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The development of children's understanding of mathematical patterns through mathematical activities

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In this paper we report how children (aged 8) developed their mathematical understanding through number tasks based on the Fibonacci sequence (Bamboo numbers) used in the context of a Substantial Learning Environment (SLE), which is designed to be mathematically rich, have a clear purpose and give opportunities to utilise mathematical thinking. The flexible nature of the SLEs makes it possible for teachers and children to explore various mathematical patterns. To capture children's activities when working within SLEs, we make particular reference to Pegg and Tall's work in 2005, and consider a theoretical framework based on the SOLO taxonomy (Biggs and Collis 1982) and the developmental process of understanding mathematical concepts. It was found that the key progression to be made through learning using our Bamboo number-based SLEs is from Multi-structural to Relational levels. It was also suggested that it is difficult for many children to understand the structural aspects of number patterns.

Keywords: Substantial Learning Environment; number patterns; mathematical activities

Introduction

Whereas Japanese mathematics education used to have a good reputation (e.g. Becker and Shimada 1997; the TIMSS study, e.g. Gonzales et al. 2009), many aspects in the learning and teaching of mathematics are now facing scrutiny and criticism. There are many areas in which it is felt there is need for improvement, e.g. students' attitudes towards mathematics, mathematical thinking and problem solving skills (Sekiguchi 2000, 112). One task of mathematics education research is to improve the current situation, and the goal of our research project is to enhance mathematical teaching and learning by developing a mathematics curriculum from primary to secondary schools in which children actively engage in rich mathematical activities and acquire a sound knowledge and understanding of mathematics (via a 'substantial mathematics curriculum'. This is influenced by the 'mathe2000 project' in Germany (see, for example, Wittmann 2001) and the design and implementation of Substantial Learning Environments (SLEs), which are teaching units intended to provide children with appropriate learning opportunities where they can explore mathematical patterns and develop mathematical thinking (Wittmann 2006).

The purpose of this paper is to evaluate two series of mathematics lessons which are designed in accordance with the idea of SLEs. We focus on children's

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mathematical activities in whole classroom lessons. Our research centres on the following three questions: Research Question 1 – how can we analyse children’s activities with SLEs in whole classroom settings?; Research Question 2 – what activities can we observe among children presented with these tasks?; and Research Question 3 – what instructional approaches are necessary to promote children’s mathematical thinking? In this paper we mainly focus on the first two questions, but provide some discussion concerning the third.

Designing Substantial Learning Environments

Wittmann’s Substantial Learning Environments

Designing mathematically rich tasks is one of the most important challenges for mathematics educators (for example, Wittmann 1981, 1995, 2001, 2006). To design a mathematical task for children, there needs to be a consensus about “what is mathematics” in order to select our fundamental theme or content for the task. Also, as Ainley, Bills, and Wilson (2005) or Ainley, Pratt and Hansen (2006) advocate, tasks for children should be purposeful to make them meaningful and encourage children to focus on these tasks. They also stated that the tasks should provide opportunities to utilise ideas, “why that idea is useful, how it can be used and what it can be used for” (Ainley, Pratt and Hansen 2006, 32). We agree with these ideas – *offering good mathematical content, purpose, and utility* – and use these as the three fundamental principles of our task design.

The pedagogical idea of the SLE proposed by Wittmann (1995) is the vehicle we have selected to achieve our goal. He states that an SLE has the following key features:

- (1) It represents key objectives, content and principles of teaching;
- (2) It is related to significant mathematical content, processes and procedures beyond this level, and is a rich source for mathematical activities;
- (3) It is flexible and can be adapted to the conditions of particular classrooms;
- (4) It incorporates mathematical, psychological and pedagogical aspects of teaching mathematics in a holistic way, and offers a wide potential for empirical research (Wittmann 1995, 365–6)

The theoretical underpinning of what makes Wittmann’s SLEs a rich source of mathematical content is the view that “mathematics is a science of patterns”. This view is expressed by several mathematicians and physicists (Hardy 1940; Sawyer 1955, 12; Feynman 1968, 314; Devlin 1998, 3) as pattern recognition and manipulation is often central to SLE activities. Let us first take as an example, the ‘number pyramid’ (for other examples of SLEs, see Wittmann, 1984, 1995, 2001), which meets our three criteria for good task design as well as Wittmann’s own desired key features. This number pyramid, derived from Pascal’s triangle, has many interesting mathematical patterns, and can be a teaching unit which has a clear *purpose* for children, giving opportunities to *utilise* various aspects of mathematical knowledge. It also facilitates the integration of research and practice in the learning and teaching of mathematics. It provides students with an interesting environment for the learning of numbers as exemplified in Figure 1.

Through completing these number pyramids, children and students can practise addition and subtraction in a challenging situation. The mathematical structure of

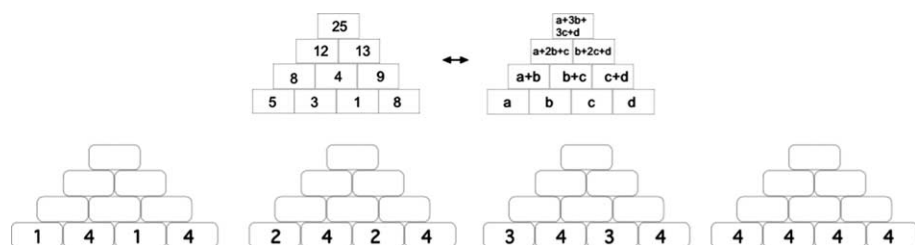


Figure 1. Number pyramids.

the pyramids (in the above ‘a–b–a–b’ type examples, where the top numbers are $4(a+b)$) can provide children with an opportunity to engage in not only basic numerical operations, but also mathematical processes such as problem solving, reasoning, communicating, etc., in which children can utilise their knowledge and understanding of numbers (Key features (1) and (2)). The design is very flexible and easy to manipulate; we can extend this problem with fractions and decimals, positive and negative numbers, and algebraic symbols can be used in secondary schools (Key feature (3)). It also provides teachers and researchers with research questions. For example, it is interesting to conduct research to examine what additional motivation students experience when undertaking ‘the number pyramid challenge’, or how children can develop calculation concepts through these problems (Key feature (4), Steinbring 2005, 133–147).

