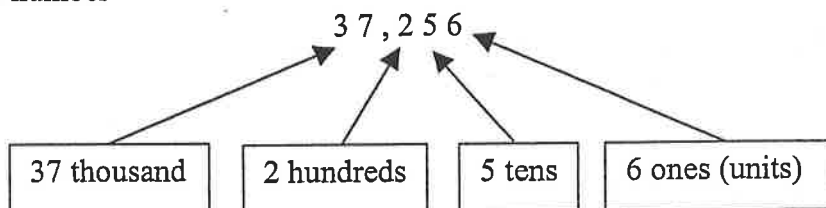


FOUNDATION MATHEMATICS MODULE

ACHA 3000

1. Number System

Consider the number



This can also be written by using column headings:

Ten Thousands	Thousands	Hundreds	Tens	Units
3	7	2	5	6

You will need to be able to read and write numbers in both numerical and word form. An important point to remember is that a million has 6 zeros.

Examples

1. 835 **Reads:** Eight hundred and thirty five
2. 57,325 **Reads:** Fifty Seven thousand, three hundred and twenty five.
3. Five thousand, three hundred and twenty **Numerical Form:** 5,020
4. Seventy five thousand and fifty two **Numerical Form:** 70,052

Using Columns Headings can be very helpful when writing numbers as words or vice versa:

Examples

5.

Thousands	Hundreds	Tens	Units
2	6	2	1

We have Two thousands
 Six Hundreds
 Two Tens i.e. Twenty
 One Unit

So the number we have is **Two thousand, six hundred and twenty one.**

6.

Ten Thousands	Thousands	Hundreds	Tens	Units
4	2	6	0	3

We have Four ten thousands, i.e. Forty thousand
 Two thousand
 Six Hundred
 Zero Tens
 Three Units

So the number we have is **Forty two thousand, six hundred and three.**

7. Write the number **seven hundred thousand six hundred and fifty four** in numerical form:

Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units
7	0	0	6	5	4

So the number is **700,654**.

Exercise

- 1) Write these numbers as words:
 - a) 8000
 - b) 33 200
 - c) 503 000
 - d) 503 100
 - e) 503 010
 - f) 503 001
 - g) 800 500 300
 - h) 909 606 303
- 2) Write these numbers in numerical form:
 - a) Five thousand
 - b) Five thousand, two hundred and thirty three
 - c) Seventy seven thousand, six hundred and five
 - d) Three hundred thousand and one
 - e) Ninety two million and three thousand

2. Mental Arithmetic

2.1. Addition

You should already know how to add numbers. If you can't remember how then here are some examples that will remind you:

Example

$$\begin{array}{r} 1\ 3 \\ +\ 2\ 1 \\ \hline 3\ 4 \end{array}$$

Method:

- 1) $3 + 1 = 4$
- 2) $1 + 2 = 3$

Example

$$\begin{array}{r} 2\ 6 \\ +\ 9\ 6 \\ \hline 1\ 2\ 2 \\ \hline 1 \end{array}$$

Method:

- 1) $6 + 6 = 12$
- 2) Write down 2 and carry 1
- 3) $2 + 9 + 1 = 12$

2.2. Subtraction

You should also be familiar with the method used for subtraction. Here are 2 examples that illustrate it:

Example

$$\begin{array}{r} 4\ 7 \\ -\ 2\ 5 \\ \hline 2\ 2 \end{array}$$

Method:

- 1) $7 - 5 = 2$
- 2) $4 - 2 = 2$

Example

$$\begin{array}{r} \cancel{10}^5\ 13 \\ -\ 4\ 5 \\ \hline 1\ 8 \end{array}$$

Method:

- 1) 3 is smaller than 5
- 2) $13 - 5 = 8$
- 3) $5 - 4 = 1$

2.3. Multiplication

Multiplying a number by 10 is easy because all the digits move one place to the left:

Examples

$$10 \times 36 = 360$$

$$10 \times 520 = 5,200$$

$$10 \times 1,100 = 11,000$$

- Multiplying by 100 moves all the digits two places to the left.
- Multiplying by 1000 moves all the digits three places to the left.
- Etc.

