

Matrices

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Matrix Types

The following MATRIX has 3 rows & 4 columns:

$$A = \begin{pmatrix} 3 & 4 & 6 & 8 \\ 2 & -1 & 3 & 0 \\ 11 & -2 & -9 & -6 \end{pmatrix} \quad \text{3 rows, 4 Columns (3 x 4)}$$

This matrix A has **ORDER (3x4)** – and so is a 3 x 4 matrix.

The **ELEMENTS** of A are: 3, 4, 6, 8, 2, -1, 3, 0, 11, -2, -9 and -6.

A **ROW MATRIX** has only one row e.g. $B = [61 \quad -11 \quad 43]$

A **COLUMN MATRIX** has only one column e.g. $C = \begin{bmatrix} 18 \\ 24 \\ 36 \end{bmatrix}$

A **NULL MATRIX** has **all** elements = ZERO

e.g. $D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

A **SQUARE MATRIX** has number of rows = number of columns

e.g. $\begin{pmatrix} 2 & -1 \\ 3 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

A **DIAGONAL MATRIX** is a matrix where all elements **not** on the diagonal are zero:

e.g. $\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \quad \text{but not} \quad \begin{pmatrix} 0 & 0 & 4 \\ 0 & 6 & 0 \\ 8 & 0 & 0 \end{pmatrix}$

A **UNIT MATRIX** is a diagonal matrix in which all diagonal elements are 1.

It is denoted as an **I**.

e.g. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{but not} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

1. Addition

To be able to add two matrices together, they must both be of the same order (meaning both the columns and rows of both matrices need to be the same). Add both corresponding elements together.

e.g. (i) $A = \begin{pmatrix} 3 & 5 \\ 6 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 7 \\ 8 & 1 \end{pmatrix}$

$$A + B = \begin{pmatrix} 3+4 & 5+7 \\ 6+8 & 2+1 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 14 & 3 \end{pmatrix}$$

(ii) $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 4 & -5 \\ 1 & 2 & -3 \end{pmatrix}$

$$A + B = \begin{pmatrix} 1+3 & 2+4 & (-3)+(-5) \\ 0+1 & -1+2 & 2-3 \end{pmatrix} = \begin{pmatrix} 4 & 6 & -8 \\ 1 & 1 & -1 \end{pmatrix}$$

(iii) $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

As the order of A $\neq$ order of B, Addition is not possible.

{Note: $A + B = B + A$ }

2. Subtraction

Just like addition, to be able to subtract two matrices, they must both be of the same order (meaning both the columns and rows of both matrices need to be the same). Subtract both corresponding elements together.

e.g. (i) $A = \begin{pmatrix} 8 & 0 & 3 \\ -7 & 5 & 1 \\ -6 & 9 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 6 & 2 & -4 \\ 5 & 3 & 6 \\ -8 & -7 & 8 \end{pmatrix}$

$$A - B = \begin{pmatrix} 8-6 & 0-2 & 3-(-4) \\ -7-5 & 5-3 & 1-6 \\ -6-(-8) & 9-(-7) & -2-8 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 7 \\ -12 & 2 & -5 \\ 2 & 16 & -10 \end{pmatrix}$$

(ii) $B - A = \begin{pmatrix} 6-8 & 2-0 & -4-3 \\ 5-(-7) & 3-5 & 6-1 \\ -8-(-6) & -7-9 & 8-(-2) \end{pmatrix} = \begin{pmatrix} -2 & 2 & -7 \\ 12 & -2 & 5 \\ -2 & -16 & 10 \end{pmatrix}$

{**Note:** $A - B \neq B - A$, as the element signs are reversed. In fact, $A - B = -1*(B - A)$ }

3. Equality

Two matrices are equal if they are of the same order and corresponding elements are equal.

e.g. If $\begin{bmatrix} 3 & y \\ x & -2 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -4 & -2 \end{bmatrix}$ then **x = -4** and **y = -6**.