Bamboo numbers as SLE

In this paper we focus on one particular SLE called ‘Bamboo numbers’. This is based on the patterns of the Fibonacci sequence, but made by a set of finite recursive relations, e.g. ‘Based on the rule 1–3–4–7, complete 2–4–6–10, 4–?–?–10, 6–?–?–10, 8–?–?–10 and so on.’ The word ‘bamboo’ is used as a metaphor for ‘growing numbers’ and ‘growing number sense in children’, because baby bamboos grow rapidly. Teaching units based on Bamboo numbers satisfy our task design principles as well as the four key features of SLEs. The *purpose* of the task is clear for children as they are simply asked to fill in the missing the numbers by using (or not using) these patterns, but they are also expected to *utilise* various ideas to investigate invisible *mathematical patterns* in these tasks. Through working on Bamboo numbers, children will notice that these numbers form some patterns, e.g. natural numbers are used, the last number is always 10, the first numbers are even, when the first number is increased by two, the second is decreased by one, etc. which are all derived from the properties of the Fibonacci sequence (Key features (1) and (2)). We can also extend these patterns by considering, for example, the following questions: Can we use decimals or fractions, but still enable the last number to be 10? If the first number is less than zero, what would happen? (Key feature (3)). Finally, children’s activities with Bamboo numbers can illuminate their mathematical thinking, which is worth studying in mathematics education research (Key feature (4), see data analysis section).

The idea of Bamboo numbers described above is not new (Pinel, 1993); indeed this type of problem solving task has been central in recent mathematics education. However, we believe the teaching units we have designed based on Bamboo numbers

can provide us with rich mathematical activities beyond simple number pattern tasks. Let us focus on the pattern 2–4–6–10, and regard the first three numbers as a 3-digit number, 246, where $2 + 4 = 6$ and $4 + 6 = 10$. As one example, the learner might notice that in adding 191 something similar emerges, i.e. $246 + 191 = 437$, and $4 + 3 = 7$, $3 + 7 = 10$. Repeat this, $437 + 191 = 628$ and so on, and one can create patterns quite easily. Moreover, in the case of the patterns which consist of four numbers, the last number does not have to be 10, and many other patterns are easily created by adding 191 (see Figure 2).

In this project, we would like children to explore not only given mathematical patterns but also, as in the scenario above, extend, discover their own patterns, and see mathematics as a ‘science of patterns’.

By using the Bamboo number concept, the following four lessons were designed for Grade 2 children (aged 7–8) based upon this SLE. The overall lessons’ aims are that children consolidate their skills in number operations, and also experience mathematical processes such as discovering mathematical patterns, creating their own patterns, and having been given the chance to extend their ‘number sense’ (Anghileri 2006). We structure the sequence by starting from an introduction to Bamboo numbers (the first and second lessons), and progressing to more complex patterns (third and fourth lessons), briefly described in Table 1.

Theoretical framework

Our intention is to evaluate our lessons in a mainly qualitative way, by considering how children engage in SLEs and discover and explore mathematical patterns. To consider the learning process more theoretically, we need to situate our theoretical position clearly. In this section we make reference to Pegg and Tall’s paper (2005), which provides us with an extremely useful summary and synthesis of existing theories about the development of mathematical concepts. According to them (469), the current frameworks describing children’s mathematical understanding can be separated into “global frameworks of long-term growth” such as the stage-theory of Piaget, van Hiele’s levels of geometric thought (1986) or Bruner’s enactive-iconic-symbolic modes (1966), and “local frameworks of conceptual growth” such as Dubinsky’s action-process-object-schema theory (1991), Sfard’s concept formations (1991), or Biggs and Collis’ (1982) Structure of the Observed Learning Outcome (SOLO) taxonomy. In the former frameworks, long term developments from modes of learning such as ikonic (sic) (mainly actions and images) to symbolic (a system of

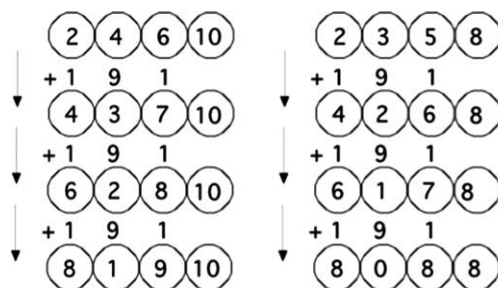
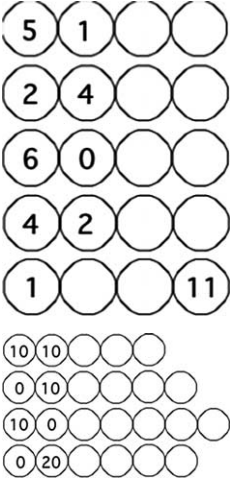
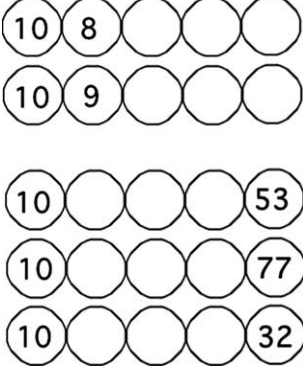
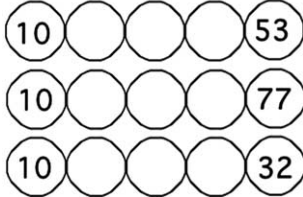


Figure 2. Adding 191 and number patterns.