Examples

$$100 \times 52 = 5200$$

$$1000 \times 62 = 62,000$$

$$10,000 \times 6 = 60,000$$

We can also handle slightly harder problems:

Examples

Calculate 300×4

Ans: We know $3 \times 4 = 12$
So $300 \times 4 = 1200$

Calculate 5000×40

Ans: We know $5 \times 40 = 200$
So $5000 \times 40 = 200,000$

2.4. Long Multiplication

Example 1

$$\begin{array}{r} 23 \\ \times 3 \\ \hline 69 \end{array}$$

Method:

- 1) $3 \times 3 = 9$
- 2) $3 \times 2 = 6$

Example 2

$$\begin{array}{r} 23 \\ \times 40 \\ \hline 920 \\ 1 \end{array}$$

Method:

- 1) Write down 0
- 2) $4 \times 3 = 12$
- 3) Write down 2 and carry 1
- 4) $4 \times 2 = 8$
- 5) $8 + 1 = 9$

Example 3

$$\begin{array}{r} 52 \\ \times 43 \\ \hline 156 \\ 2080 \\ \hline 22136 \end{array}$$

Method:

- 1) $3 \times 2 = 6$
- 2) $3 \times 5 = 15$
- 3) Write down 0
- 4) $4 \times 2 = 8$
- 5) $4 \times 5 = 20$
- 6) Calculate $156 + 2080$

Example 4

$$\begin{array}{r} 78 \\ \times 53 \\ \hline 234 \\ 3900 \\ \hline 41134 \end{array}$$

Method:

- 1) $3 \times 8 = 24$
- 2) Write down 4 and carry the 2
- 3) $3 \times 7 = 21$
- 4) $21 + 2 = 23$
- 5) Write down 0
- 6) $5 \times 8 = 40$
- 7) Write down 0 carry the 4
- 8) $5 \times 7 = 35$
- 9) $35 + 4 = 39$
- 10) $234 + 3900 = 4134$

2.5. Short Division

Example

Calculate $861 \div 3$

$$\begin{array}{r} 287 \\ 3 \overline{) 861} \end{array}$$

Method

- 1) 3 into 8 goes twice with 2 left over.
- 2) 3 into 26 goes 8 times with 2 left over.
- 3) 3 into 21 goes 7 times with 0 left over.

Example

Calculate $123 \div 7$

$$\begin{array}{r} 17 \text{ r}4 \\ 7 \overline{) 123} \end{array}$$

Method

1. 7 doesn't go into 1.
2. 9 into 12 goes once with 5 left over.
3. 7 into 53 goes 7 times with 4 left over.

2.6. Long Division

Example

Calculate $325 \div 21$ using long division.

$$\begin{array}{r} 15 \text{ r}10 \\ 21 \overline{) 325} \\ \underline{21} \\ 115 \\ \underline{105} \\ 10 \end{array}$$

Method:

- 1) 21 doesn't go into 3.
- 2) 21 goes into 32 once.
- 3) $1 \times 21 = 21$.
- 4) $32 - 21 = 11$.
- 5) Bring down the 5.
- 6) 21 goes into 115 five times
- 7) $5 \times 21 = 105$
- 8) $115 - 105 = 10$

Example

Calculate $702 \div 33$ using long division.

$$\begin{array}{r} 21 \text{ r}9 \\ 33 \overline{) 702} \\ \underline{66} \\ 42 \\ \underline{33} \\ 9 \end{array}$$

Method:

- 1) 33 doesn't go into 7.
- 2) 33 goes into 70 twice.
- 3) $2 \times 33 = 66$.
- 4) $70 - 66 = 4$.
- 5) Bring down the 2.
- 6) 33 goes into 42 once.
- 7) $1 \times 33 = 33$
- 8) $42 - 33 = 9$

3. Using Decimals

Doing calculations using numbers that involve decimal points is no harder than earlier as long as you remember to keep the decimal point in the same place.

3.1. Simple Calculations

$$\begin{array}{llll} 10 \times 3.653 & = & 36.53 & \text{[Everything moves 1 place to the left]} \\ 100 \times 3.653 & = & 365.3 & \text{[Everything moves 2 places to the left]} \\ 1000 \times 3.653 & = & 3653 & \text{[Everything moves 3 places to the left]} \\ 10,000 \times 3.653 & = & 36,530 & \text{[Everything moves 4 places to the left]} \end{array}$$

$$\begin{array}{llll} 370 \div 10 & = & 37 & \text{[Everything moves 1 place to the right]} \\ 370 \div 100 & = & 3.7 & \text{[Everything moves 2 places to the right]} \\ 370 \div 1000 & = & 0.37 & \text{[Everything moves 3 places to the right]} \\ 370 \div 10000 & = & 0.037 & \text{[Everything moves 4 places to the right]} \end{array}$$

Multiplying by 0.1 moves everything 1 place to the right.