4. Multiplication by a Scalar

(i.e. a constant)

Multiply each element by the scalar.

e.g. $A = \begin{bmatrix} 4 & 6 \\ -3 & -2 \end{bmatrix}, \quad 3A = \begin{bmatrix} 12 & 18 \\ -9 & -6 \end{bmatrix}, \quad -A = \begin{bmatrix} -4 & -6 \\ 3 & 2 \end{bmatrix}$

5. Multiplication of 2 Matrices

The number of columns in the first matrix must be equal to the number of rows in the second matrix.

e.g. $A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}$

Take each row in turn, with each column and multiply corresponding elements

e.g.
$$AB = \begin{bmatrix} \text{Row1, Column1} & \text{Row1, Column2} \\ (3 * 1) + (1 * 3) + (2 * 2) & (3 * 2) + (1 * 1) + (2 * 6) \\ \text{Row2, Column1} & \text{Row2, Column2} \\ (4 * 1) + (1 * 3) + (3 * 2) & (4 * 2) + (1 * 1) + (3 * 6) \end{bmatrix}$$
$$= \begin{bmatrix} 3 + 3 + 4 & 6 + 1 + 12 \\ 4 + 3 + 6 & 8 + 1 + 18 \end{bmatrix} = \begin{bmatrix} 10 & 19 \\ 13 & 27 \end{bmatrix}$$

In this example, we can also compute BA

e.g.
$$BA = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 4 & 1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} (1 * 3) + (2 * 4) & (1 * 1) + (2 * 1) & (1 * 2) + (2 * 3) \\ (3 * 3) + (1 * 4) & (3 * 1) + (1 * 1) & (3 * 2) + (1 * 3) \\ (2 * 3) + (6 * 4) & (2 * 1) + (6 * 1) & (2 * 2) + (6 * 3) \end{pmatrix}$$
$$= \begin{bmatrix} 3 + 8 & 1 + 2 & 2 + 6 \\ 9 + 4 & 3 + 1 & 6 + 3 \\ 6 + 24 & 2 + 6 & 4 + 18 \end{bmatrix} = \begin{bmatrix} 11 & 3 & 8 \\ 13 & 4 & 9 \\ 30 & 8 & 22 \end{bmatrix}$$

6. Multiplication by the Unit Matrix

e.g. $A = \begin{pmatrix} 2 & 1 & 7 \\ 3 & 5 & -4 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$$AI = \begin{pmatrix} 2 & 1 & 7 \\ 3 & 5 & -4 \end{pmatrix}$$

7. Transposition

Interchange the rows with the columns

e.g. (i) $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad A^T = \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \\ 7 & 8 & 9 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & -4 & 7 \\ 2 & -5 & 8 \\ 3 & -6 & 9 \end{pmatrix}$

{**Note:** easier to swap mirror images through the diagonal.}

8. Determinant of a Square Matrix

For the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is:

$$\det A \text{ or } |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a.d - b.c$$

e.g. $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}, \quad |A| = \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}$

$$\begin{aligned} &= 4 * 3 - 1 * 2 \\ &= 12 - 2 \\ &= 10 \end{aligned}$$

For the matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

$$\det A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and is computed as}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

i.e: whichever element you are considering, blank off the row and the column containing that element and write down the remaining elements as a 2x2 determinant.

(i) e.g.

$$\begin{aligned}
 |A| &= \begin{vmatrix} 1 & 4 & 2 \\ 3 & 8 & 7 \\ 1 & 4 & 6 \end{vmatrix} = 1 \begin{vmatrix} 8 & 7 \\ 4 & 6 \end{vmatrix} - 4 \begin{vmatrix} 3 & 7 \\ 1 & 6 \end{vmatrix} + 2 \begin{vmatrix} 3 & 8 \\ 1 & 4 \end{vmatrix} \\
 &= 1(48 - 28) - 4(18 - 7) + 2(12 - 8) \\
 &= 1 * 20 - 4 * 11 + 2 * 4 \\
 &= 20 - 44 + 8 \\
 &= -16
 \end{aligned}$$

(ii) e.g.

$$\begin{aligned}
 \text{If } B &= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \det B = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \\
 &= 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\
 &= 1(45 - 48) - 2(36 - 42) + 3(32 - 35) \\
 &= -3 - 2(-6) + 3(-3) \\
 &= -3 + 12 - 9 \\
 &= 12 - 12 \\
 &= 0
 \end{aligned}$$

For any element, the determinant obtained by blanking off its row and column is known as the **MINOR** of that element.