Table 1. Lesson structure, objectives and activities.

Lesson	Main problem	Objectives and activities
1		This lesson is an introductory one in which children undertake exercises with relatively easy patterns (see left).
2		This lesson is a further introduction but with some extension. For example, in the left pattern, some children might notice a pattern 3–3–6–9 is missing.
3		This lesson is designed so that children can freely explore Bamboo numbers.
4		In the fourth lesson, children will undertake more challenging tasks in which they have to apply both vertical and horizontal patterns to complete Bamboo numbers effectively.

symbols) are considered, and the latter are related to more specific areas of cognitive development such as counting, number operations and so on (Pegg and Tall, 2005, 468–9). While our ultimate goal is to construct a global framework which could be applied to the development of children's mathematical concepts by using SLEs from primary to secondary schools, and from sensori-motor, enactive and iconic to more symbolic and formal modes, at the moment we are still experimenting with individual SLEs and designing series of only 4–6 consecutive lessons. Thus, at this stage, ideas from local frameworks are more useful than global ones, and we particularly focus on Biggs and Collis' SOLO taxonomy.

The SOLO taxonomy suggests that in general our understanding develops through the following levels: Pre-structural (little or no response), Uni-structural (responding to one feature), Multi-structural (responding to more than one feature), Relational (start making relationship between features) and Extended abstract (going to beyond what is given). The SOLO taxonomy is not limited to mathematical concepts and there are some weaknesses (Watson, 2007), nevertheless we believe that this model is useful to analyse children's activities with SLEs observed in whole classroom situations, because the level descriptions fit the learning process of the SLEs: in general, mathematical structures and patterns within the SLEs are presented 'invisibly', but this invisibility encourages children to explore the tasks (Steinbring, 2001). Within the learning process, children gradually notice patterns or structure by first completing individual tasks (Uni-Structural) and then comparing several tasks (Multi-Structural), and finally, they might see and understand 'invisible' structures (Relational) and start exploring further by, for example, creating their own problems (Extended abstract).

Currently, we are focusing on SLEs in primary schools where most children are in the concrete symbolic mode of learning (Pegg 2003, 242; Pegg and Tall 2005, 468). At this stage of learning, children learn mathematical concepts mainly with a system of symbols in the written language (e.g. 1, 2, 10, 99, +, −, =, etc.) and two Uni-Multi-Relational (UMR) cycles of SOLO can be utilised (see below) to provide more detailed analysis of the development of mathematical concepts in this mode (*ibid.*, 470). The UMR cycles can be applied to various local cognitive developmental theories, such as Dubinsky's action-process-object-schema theory in mathematics education (See Table 3 in *ibid.*, 453). We apply this idea in the context of Bamboo numbers; children's responses are considered using the SOLO taxonomy and two UMR cycles. Note, in this paper the SLE-UMR cycles we describe should not be confused with the generally recognised UMR cycles (the word cycle is used in the sense of Pegg and Tall, i.e. the three levels, Uni-Structural, Multi-Structural and Relational, are taken together as a sequence). In the initial cycle, children first try to grasp the mathematical structure of one Bamboo number such as 1–3–4–7 (Uni-Structural-1), then they try another or more Bamboo numbers such as 2–3–5–8 or 2–4–6–10 (Multi-Structural-1), and finally they create their own number patterns, or extend number patterns such as 1–2–3–5–8–13 and so on (Relational-1). Now they at least understand the 'structure' of individual sequences, and are ready to embark upon the second cycle. For this next stage, children might be given further number sequences (e.g. Table 1 Lesson 2), and they might notice vertical relationships such as the second numbers are increased by 1, and so on (Uni-Structural-2). These observations are tested, refined and confirmed by studying more examples of Bamboo numbers and their relationships (Multi-

Structural-2) (e.g. Table 1 Lesson 3). By the end of the Multi-Structural-2 level, children start utilising the multiple patterns they have identified more effectively to solve more challenging questions. For example, it is possible to complete $10-?-?-?-53$ by trial and error methods (Multi-Structural-2), but if one uses the relationship between the second and last numbers more effectively, then the second number will be $11 = 9 + 2$ as the last number is increased by $6 = 3 + 3$ for $10-?-?-?-53$ (e.g. Table 1 Lesson 4), and if a child can use this strategy confidently, then this is considered as a Relational-2 response because some form of reasoning is observed. If a child completely grasps the patterns and relationships between the number sequences, then this is the 'Extended abstract level'. These children will be able to create general patterns by themselves, for example extending the patterns above, devising their own problems by using other numbers (including increasing and decreasing the number of circles) or investigating further the properties of Bamboo numbers, etc. At this level, children may start investigating more formally why adding 191 gives you the answer for some Bamboo numbers (Figure 2).