Multiplying by 0.01 moves everything 2 places to the right.

Multiplying by 0.001 moves everything 3 places to the right.

Examples

$$\begin{array}{llll} 0.1 \times 360 & = & 36 & \\ 0.01 \times 360 & = & 3.6 & \end{array} \qquad \begin{array}{llll} 0.001 \times 360 & = & 0.36 & \\ 0.0001 \times 360 & = & 0.036 & \end{array}$$

Slightly harder examples include:

Examples

$$\begin{array}{llll} \text{Calculate} & 0.2 \times 5 & & \\ \text{We know} & 2 \times 5 & = & 10 \\ \text{So} & 0.2 \times 5 & = & 1 \end{array} \qquad \begin{array}{llll} \text{Calculate} & 0.003 \times 7 & & \\ \text{We know} & 3 \times 7 & = & 21 \\ \text{So} & 0.003 \times 7 & = & 0.021 \end{array}$$

3.2. Addition

Examples

1. Add $2.7 + 3.9 + 14.65$

$$\begin{array}{r} 2.70 \\ 3.90 \\ 14.65 \\ \hline 21.25 \\ \hline \end{array}$$

2. Add $2.95 + 3.7 + 13$

[Note that 13 can be written 13.00 and 3.7 can be written 3.70]

$$\begin{array}{r} 2.95 \\ 3.70 \\ 13.00 \\ \hline 19.65 \\ \hline \end{array}$$

3.3. Multiplication

Examples

$$\begin{array}{r} 50.2 \\ \times \quad 3 \\ \hline 150.6 \end{array}$$

$$\begin{array}{r} 3.64 \\ \times \quad 5 \\ \hline 18.20 \\ \quad 3 \quad 2 \end{array}$$

3.4. Division

When we do divisions with decimals we keep on going until there is no remainder left. If necessary we can add extra zeros onto the end of the number as long as we are to the right of the decimal point.

Example

3.96 could be written as 3.960
or 3.9600
or 3.96000 etc.

Examples

1) Work out $5.36 \div 4$

Ans:
$$4 \overline{) 5.36} = 1.34$$

2) Work out $96.2 \div 5$

Ans:
$$5 \overline{) 96.20} = 19.24$$

Method: 5 goes into 9 once with 4 left over.
5 goes into 46 nine times with 1 left over.
5 goes into 12 twice with 2 left over.
We now need to add an extra 0!!!!
5 goes into 20 four times exactly.
Since there is nothing left over we are finished.

3) Work out $1.0 \div 4$

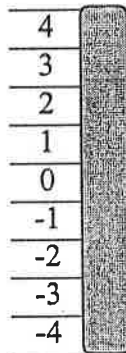
Ans:
$$4 \overline{) 1.00} = 0.25$$

Method: 4 goes into 1 zero times with 1 left over.
4 goes into 10 twice with 2 left over.
We now need to add an extra 0!!!
4 goes into 20 five times exactly.
Since there is nothing left over we are finished.

4. Negative Numbers

4.1. Introduction

You will probably have seen negative numbers when looking at thermometers or perhaps even your bank balance! If the temperature falls below 0 then we have negative temperatures. If your bank balance falls below 0 then you have a negative balance.



Exercise

- 1) Put these temperatures in order (lowest first)
 - a) 2°C , -2°C , -5°C
 - b) -7°C , -4°C , -10°C
- 2) What is the difference in temperature between
 - a) 3°C and -4°C
 - b) 12°C and -5°C
 - c) -1°C and -13°C
 - d) 1°C and -7°C

Note that if you are taking away a minus number you are actually adding the number. This is because **TWO NEGATIVES MAKE A POSITIVE!!!**

- 3) Calculate
 - a) $3 - (-4)$
 - b) $3 + (-4)$
 - c) $-5 - 7$
 - d) $-5 - (-7)$
 - e) $-5 + (-7)$

4.2. Multiplying Negative Numbers

To work out multiplication problems involving negative numbers is no harder than with positive numbers. You simply ignore the signs and do the calculation as normal. Then use the following table to work out what sign your answer should be:

+	$\times$	+	=	+ve answer
+	$\times$	-	=	-ve answer
-	$\times$	+	=	-ve answer
-	$\times$	-	=	+ve answer

Examples

1) Calculate 5×-3

We know $5 \times 3 = 15$

From the table our answer should be negative.