When taken with the varying sign $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ for 3x3 determinants then that element is known as a co-factor.

e.g. using B above, for b_{32} , (i.e element (3,2), element at row 3, column 2)

$$\text{minor of } b_{32} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6 - 12 = -6$$

$$\therefore \text{the cofactor of } b_{32} = - \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -(6 - 12) = -(-6) = 6$$

9. The Adjoint of a Square Matrix

The transpose of the matrix of co-factors of each element of A is called the adjoint of A and is denoted by $\text{adj}(A)$.

(i) For a 2x2 matrix, e.g. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The co-factors in this case are the single values left when deleting the row and column of the element. The sign table in this case is $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$

$$\text{Matrix of cofactors} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \therefore \text{adj } A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

e.g. $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$; Matrix of co-factors $\begin{pmatrix} +3 & -2 \\ -1 & +4 \end{pmatrix}$; $\text{adj} = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$;

and for reference, $\det A = \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 12 - 2 = 10$

(ii) For a 3x3 matrix,

$$\text{If } A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}; \text{ Matrix of co-factors} = \begin{bmatrix} + \begin{vmatrix} e & f \\ h & i \end{vmatrix} & - \begin{vmatrix} d & f \\ g & i \end{vmatrix} & + \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ - \begin{vmatrix} b & c \\ h & i \end{vmatrix} & + \begin{vmatrix} a & c \\ g & i \end{vmatrix} & - \begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ + \begin{vmatrix} b & c \\ e & f \end{vmatrix} & - \begin{vmatrix} a & c \\ d & f \end{vmatrix} & + \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

And the adjoint of A is the transpose of the matrix of cofactors:-

$$\text{adj } A = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & - \begin{vmatrix} b & c \\ h & i \end{vmatrix} & + \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ - \begin{vmatrix} d & f \\ g & i \end{vmatrix} & + \begin{vmatrix} a & c \\ g & i \end{vmatrix} & - \begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & - \begin{vmatrix} a & b \\ g & h \end{vmatrix} & + \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

e.g. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$; Matrix of co-factors = $\begin{bmatrix} + \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} & - \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} & + \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} \\ - \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & + \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} \\ + \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} & - \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}$

$$\text{i.e. Matrix of co-factors} = \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix} \quad \therefore \text{adj } A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

10. The Inverse of a Matrix

If A & B are square matrices, and the matrix product $AB = I$ (where I is the unit matrix – often called the identity matrix- then B is called the **inverse** (or reciprocal) of A , denoted A^{-1} thus $AA^{-1} = I$, and it can be shown that $A^{-1} = \frac{\text{adj } A}{\det A}$

$$\text{e.g. (i) } A = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$$

$$\text{Matrix of co-factors} = \begin{pmatrix} +5 & -4 \\ -2 & +3 \end{pmatrix}; \quad \text{adj} = \begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix};$$

$$\det A = \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 15 - 8 = 7 \quad \therefore A^{-1} = \frac{\text{adj } A}{\det A} = \frac{1}{7} * \begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 5/7 & -2/7 \\ -4/7 & 3/7 \end{pmatrix}$$