While the above theoretical discussion gives us insight into how we could classify children's progress within SLEs, it is also important to consider the *transition process* from, for example, Multi-Structural-2 to Relational-2 levels, focusing on what cognitive process is involved in each transition. While children at Multi-Structural-2 level could recognise more than one pattern in Bamboo numbers, they would not necessarily be able to use these to solve more challenging questions. To consider how children make the leap to use the patterns they have discovered, we shall exploit theoretical ideas regarding the development of concepts which are related to Piaget's theory of reflective abstraction, "drawn from the general coordination of actions or of operations" (Piaget, 1980, 89–97). Following Piaget's school of thought, Sfard (1991) points out that mathematical concepts are interpreted as 'operational' and 'structural'. A rational number $3/2$ has a static structure, and at the same time, this is dynamic and operational, as 3 is divided by 2. She also points out that 'operational' conceptions are formed prior to 'structural' ones within the learning process, i.e. the structure of $3/2$ is understood by children only after the operational conception is formed. In the process of concept formations, 'interiorisation', 'condensation' and 'reification' are involved. Similarly, Dubinsky (1991) considers that the interiorisation and encapsulation between objects and processes play key roles in cognitive development of mathematical concepts (107). These theories offer a useful framework for analysing children's transitions between different levels of understanding within the SOLO taxonomy.

We focus on Sfard's idea in the context of Bamboo numbers to describe the developmental process. For example, after children grasp the individual structure of Bamboo numbers (Relational-1), they then study several Bamboo numbers together (e.g. Table 1 Lesson 2). By completing number patterns and making mistakes (Uni-Structural-2), children gradually observe and grasp the relationship between numbers, often without any calculation (interiorisation). Then they undertake further Bamboo numbers tasks, repeatedly study the relationships, and become increasingly familiar with these patterns, i.e. 'condensation', which Sfard describes as "a period of "squeezing" lengthy sequences of operations into more manageable units" (19). At this stage, children have learned to quickly recognise vertical patterns among Bamboo numbers and they should further understand more complex patterns. Finally, the whole structure of Bamboo numbers is suddenly seen as

“something familiar in a totally new light” (19), i.e. ‘reification’. In the case of Bamboo numbers, children see the entire structure, and utilise mathematical patterns to solve $10-?-?-?-53$ more effectively.

In summary, our analysis of children’s activities within SLE is framed by the following theoretical ideas (see Table 2).

While our main interests are children’s mathematical activities and understanding, the role of the teacher in managing the SLEs should not be ignored. In fact, research evidence has suggested that teachers’ instructional interventions can promote children’s learning. For example, Henningsen and Stein (1997) found that, in addition to the appropriateness of the tasks they set, supportive actions from teachers can influence the level of children’s engagements. Mason (1998) notes that “the role of a teacher is to create conditions in which students experience a corresponding shift in the structure of their attention, in which they become aware of acts and facts of which they were previously unaware” (244). Stylianides and Stylianides (2009) propose a theoretical construct which they call ‘conceptual awareness pillars’ “to describe instructional activities that aim to direct students’ attention to their conceptions about a particular mathematical topic” (322). They maintain that careful integrations of conceptual awareness pillars can raise learners’ awareness of mathematical concepts. Similarly, in our research, teachers encourage children to pay attention to mathematical patterns and structures in the SLEs in various forms (interaction with individual/group of children, whole class situation, etc.).

Research design and context

Our research design is based on the traditional approach for professional development in Japan, Lesson Studies, where teachers work in small teams, carefully and collaboratively crafting lesson plans and then analysing the outcomes (Lewis, 2002). After designing the SLEs and lesson plans, pilot lessons are implemented, and then these lessons are evaluated by teachers and researchers. We carried out our research using SLEs mainly in a school attached to a university. Children’s attainment levels are generally average to high, and they particularly enjoy challenging tasks in mathematics. Most teachers had about 10 years experience, and they also enjoyed sharing ideas with university researchers. In our research project, pilot lessons were mainly recorded by observation with video cameras and field notes. These pilot lessons were analysed by researchers, with teachers and research students often joining the data analysis process. We regularly watched recorded lessons, reflected on what happened and made notes as to how lessons could be improved, following which revised lessons were implemented, of which one example will be outlined below. So far we have undertaken more than 50 lessons with SLEs since 2004.

In this paper, we report our findings from lessons with Bamboo numbers which were mentioned in the previous section in Table 1. To develop and test the theoretical framework, the research process is adopted from a design experiment (Cobb et al, 2003) in which a project team design, implement and analyse experiments in several cycles. First 4 pilot lessons were implemented for Grade 2 children (aged 7–8, 20 boys and 20 girls) in May 2007. These lessons were rather successful, as our observations and field notes suggest that children really enjoyed the tasks, discovered many mathematical patterns, and had productive discussions around how to utilise these

patterns to solve more challenging tasks. Nevertheless, we wondered why some children still could not solve the tasks presented in the fourth lesson despite their experience in the three previous lessons and discussions about the patterns visible amongst Bamboo numbers. To investigate this issue further, we used the same four consecutive lessons in a different class (20 boys and 20 girls; this test became the first main study) a week after the pilot lessons. This time, lessons were recorded by a video camera to capture whole classroom interactions, as we felt field notes were not sufficient to consider the issue. The teacher, who had more than 15 years of teaching experience, did not at this stage consider any theoretical ideas when he designed the 4 lessons. Again, we obtained very similar results to those from the pilot lessons. Following discussion while watching these lessons again with the teacher, we decided to use the SOLO taxonomy to differentiate various children's activities. This initial analysis led us to consider that the key progress we desired children to make would be from Multi-Structural-2 to Relational-2 levels. We then designed a similar series of four lessons for a different class of Grade 2 children (again aged 7–8, 20 boys and 20

Table 2. Theoretical framework for analysis.