So $5 \times -3 = -15$

2) Calculate -7×3

We know $7 \times 3 = 21$

From the table our answer should be negative.

So $-7 \times 3 = -21$

3) Calculate -5×-6

We know $5 \times 6 = 30$

From the table our answer should be positive.

So $-5 \times -6 = 30$

Note: When the signs are different the answer is negative, but when they are the same the answer is positive.

4.3. Dividing Negative Numbers

To do these calculations first ignore the signs and then use the following table to work out what sign your answer should take:

+	÷	+	=	+ve answer
+	÷	-	=	-ve answer
-	÷	+	=	-ve answer
-	÷	-	=	+ve answer

Example

1) Work out $-9 \div 3$

We know $9 \div 3 = 3$

From the table the answer should be negative.

So $-9 \div 3 = -3$

2) Work out $-10 \div -5$

We know $10 \div 5 = 2$

From the table the answer should be positive.

So $-10 \div -5 = 2$

4.4. Using a Calculator with Negative Numbers

To enter negative numbers on your calculator you need to press the +/- key on your calculator. Whether you need to press this button before or after you enter the number will depend on your calculator.

Exercise

Redo all the exercises and examples in this section using a calculator.

5. Rounding Off

5.1. Nearest 10, 100, 1000, etc.

Examples

37 to the nearest 10 is 40	--- 37 is closer to 40 than 30.
537 to the nearest 100 is 500	--- 537 is closer to 500 than 600.
537 to the nearest 10 is 540	--- 537 is closer to 540 than 530.

If we are **exactly** half way between two numbers the convention is to **round up!!**

3350 to the nearest 100 is 3400	--- 3350 is half way between 3300 and 3400
7500 to the nearest 1000 is 8000	--- 7500 is half way between 7000 and 8000

Exercise

- 1) Write these numbers to the nearest 10:
a) 4624 b) 624 c) 58
- 2) Write these numbers to the nearest 100:
a) 4624 b) 624 c) 58
- 3) Write these number to the nearest 1000:
a) 4624 b) 4500 c) 850

5.2. Decimal Places

Writing numbers to a certain number of decimals is easy as long as you remember the following 3 steps:

Imagine you wanted to write a number to 3 decimal places.

- 1) Locate the 3rd decimal place.
- 2) Look at the next digit to the RIGHT.
- 3) If it's 5 or more, then we **ROUND UP** the 3rd decimal place. If it's 4 or less then we leave the 3rd decimal place as it is.

Example

- 1) Write the following to 3 decimal places:
 - a) 7.432543
The number in the 3rd decimal place is 2.
In the 4th decimal place we have 5, which of course is **5 or more**.
We round up the 2 to 3
Hence we have 7.433 (3 d.p.)
 - b) 0.001232
The number in the 3rd decimal place is 1.
In the 4th decimal place we have 2, which is **4 or less**.
We leave the 1 as it is.
Hence we have 0.001 (3 d.p.)

- 2) Consider the number 45.319461
- To 5 decimal places 45.31946 (5 d.p.)
 - To 4 decimal places 45.3195
 - To 3 decimal places 45.319
 - To 2 decimal places 45.32
- 3) Write the following numbers to 4 decimal places.
- 4.3014643 **Ans:** 4.3015 (4 d.p.)
 - 10.210081 **Ans:** 10.2101 (4 d.p.)
 - 11.234956

The number in the 4th decimal place is 9.
 In the 5th decimal place we have 5, which is **5 or more**.
 We round the 9 up to 10.
 Write down 0 carry the 1. $1 + 4 = 5$
 Hence we have 11.2350 (4 d.p.)

5.3. Significant Figures

The 1st significant figure of any number is the first digit which isn't zero.

Example

The 1st significant figure of 3.562 is 3
 The 1st significant figure of 0.023 is 2

The 2nd, 3rd, 4th, etc. significant figures follow immediately from the 1st **regardless of whether they are zero or not**.

Example

Consider 0.054032

The 1st significant figure is 5
 The 2nd significant figure is 4
 The 3rd significant figure is 0
 The 4th significant figure is 3
 The 5th significant figure is 2

To write numbers to a certain number significant figures you follow the following 3 steps:

Imagine you wanted to write a number to 4 significant figures.

