 0 & 0 \\ -0.64 & 0.48 & 1.64 & -0.48 & -1 & 0 \\ 0.48 & -0.36 & -0.48 & 0.36 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 10^7 \text{ N/m}^2$$

~~XXXXXXXXXX~~

$$F = K\delta$$

$$\begin{pmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{pmatrix} = [K]_G \begin{pmatrix} \delta_{1x} \\ \delta_{1y} \\ \delta_{2x} \\ \delta_{2y} \\ \delta_{3x} \\ \delta_{3y} \end{pmatrix}$$

$$\begin{pmatrix} R_{1x} \\ R_{1y} \\ 0 \\ -1000 \\ R_{3x} \\ R_{3y} \end{pmatrix} = [K] \begin{pmatrix} 0 \\ 0 \\ \delta_{2x} \\ \delta_{2y} \\ 0 \\ 0 \end{pmatrix}$$

← INSERT BCs

$$\begin{pmatrix} R_{1x} \\ R_{1y} \\ 0 \\ -1000 \\ R_{3x} \\ R_{3y} \end{pmatrix} = \begin{bmatrix} 0.64 & -0.48 & -0.64 & -0.48 & 0 & 0 \\ -0.48 & 0.36 & 0.48 & -0.36 & 0 & 0 \\ -0.64 & 0.48 & 1.64 & -0.48 & -1 & 0 \\ 0.48 & -0.36 & -0.48 & 0.36 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times 10^7 \begin{bmatrix} 0 \\ 0 \\ \delta_{2x} \\ \delta_{2y} \\ 0 \\ 0 \end{bmatrix}$$

From Row (3):

$$0 = 0 + 0 + 1.64 \times 10^7 \delta_{2x} - 0.48 \times 10^7 \delta_{2y} + 0 + 0$$

$$0 = 1.64 \times 10^7 \delta_{2x} - 0.48 \times 10^7 \delta_{2y} \quad *$$

From Row (4):

$$-1000 = 0 + 0 + 0.48 \times 10^7 \delta_{2x} + 0.36 \times 10^7 \delta_{2y} + 0 + 0$$

$$-1000 = 0.48 \times 10^7 \delta_{2x} + 0.36 \times 10^7 \delta_{2y} \quad * *$$

• FROM *

$$1.64 \times 10^7 \delta_{2x} = 0.48 \times 10^7 \delta_{2y}$$

$$\underline{\underline{\delta_{2x} = 0.29 \delta_{2y}}}$$

• FROM **

$$-1000 = -0.48 \times 10^7 (0.29 \delta_{2y}) + 0.36 \times 10^7 \delta_{2y}$$

$$-1000 = -0.1392 \times 10^7 \delta_{2y} + 0.36 \times 10^7 \delta_{2y}$$

~~$$1.2 \times 10^7 \delta_{2y}$$~~

$$-0.453 \times 10^{-4} \text{ m} = \delta_{2y}$$

~~$$1.00 \times 10^7 \delta_{2y}$$~~

$$\underline{\underline{-0.453 \mu\text{m} = \delta_{2y}}}$$

$$\delta_{2x} = 0.29 \delta_{2y}$$

$$= 0.29 \times -0.453$$

$$= \underline{\underline{-0.131 \mu\text{m}}}$$

TO FIND STRESS/STRAIN WE MUST REVERT TO
LOCAL COORDINATES

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$$\begin{pmatrix} \delta_{2x} \\ \delta_{2y} \end{pmatrix}^L = \begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix} \begin{pmatrix} -0.131 \\ -0.453 \end{pmatrix}$$

$$\begin{aligned} \delta_{2x}^L &= 0.8 \times (-0.131) + (-0.6)(-0.453) \\ &= \underline{\underline{0.167 \text{ mm}}} \end{aligned}$$

$$\begin{aligned} \delta_{2y}^L &= (0.6)(-0.131) + (0.8)(-0.453) \\ &= \underline{\underline{-0.441 \text{ mm}}} \end{aligned}$$

$$\therefore \underline{\underline{\delta_{2x}^L = 0.167 \times 10^{-3}}}$$

$$\epsilon^{(1)} = \frac{\delta L}{L} = \frac{0.167 \times 10^{-3}}{1} = \underline{\underline{167 \times 10^{-6}}}$$

$$\sigma^{(1)} = E \epsilon^{(1)} = 200 \times 10^9 \times 167 \times 10^{-6} = \underline{\underline{33.4 \text{ MPa}}}$$