Learning stages with Bamboo numbers	Levels of understanding The SOLO taxonomy	Conceptual development
Before studying Bamboo numbers	Pre-structural	
Individual Bamboo numbers	Uni-Structural-1: Recognising patterns in one Bamboo number example.	Children gradually observe/ grasp relationship between numbers-often without any calculation (interiorisation).
	Multi-Structural-1: Recognising patterns of several examples.	
	Relational-1: Grasping the structure of individual Bamboo numbers.	
Multiple Bamboo numbers	Uni-Structural-2: Recognising patterns in one example of multiple Bamboo numbers.	Children become more familiar with patterns (condensation).
	Multi-Structural-2: Recognising patterns in several examples.	
	Relational-2: Grasping the whole structure of multiple Bamboo numbers	Children see entire structure of Bamboo number, and can utilise mathematical patterns (reification).
After studying Bamboo numbers	Extended abstract: Creating new problems with multiple Bamboo numbers, exploring patterns in more formal ways, etc.	

girls) in May 2009 to research this issue (the second main study). In these lessons, two video cameras were used to obtain more information about children's activities – one was for whole classroom interaction and the other for one child who was randomly chosen from the class. The teacher who taught the four lessons in 2007 was aware of the SOLO taxonomy, and tried hard to make children pay attention to patterns and utilise them to solve challenging tasks. Again, after the lessons thorough evaluation of the lessons was done by the teacher and researchers. Also, more theoretical ideas were reviewed, to give more detailed analysis of children's activities and development of mathematical concepts, as discussed in the previous section.

Data analysis

Children's Multi-Structural responses in the second cycle

The rule of Bamboo numbers was easily grasped by all children (Uni-Structural-1). After examining, completing and checking individual patterns (Multi-Structural-1 and Relational-1), they naturally started to pay attention to horizontal and vertical relationships (Uni-Structural-2). For example, in the second lesson of the first main study, children started recognising the relationship between numbers vertically, comparing horizontal relationships, and could create another pattern '3–3–6–9' (Multi-Structural-2) (see Figure 3).

At this point, it is clear that children are at least in the process of interiorisation and condensation, as they could predict the pattern 3–3–6–9 by just observing the completed number patterns without doing any calculations, and they used the given and completed Bamboo numbers effectively to do this, i.e. some structural aspects are interiorised and condensed.

In the third lesson of the first main study, the task below was given to children to provide opportunities for them to look for patterns not only horizontally but also vertically, to encourage their further understanding about the relationships between Bamboo numbers. Children first undertook the task individually, and found they could extend the patterns to the left by looking at the numbers diagonally, as shown in Figure 4. By the end of this task, they found that the patterns are actually the same. The evidence for their Multi-Structural-2 level of understanding can be found in their discussion.

The extract above suggests that children were flexibly using vertical and horizontal relationships to create and extend mathematical patterns dynamically. Similar responses were also observed in the second main study. Again these responses

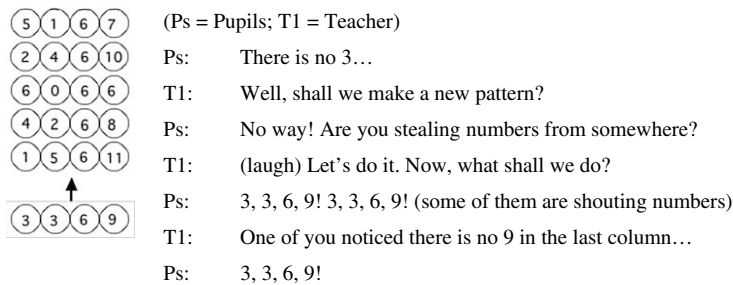


Figure 3. What children noticed in the second lesson.

from children indicate that some structures of Bamboo numbers have become interiorised and condensed in their minds, as they quickly stated what they discovered by simply observing the completed patterns.

Children's Relational responses in the second cycle

Children's statements recorded during the lessons indicate that they clearly advanced to the Multi-Structural-2 level of response. It was also encouraging that children could see numbers flexibly and use their knowledge of multiplication to describe the patterns they noticed (See Figure 5. Bold numbers were completed by children). However, it seems that they did not fully understand the relationship between Bamboo numbers, i.e. "if the first number increased by 1, then the last number is also increased by 1", although one child stated that "The right ones are 4-4-8-12 and the left ones are 5-4-9-13 and the difference between 4 and 5 is 1, and that's the same difference as between 8 and 9, and 12 and 13", which might be considered as an early Relational-2 response (Figure 5, left).

By the end of the second lesson of the first main study, most children in the class still stayed at the Multi-Structural-2 level, as it was observed that some struggled to complete the last two patterns in Figure 6 ('9-?-?-17' and '11-?-?-17'), i.e. these children could not use the multi-structure (the vertical and horizontal structure) to solve problems, which indicates reification has yet to emerge. However, some children showed progress from Multi-Structural-2 to the early Relational-2 level through their pattern recognition and interactions with the teacher, and some children clearly indicated that they were working with both the vertical and horizontal patterns to



- Ps: There are three 7s (looking at the numbers diagonally)! And it's the same for 12 as well!
- Pupil K: If we put 5 next to 7, then we also have three 5s!
- T1: I think K made a really good point. If there is 5 next to 7. Ah ha! I thought K would say this, so I have actually prepared... (unfolds paper to reveal more circles on the left)
- Ps: You already did it?
- T1: (laugh) I have been waiting for this, and if there is 5 next to 7, then ...
- Pupil H: We can add 2s!
- T1: And do you notice anything else?
- Pupil B: We can also have 0...
- Pupil D: If we add 0s, then we have three 0s.
- Ps: I know something else!
- Pupil E: There is 10 here, and if we add two 0s, then we also have three 10s.
- T1: Can we add 10 here?
- Ps: Yes!
- T1: Are you sure?
- Ps: Yes (checking the horizontal relationship)!