- 1) Locate the 4th significant figure.
- 2) Look at the next digit to the RIGHT.
- 3) If it's 5 or more, then we **ROUND UP** the 4th significant figure. If it's 4 or less then we leave the 4th significant figure as it is.

Example

	4 SF	3 SF	2 SF	1 SF
1) 54.7651	54.77	54.8	55	50
2) 17.0067	17.01	17.0	17	20
3) 0.0045902	0.004590	0.00459	0.0046	0.005
4) 30895.4	30900	30900	31000	30000

6. BODMAS

Consider the calculation $4 + 2 \times 5$

By using brackets we can change what the answer is:

$$(4 + 2) \times 5 = 6 \times 5 = 30 \quad \text{OR} \quad 4 + (2 \times 5) = 4 + 10 = 14$$

But which one is $4 + 2 \times 5$? Is it $(4 + 2) \times 5$ OR $4 + (2 \times 5)$?

To answer this question we need to know the BODMAS rule:

Brackets Or Division and Multiplication then Addition and Subtraction

So in $4 + 2 \times 5$ we do the multiplication before the addition:

$$4 + 2 \times 5 = 4 + 10 = 14$$

Example

1) Calculate

a) $4 + (8 \div 2)$

b) $(4 + 8) \div 2$

c) $4 + 8 \div 2$

Answer:

a. Brackets first: $4 + (8 \div 2) = 4 + 4 = 8$

b. Brackets first: $(4 + 8) \div 2 = 12 \div 2 = 6$

c. Division first: $4 + 8 \div 2 = 4 + 4 = 8$

Now consider $\frac{10}{2+3}$

When you need to work out fractions you should

- **work out the top**
- **work out the bottom**
- **divide your answers**

So for the above:

Top	=	10	Bottom	=	$2 + 3$
				=	5
Answer	=	$10 \div 5$			
	=	2			

Example

$$\frac{12+3}{5-2}$$

$$\text{Top} = 12 + 3 = 15$$

$$\text{Bottom} = 5 - 2 = 3$$

$$\text{Answer} = 15 \div 3 = 5$$

It is also possible to work out such problems by using brackets. We use brackets to tell the calculator that we want to divide **all of the top** by **all of the bottom**. For the above example we would enter:

$$(1 2 + 3) \div (5 - 2) = 5$$

If you didn't enter brackets then because your calculator knows the BODMAS rule it would do division first and work out:

$$12 + \frac{3}{5} - 2 = 12 + 0.6 - 2 = 10.6$$

But this is not what we wanted so you **must use brackets**.

Example

1. Use the calculator to calculate the following:

a) $\frac{54}{11+7}$

b) $\frac{5 \times 28}{22+18}$

c) $\frac{13.6 - 5.1}{7 - 2}$

d) $(3.2 - 1.5) \times (4.1 - 1.6)$

2. Write down the keys you would press to calculate $\frac{5+7}{6-2}$

7) Estimating

Imagine you wanted to get an estimate for

$$\frac{56.5 \times 3}{17.3 + 42.5}$$

First write each of the numbers to 1 significant figure:

17.3	→	20
42.5	→	40
56.5	→	60
3	→	3

Now

$$\frac{56.5 \times 3}{17.3 + 42.5} \approx \frac{60 \times 3}{20 + 40} = \frac{180}{60} = \frac{18}{6} = 3$$

So we estimate the answer to be 3.

The symbol $\approx$ means **approximately equal to**.

Example

Estimate $11.52 + 13.2 + 37.35$

Round each number to 1 significant figure

11.52	→	10
13.2	→	10
37.35	→	40

So $11.52 + 13.2 + 37.35 \approx 10 + 10 + 40 = 60$

Example

Estimate $135.23 \times 25.36 \times 37.52$

Round each number to 1 significant figure:

135.23	→	100
25.36	→	30
37.52	→	40

So $135.23 \times 25.36 \times 37.52 \approx 100 \times 30 \times 40 = 3000 \times 40 = 120000$

8. Factors and Prime Factors

8.1. Multiples

The **multiples** of a number are just its multiplication table. For example the multiples of 15 are 15 30 45 60 75 90 105 etc...

Exercise

1. Write down six multiples of 13.
2. Which of the following are multiples of 3: 12 13 14 15

8.2. Factors

The **factors** of a number are all those numbers that divide the number exactly, i.e. leave no remainder.

Example

The factors of 12 are 1 2 3 4 6 12

To find the factors of 12 consider which numbers multiply together to make 12:

$$1 \times 12 = 12 \qquad 2 \times 6 = 12 \qquad 3 \times 4 = 12$$

Note that for any even number 2 is **always** a factor.