To check:

$$\begin{aligned} AA^{-1} &= \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 5/7 & -2/7 \\ -4/7 & 3/7 \end{pmatrix} \\ &= \begin{bmatrix} (3 * \frac{5}{7}) + (2 * \frac{-4}{7}) & (3 * \frac{-2}{7}) + (2 * \frac{3}{7}) \\ (4 * \frac{5}{7}) + (5 * \frac{-4}{7}) & (4 * \frac{-2}{7}) + (5 * \frac{3}{7}) \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\text{e.g. (ii) } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}; \text{ Matrix of co-factors} \begin{bmatrix} + \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} & - \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} & + \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \\ - \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} & + \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \\ + \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & - \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix}$$

$$\text{Matrix of co-factors} = \begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix} \quad \therefore \text{adj } A = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix} = 1 \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 3 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \\ &= -7 + 2 + 3 = -2 \end{aligned}$$

$$A^{-1} = \frac{\text{adj } A}{\det A} = \frac{-1}{2} \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

10a. Exercise

If $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 7 \\ 1 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 & 0 \\ 6 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

Find

(i) $A + B$ (ii) $A - B$ (iii) AB (iv) BA (v) $6A$ (vi) $-3B$ (vii) $B - A$

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ -5 & -6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & -5 \\ 2 & 4 & -6 \end{pmatrix}$

Find

(i) $A + B$ (ii) $A - B$ (iii) AB (iv) BA

If $\begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ 21 \end{pmatrix}$

Find a, b

$$[a=2, b=3]$$

Find the inverse of the following matrices:

(i) $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ $\left[\frac{1}{5} \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix} \right]$

(ii) $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ $\left[\begin{pmatrix} 0.3 & -0.1 \\ -0.2 & 0.4 \end{pmatrix} \right]$

(iii) $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$ $\left[\frac{1}{14} \begin{pmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{pmatrix} \right]$

(iv) $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ $\left[\frac{1}{5} \begin{pmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{pmatrix} \right]$

$$(v) \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

11. Matrix solutions for Simultaneous Equations

e.g. Solve the equation

$$3x + 2y = 12 \quad \text{_____} \quad \textcircled{1}$$

$$4x + 5y = 23 \quad \text{_____} \quad \textcircled{2}$$

We can write these equations in matrix form as follows:

$$\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 23 \end{pmatrix}$$

This may be shown by multiplying out the 2 matrices on the LHS of the equation

i.e. row1 * col1 and row2 * col1 (as per sheet 38, item 5)

$$\begin{bmatrix} (3x + 2y) \\ (4x + 5y) \end{bmatrix} = \begin{bmatrix} 12 \\ 23 \end{bmatrix}$$

And since for 2 matrices to be equal, all corresponding elements must be equal, we obtain

$$3x + 2y = 12$$

$$4x + 5y = 23$$

Proving that matrix representation of these equations using matrix multiplication is valid.

Now we shall solve the equations $\textcircled{1}$ and $\textcircled{2}$ given above:

$$\text{Let } A = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad K = \begin{pmatrix} 12 \\ 23 \end{pmatrix}$$

We can represent the matrix equation above as

$$AX = K$$

If we multiply both sides by A^{-1} we get

$$A^{-1}AX = A^{-1}K$$

But $A^{-1}A = I$ (the unit matrix)

$$\therefore IX = A^{-1}K$$

And since $IX = X$ (multiplying a matrix by the unit matrix does not change it)

$$\underline{X = A^{-1}K}$$

This equation is used to solve the simultaneous equations.

11a. Simultaneous Equations

The steps to find the solution of simultaneous equations are:

- i) Create a matrix A from the coefficients of the variables
- ii) Derive $\text{adj } A$
- iii) Evaluate $\det A$
- iv) Evaluate $A^{-1} = \frac{\text{adj } A}{\det A}$
- v) Evaluate $A^{-1}K$ where K is the column matrix of number terms
- vi) Read off the solutions using $X = A^{-1}K$

e.g. ①:

Solve: $3x + 2y = 12$

$$4x + 5y = 23$$

(i) $A = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$

(ii) Matrix of co-factors = $\begin{pmatrix} +5 & -4 \\ -2 & +3 \end{pmatrix}$

$$\text{adj } A = \begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}$$

(iii) $\det A = \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 15 - 8 = 7$

(iv) $A^{-1} = \frac{1}{7} \begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix}$