Figure 4. What children noticed in the third lesson.

complete the last two, i.e. the second numbers are 7, 6 and 5, so the last two should be 4 and 3. Also, the third numbers are 11, 12 and 13, so the next numbers should be 14 and 15. One child (Pupil T below) said that she first found these numbers vertically and then checked these numbers also “worked” horizontally, as described in Figure 6.

The difference between Multi-Structural-2 (e.g. Pupils F or O above) and Relational-2 (e.g. Pupil T) levels can be described as follows, i.e. children at Multi-Structural-2 level (Figure 7 left) are aware of the vertical and horizontal patterns, but cannot use this to complete the pattern “9-?-?-17”, but those at Relational-2 can utilise these interiorised and condensed mathematical structures in their mind to complete the remaining missing numbers (see Figure 7).

We shall present another example of a Relational-2 level of response in which it is evident that children started focusing on the vertical and horizontal relationships to solve 2-?-?-34. In the fourth lesson of the second main study, a pupil (pupil R) extended the patterns and created new problems such as 2-?-?-13 or 2-?-?-40, which indicates that this child used vertical and horizontal structures of the problem very effectively to grasp and extend the entire structure (see Figure 8).

Difficulties in transition between Multi-Structural-2 and Relational-2

The children’s recognition of patterns in general encouraged us to give a further challenging task in the fourth lesson in both the main studies. Our hypothesis was that children would be able to complete the task effectively if they had a good understanding about the relationship between the previous sequences of numbers, e.g. “If the first number is increased by 1, then what would happen to the other numbers? If the second number is increased by 1, then . . .” By using this relationship, the pattern 10-?-?-53 can be solved if one notices that 47 (the sum of the previous row) $+ 6 = 53$, so in order to get the additional 6 required to reach 53, the second

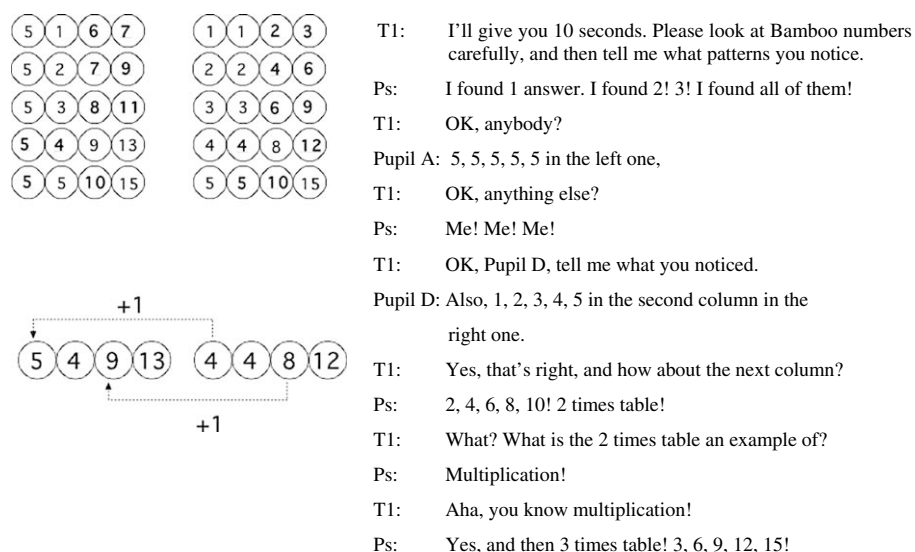


Figure 5. Children’s Multi-Structural-2 responses of Bamboo numbers.



- T1: It seems that you had trouble with the last two... How did you solve this? Did you try several numbers?
- Pupil F: I tried several numbers...
- Pupil N: I did it in my head...
- T1: I heard someone say that it is easy if you notice 'something'... So what is 'something'?
- Pupil N: A pattern!
- T1: A pattern?
- Pupil O: I don't think a pattern would help me...
- T1: Pattern! Pattern! So if we want to complete the last two, it is easy if you notice something, and what do you notice?
- Pupil T: I know! I know!
- T1: Ok, what did you notice?
- Pupil T: 7-6-5 ... (indicating the 2nd column), and then 4, 3...
- T1: She focused on the 2nd column, and 7-6-5 and then 4, 3...

Figure 6. Children's Multi-Structural-2 and early Relational-2 responses.

number should be increased by not 1, but 2: $9 + 2 = 11$ etc. In other words, our hypothesis was that the transition between Multi-Structural-2 and Relational-2 would be smooth after three lessons focusing on Bamboo numbers. However, evidence from both main studies suggested that this is not easy for many children.

For example, in the fourth lesson of the first main study, the first five patterns were completed easily by all children, and their findings were shared and discussed. They noticed that "the last numbers increased by 3", "the second column is 5, 6, 7, 8 and 9", etc. They also noticed that "the third column is increased by 1" and "the fourth column is increased by 2". After sharing these findings, children undertook the last three tasks. From what children stated and shared, we expected that many of them would soon utilise the patterns which they discovered, i.e. progress to the

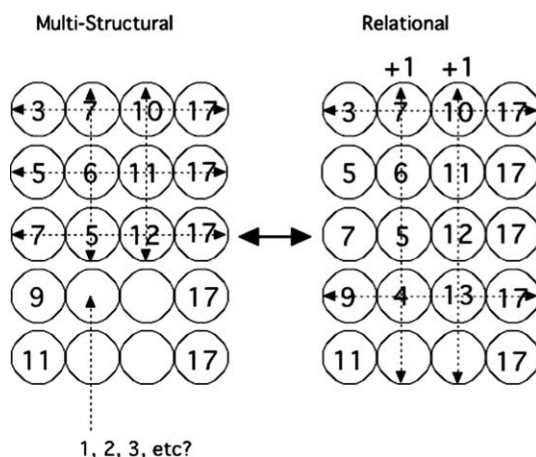
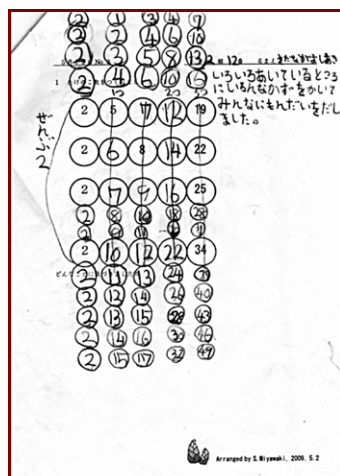
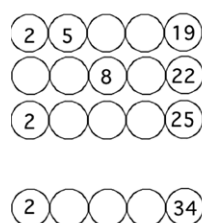


Figure 7. Multi-Structural-2 and early Relational-2 levels.