8.3. Prime Factors

You should notice that for any number

- 1 is a factor of the number
- The number itself is a factor

Some numbers such as 7 only have these factors. That is 7 only has the factors 1 and 7. We call such number **prime numbers**. Note that 0 and 1 are **NOT** prime numbers!!!

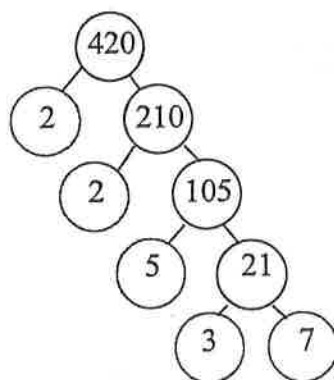
Examples

3, 5, 7, 11 are all prime numbers.

Any number can be written as a product of its prime factors by a factor tree:

Example

Write 420 as a product of its prime factors:



So we can write $420 = 2 \times 2 \times 5 \times 3 \times 7$

Exercises

1. Write the following as a product of their prime factors:

a. 990

b. 160

8.4. Lowest Common Multiple (LCM)

The lowest common multiple of a group of numbers is the smallest number that they all divide. To find the lowest common multiple following the steps below:

- Write down the multiples of each number.
- The smallest number that occurs in all lists is the lowest common multiple.

Example

1. Find the lowest common multiple of 2 and 3:

Multiples of 2 are:	2	4	6	8	10	12	14	...
Multiples of 3 are:	3	6	9	12	15	18	21	...

The LCM of 2 and 3 is 6

2. Find the lowest common multiple of 2 and 6:

Multiples of 2 are:	2	4	6	8	10	12	14	...
Multiples of 6 are:	6	12	18	24	30	36	42	...

The LCM of 2 and 6 is 6.

Exercise

1. Find the LCM of 8 and 9.

2. Find the LCM of 7 and 9.

8.5. Highest Common Factor (HCF)

The highest common factor of a group of numbers is the biggest number that divides all of them exactly. To find the highest common factor just follow the steps below:

- Write down the factors of each of the numbers.
- The biggest number that occurs in all lists is the highest common factor.

Example

Find the HCF of 36, 54 and 72:

Factors of 36 are:	1	2	3	4	6	9	12	18	36
Factors of 54 are:	1	2	3	6	9	18	27	54	
Factors of 72 are:	1	2	3	4	6	8	9	12	18

So the HCF of 36, 54 and 72 is 18.

Exercises

Find the HCF of

1. 10 and 12

2. 15 and 30

3. 10 and 1

9. Fractions

9.1. Kinds of Fractions

If a cake is cut into 8 pieces and you eat 2, then you've eaten $\frac{2}{8}$ of the cake.

The bottom number in a fraction is called the **denominator**.

The top number in a fraction is called the **numerator**.

The **denominator** tells us how many equal pieces the whole is divided into whilst the **numerator** tells us how many pieces we are considering. If the **numerator** is bigger than the **denominator** we say the fraction is top heavy.

Example

$\frac{11}{10}$ is top heavy. The whole is split into 10 equal pieces and we have taken 11 of them. This means we have **1 whole** and $\frac{1}{10}$. Hence we can write $\frac{11}{10} = 1\frac{1}{10}$.

Example

Write $6\frac{3}{4}$ as a top heavy fraction.

Here we have 6 wholes and 3 quarters. How many quarters is this in total?

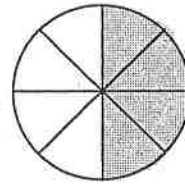
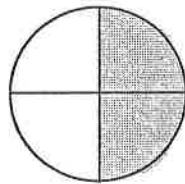
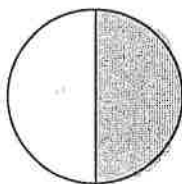
Each whole is 4 quarters and we have 6 wholes $\rightarrow 6 \times 4 = 24$ quarters.

Hence in total we have $24 + 3 = 27$ quarters.

So $6\frac{3}{4} = \frac{27}{4}$

9.2. Equivalent Fractions

Consider if I gave you $\frac{1}{2}$ a cake. You may say you've been given $\frac{2}{4}$ or even $\frac{4}{8}$ of the cake.



We say $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$ are **equivalent fractions** because they represent the same amount. To find equivalent fractions we multiply the top and bottom of a fraction by the same number.