(v) $A^{-1}K = \frac{1}{7} \begin{pmatrix} 5 & -2 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 12 \\ 23 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 5 * 12 & -2 * 23 \\ -4 * 12 & +3 * 23 \end{pmatrix}$
$$= \frac{1}{7} \begin{pmatrix} 60 & -46 \\ -48 & +69 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 14 \\ 21 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

(vi) $X = A^{-1}K$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \therefore x = 2, y = 3$$

11b. Matrix Solutions

e.g. ②:

Solve:

$$4x + y + 2z = 4 \quad \text{_____} \quad \textcircled{1}$$

$$2x - y - 4z = 4 \quad \text{_____} \quad \textcircled{2}$$

$$3x + 8y - z = 20 \quad \text{_____} \quad \textcircled{3}$$

$$(i) \quad A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & -1 & -4 \\ 3 & 8 & -1 \end{pmatrix}$$

(ii) matrix of co-factors

$$\begin{aligned} &= \begin{pmatrix} + \begin{vmatrix} -1 & -4 \\ 8 & -1 \end{vmatrix} & - \begin{vmatrix} 2 & -4 \\ 3 & -1 \end{vmatrix} & + \begin{vmatrix} 2 & -1 \\ 3 & 8 \end{vmatrix} \\ - \begin{vmatrix} 1 & 2 \\ 8 & -1 \end{vmatrix} & + \begin{vmatrix} 4 & 2 \\ 3 & -1 \end{vmatrix} & - \begin{vmatrix} 4 & 1 \\ 3 & 8 \end{vmatrix} \\ + \begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix} & - \begin{vmatrix} 4 & 2 \\ 2 & -4 \end{vmatrix} & + \begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 33 & -10 & 19 \\ 17 & -10 & -29 \\ -2 & 20 & -6 \end{pmatrix}, \quad \therefore adj = \begin{pmatrix} 33 & 17 & -2 \\ -10 & -10 & 20 \\ 19 & -29 & -6 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (iii) \quad \det A &= \begin{vmatrix} 4 & 1 & 2 \\ 2 & -1 & -4 \\ 3 & 8 & -1 \end{vmatrix} \\ &= 4(1 - (-32)) - 1(-2 - (-12)) + 2(16 - (-3)) \\ &= 4(33) - 1(10) + 2(19) = 160 \end{aligned}$$

$$(iv) \quad A^{-1} = \frac{adj A}{\det A} = \frac{1}{160} \begin{pmatrix} 33 & 17 & -2 \\ -10 & -10 & 20 \\ 19 & -29 & -6 \end{pmatrix}$$

$$(v) \quad A^{-1}K = \frac{1}{160} \begin{pmatrix} 33 & 17 & -2 \\ -10 & -10 & 20 \\ 19 & -29 & -6 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix} = \frac{1}{160} \begin{pmatrix} 160 \\ 320 \\ -160 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$(vi) \quad X = A^{-1}K, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\therefore x = 1, y = 2, z = -1$$

11c. Exercise

Solve:

$$\begin{array}{ll} \text{(i)} & 7x - 4y = 37 \\ & 6x + 3y = 51 \end{array} \qquad [x = 7, y = 3]$$

$$\begin{array}{ll} \text{(ii)} & 2x + 7y = 11 \\ & 5x + 3y = 13 \end{array} \qquad [x = 2, y = 1]$$

$$\begin{array}{ll} \text{(iii)} & x - y + 4z = 5 \\ & 2x + 3y + z = 1 \\ & x + 6y - 2z = 0 \end{array} \qquad [x = -2, y = 1, z = 2]$$

$$\begin{array}{ll} \text{(iv)} & 2x + y - z = 1 \\ & 5x + 2y - 2z = 3 \\ & 3x + 4y - 3z = 2 \end{array} \qquad [x = 1, y = 2, z = 3]$$

$$\begin{array}{ll} \text{(v)} & 2x + y + z = 7 \\ & 3x - y + 3z = 16 \\ & 2x - 3y + 4z = 17 \end{array} \qquad [x = 3, y = -1, z = 2]$$