I added various numbers in the empty spaces, and gave my problems to others.

Figure 8. Pupil R's worksheet.

Relational-2 level of understanding about Bamboo numbers. Not all children, however, used the patterns; several just tried some numbers randomly, i.e. these children could not apply the patterns which they had just shared in the class. At the end of the lesson, the teacher gave a final hint, as described in Figure 9.

Here, the teacher clearly tried to bring children's attention to the vertical and horizontal patterns of Bamboo numbers, and get them to use this to complete the rest of the row ($10-?-?-?-53$), but did this work? By the end of the lesson, 18 of 40 children could complete all patterns, and 7 managed to complete all except the last two, $10-?-?-?-77$ and $10-?-?-?-32$. The others were unable to complete these last two patterns. Considering the age of children and difficulty of the tasks, what we have observed from the lessons does suggest some positive results, i.e. the design and organisation of this series of lessons could encourage some children (62.5% of the class) to advance from Multi-Structural-2 to Relational-2 levels in their understanding of Bamboo numbers. However, since 100% of children could advance to Multi-Structural-2 level, our observation implies that advancement from the Multi-Structural-2 to Relational-2 level is a big step for many children.

Based on this experience, we posed similar tasks in the second main study. This time the teacher focused on encouraging the children to pay attention to the vertical and horizontal relationships of Bamboo numbers. However, while almost all children could complete the first three patterns (Figure 10, left), some of them struggled to complete the pattern $2-?-?-?-34$ and could not use the vertical relationship of the last numbers (increased by 3). Their approach was, in general, to use the space between $2-?-?-?-25$ and $2-?-?-?-34$, and typically they tried 8 or 9, but did not notice the fact that as the difference of the last number is $34-25=9=3\times 3$, the second number should be increased by 3. One child's process of problem solving was summarised in Figure 10, which is evidently an example of a Multi-Structural-2 level of response, i.e. this child can use horizontal and vertical relationships. As this child quickly completed the first three patterns, it can be considered that structural aspects of Bamboo numbers are interiorised and condensed. However, this child is not aware of the relational structure of Bamboo numbers, and cannot use these relationships effectively to solve problems,

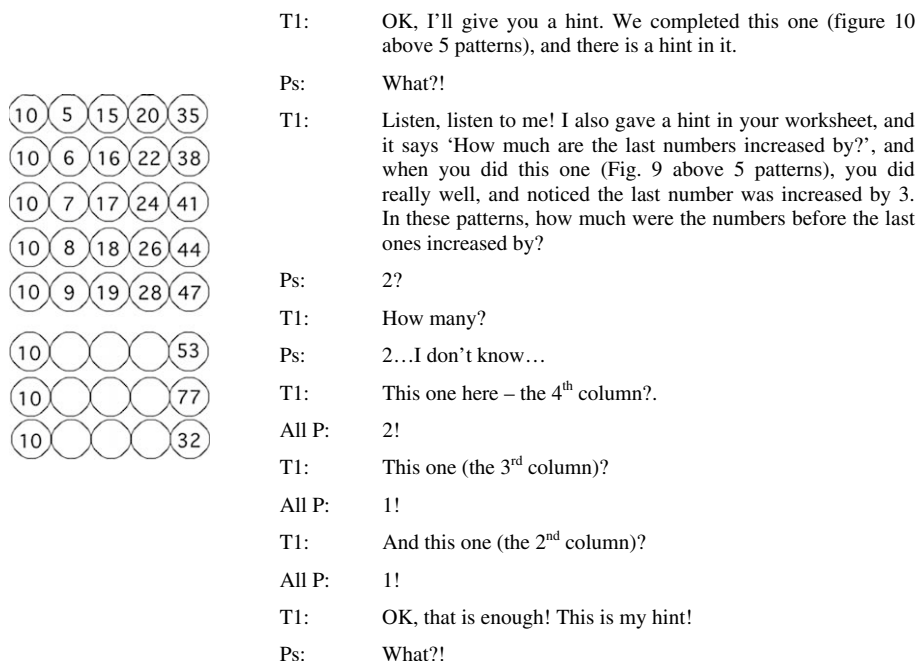


Figure 9. Teacher and children's interaction.

i.e. this child is not in the process of reification. In the process illustrated below, the child went through five stages: Status 1 – starting point; Status 2 – added 5 circles and wrote (1) 6, 7, 8, 9, (2) 7, 9, 10, 11 and (3) 12, 14, 16, 18, 20; Status 3 – (4) deleted 20, and then (5) wrote 28; Status 4 – (6) wrote 10 and 12; Status 5 – (7) deleted 10 and 12 (very confused), and then changed 12 to 17.