Example

Find two equivalent fractions of $\frac{1}{4}$

Multiplying top and bottom by 2 yields: $\frac{2}{8}$

Multiplying top and bottom by 3 yields: $\frac{3}{12}$

9.3. Adding Fractions

It is very easy to add fractions if they have the **SAME** denominator:

Example

$$\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$$

But how would you work out $\frac{1}{5} + \frac{2}{3}$?

The idea is to replace the fractions by equivalent fractions that have the same denominator. We use the following method:

LCM of 5 and 3 is 15 --- so we look for equivalent fractions with 15 in the denominator.

$$\frac{1}{5} = \frac{3}{15} \quad (\text{Multiplying top and bottom by 3})$$

$$\frac{2}{3} = \frac{10}{15} \quad (\text{Multiplying top and bottom by 5})$$

$$\text{Hence } \frac{1}{5} + \frac{2}{3} = \frac{3}{15} + \frac{10}{15} = \frac{13}{15}$$

Exercise

Calculate the following without the use of a calculator:

$$1. \frac{1}{2} + \frac{4}{2} \qquad 2. \frac{1}{3} + \frac{2}{6} \qquad 3. \frac{1}{7} - \frac{1}{8} \qquad 4. \frac{2}{10} + \frac{3}{4}$$

9.4. Multiplying and Dividing Fractions

Multiplying fractions is easier still. You simply multiply the numerators and then multiply the denominators:

Example

$$\frac{4}{2} \times \frac{2}{5} = \frac{8}{10}$$

People think that dividing fractions is more difficult, but all you do is turn the fraction you are dividing by **upside down** and then multiply instead.

Example

$$\text{To calculate } \frac{5}{8} \div \frac{3}{6} \qquad \rightarrow \qquad \text{We work out } \frac{5}{8} \times \frac{6}{3} = \frac{30}{24} = 1\frac{6}{24}$$

CAUTION: MAKE SURE IT IS THE FRACTION YOU ARE DIVIDING BY THAT YOU TURN UPSIDE DOWN!!!!!!!

Exercise

1. $\frac{2}{3} \times \frac{5}{8}$

2. $\frac{4}{13} \div \frac{2}{5}$

3. $\frac{1}{3} \div \frac{2}{3}$

4. $\frac{2}{3} \times \frac{4}{3} \div \frac{1}{5}$

9.5. Changing Fractions into Decimals

To change a fraction into a decimal you simply **divide the top by the bottom**.

Example

To change $\frac{3}{5}$ into a decimal we work out $3 \div 5 = 0.6$

So $\frac{3}{5} = 0.6$

Example

To change $4\frac{1}{5}$ into a decimal we do the following:

- $\frac{1}{5}$ as a decimal is given by $1 \div 5 = 0.2$
- Now add the 4 wholes to it $4 + 0.2 = 4.2$

So $4\frac{1}{5} = 4.2$

Exercise

Write the following fractions as decimals:

1. $\frac{3}{5}$

2. $\frac{4}{10}$

3. $\frac{10}{20}$

9.6. Calculating a Fraction of a Quantity

Imagine you wanted to work out $\frac{3}{4}$ of £6 to do this we calculate $\frac{3}{4} \times 6$

$$\frac{3}{4} \times \frac{6}{1} = \frac{18}{4} = 4.5 = \text{£}4.50$$

Exercise

1. Calculate $\frac{5}{8}$ of £24

2. Calculate $\frac{3}{5}$ of 10

What if you wanted to calculate $3\frac{3}{4}$ of £6? Do it in stages:

- Calculate $\frac{3}{4}$ of £6 $\frac{3}{4} \times 6 = 4.5$
- Calculate 3 lots of £6 $3 \times 6 = 18$
- So $3\frac{3}{4}$ of £6 will be $18 + 4.5 = \text{£}22.50$

Exercise

1. Calculate $2\frac{3}{8}$ of £24

2. Calculate $7\frac{4}{10}$ of £10

10. Percentages

10.1. Finding Percentages 1

Percentage means **per hundred**.