At the end of the lesson the class teacher invited Child R to pose an extended problem for the others, which is $2-?-?-?-43$. Again, interactions observed indicate that many of them still did not know how to solve this problem effectively by using the vertical and horizontal patterns which they just had discussed. By the end of the lesson, while all children could manage to complete $2-?-?-?-34$, in the second main study only 8 of 40 children could complete $2-?-?-?-43$. This again indicates how difficult it is for children to grasp the entire structure of Bamboo numbers.

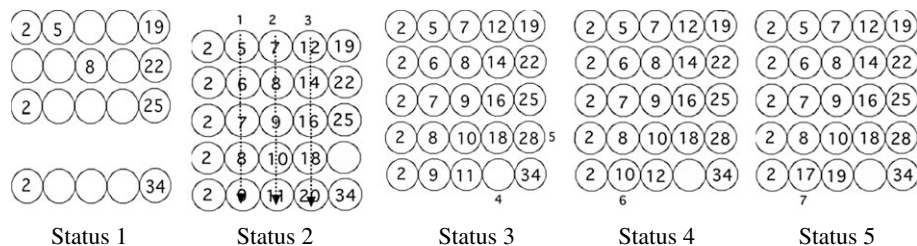


Figure 10. One child's problem solving process.

Discussion

In summary, by way of answer to our Research Question 2, “what activities can we observe among children presented with these tasks?”, children engaged with Bamboo numbers tasks found many mathematical patterns, and extended and created new patterns (evidence exemplified in Figure 3 or 4). In so doing, children experienced the process of interiorisation and condensation as, becoming familiar with various patterns among Bamboo numbers as the lessons progressed. Some of them could *utilise* the relationships to solve more challenging problems (Figures 7 and 8), which suggests that these children were at the very last stage of the second cycle of the SLEs, i.e. Relational-2 level. They also undertook a considerable number of calculations through their trial and error activities, which contributed to the consolidation of their calculation skills and number sense (Figure 5).

The theoretical ideas also provided a useful guide to which level of thinking children had attained, and gave an indication of the criteria for successful mathematical problem solving in whole classroom situations, i.e. reification. In particular, UMR cycles and the process of the development of mathematical concepts offer detailed and useful insights into children’s activities with the SLEs (This is an answer to Research Question 1). It is now clear that the lessons designed with our Bamboo Numbers SLEs and the teacher’s instructions were effective in helping children progress from Uni-Structural-2 to Multi-Structural-2 levels of the second cycle, but the critical area of progression in this SLE was from Multi-Structural-2 to Relational-2 levels, and in this respect the lesson design and teacher’s instructions might not be sufficient to enable children to complete this step. In what follows, we shall extend our discussion by focusing on this issue, in particular children’s difficulties in the process of reification, and consider some ideas for more effective instructional approaches (related to Research Question 3).

As we have seen, e.g. in Figures 3–5, children in our study were often stuck in the process of interiorisation and condensation in the development of their understanding of the structure of Bamboo numbers. It seems, however, that they did not reify the entire structure of Bamboo numbers (e.g. Figures 9 and 10). In the process of reification, it is considered that children should be able to see, either suddenly or gradually, the multi-structure of Bamboo numbers from completely different points of view. If this kind of ‘enlightenment’ had happened to, for example, the child in Figure 10, then s/he could have completed the task without any confusion, but the thought process we can discern from this child’s responses clearly indicates that this is not the case. This reification ‘enlightenment’ for children seems much more complicated than interiorisation and condensation. After the pilot lessons and the first main study, in both of which children could not reach Relational-2 level of understanding of Bamboo numbers, we considered that it would be important to give teachers and children opportunity to reflect on their mathematical activities by considering the effectiveness of instructive supports. In the first main study, this was mainly done by reflective discussions, using observations from the completed Bamboo numbers. In the second main study, considering the teacher’s role in directing children’s attention to structural aspects (Mason 1998; Stylianides and Stylianides 2009), we particularly tried to encourage children to reflect upon what patterns they could recognise. This approach worked to some extent, as many children started noticing vertical patterns and tried to use these to solve problems,

but as the case of Figure 10 suggests, the children's problem solving strategies were far from what we expected. In the end, this kind of 'reflection' did not function as what we considered to be 'reflective abstraction'.

To trigger children's reification process, additional instructional approaches should be considered. One idea is that children can revisit Bamboo number tasks in their higher grades in primary school, and this is possible thanks to the flexible nature of the SLEs. As children experience the UMR cycles of the SLEs again and again, the 'enlightenment' might happen to them and they might suddenly see the whole structure of Bamboo numbers. Also, Sfard (1991) suggests names, symbols, graphs, and other representations (21) may have a role in helping this process, but the use of graphs might be too early for 7 or 8 year-olds. Other representations of Bamboo numbers can be considered, for example, by using more visual representations. For example, Harries and Barmby (2007) and Barmby et al (2009) studied various representations of numbers, and how array representations facilitate children's understanding of multiplication. The manipulation of concrete objects which are closely related to mathematical structure can also encourage learners' reflective abstraction based on the 'operative principle' (Wittmann 1985). One problem is that it is difficult to visualise the structural relationships of Bamboo numbers. Alternatively, the use of technology might help children's thinking process. For example, Ainley, Bills, and Wilson (2005) used a spreadsheet to transform the number pyramids to a more accessible and interesting representation for children. As Bamboo numbers have recurring number patterns, it is possible to design a learning environment for children based on a spreadsheet. In particular, for children who have already been in the process of interiorisation and condensation, it could be a powerful tool which would encourage them to concentrate more on the structural aspects of Bamboo numbers and hence the reification process might be encouraged. In future research, in addition to designing and implementing more lessons based on SLEs, we would like to continue to consider which instructional approaches and representations would particularly contribute to children's reification process in terms of their understanding of Bamboo numbers and other SLEs.

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