Example

$$30\% \text{ means } \frac{30}{100}$$

$$55\% \text{ means } \frac{55}{100}$$

$$79\% \text{ means } \frac{79}{100}$$

To work out 30 % of 200 you do the following:

$$30\% \text{ of } £200 \text{ is } \frac{30}{100} \times 200 = 60$$

Exercise

Calculate

1. 37% of £600

2. 40% of £1.50

3. 93% of 1200

Example

My car is worth £7000 when I purchase it. It's value falls by 10% each year. What will it be worth after one year?

$$\begin{aligned} \text{Ans: } 10\% \text{ of } £7000 &= \frac{10}{100} \times 7000 = £700 \\ \text{Value after 1 year} &= 7000 - 700 \\ &= £6300 \end{aligned}$$

Exercise

If my house is worth £70,000 and over a period of time its value increase by 20%. What will it now be worth?

10.2. Comparing Fractions, Percentages and Decimals

We know how to change fractions into decimals, but it is just as easy to change decimals into percentages.

To change decimals into percentages you MULTIPLY by 100

To change percentages into decimals you DIVIDE by 100.

Example

1. What is 0.36 as a percentage?
 $0.36 \times 100 = 36\%$

2. What is 59% as a decimal?
 $59 \div 100 = 0.59$

Exercise

1. Convert 45%, 73% and 22% into decimals
2. Convert 0.55, 0.88 and 0.01 into percentages

Hence we can now change fractions into decimals and then into percentages. When you are asked to compare fractions, percentages and decimals you must first convert them all into the same type.

Example

Place the following in order from smallest to largest:

$\frac{1}{4}$ 0.2 23%

Firstly we convert them all to decimals:

$$\frac{1}{4} \rightarrow 1 \div 4 = 0.25$$

$$0.2 \rightarrow 0.2$$

$$23\% \rightarrow 23 \div 100 = 0.23$$

It is now easy to put them in order:

0.2 23% $\frac{1}{4}$

Exercise

Place the following in ascending order:

1. $\frac{3}{8}$ 0.41 36%

2. $\frac{7}{8}$ 0.8 87%

10.3. Finding Percentages 2

Imagine in an examination you had 23 out of 50. What percentage have you had?

To work this out we calculate

- $\frac{23}{50} = 0.46$
- $0.46 \times 100 = 46\%$

Exercise

What percentage is

1. 24 out of 50

2. 56 out of 90

3. 43 out of 76

10.4. Percentage Increase or Decrease

Imagine you own a property whose value has increased by 20% to £72,000. How do we find what it was originally worth?

Let's think of the original price as being 100% so if it has increased by 20% we now have 120%:

$$£72,000 = 120\%$$

Divide by 120 to find out what 1% is:

$$£600 = 1\%$$

Multiply by 100 to discover the original value:

$$£60,000 = 100\%$$

Exercise

After an increase of 10% my house is worth £55,000. What was it originally worth?

11. Simple and Compound Interest

11.1. Simple Interest

In a simple interest account the interest each year is calculated on the initial amount invested not on your current balance (i.e. unlike a bank account).

Example

Say we invest £50 in a simple interest account that pays 2% interest per annum:

$$\begin{array}{ll} \text{Interest given at end of 1}^{\text{st}} \text{ year:} & 2\% \text{ of } £50 = \frac{2}{100} \times 50 = 0.02 \times 50 = £1 \\ \text{Balance after 1}^{\text{st}} \text{ year:} & 50 + 1 = £51 \end{array}$$

$$\begin{array}{ll} \text{Interest given at end of 2}^{\text{nd}} \text{ year:} & 2\% \text{ of } £50 = \frac{2}{100} \times 50 = 0.02 \times 50 = £1 \\ \text{Balance after 2}^{\text{nd}} \text{ year:} & 51 + 1 = £52 \end{array}$$

$$\begin{array}{ll} \text{Interest given at end of 3}^{\text{rd}} \text{ year:} & 2\% \text{ of } £50 = \frac{2}{100} \times 50 = 0.02 \times 50 = £1 \\ \text{Balance after 3}^{\text{rd}} \text{ year:} & 52 + 1 = £53 \end{array}$$

Because the interest paid is the same each year it is easy to work out the amount of interest paid over a period of time. For example over 61 years we would have received $71 \times 1 = £71$ in interest.

Exercise

I invest £3000 in a simple interest account that pays 5% per annum. How much interest will I have received after 10 years? What will be my balance after 10 years?

11.2. Compound Interest

Compound interest works in the same way as most bank accounts. The interest at the end of each year is calculated on your current balance and **not** the initial amount invested.

Example

I invest £250 in a compound interest account at 5% p.a.:

$$\begin{array}{ll} \text{Interest given at end of 1}^{\text{st}} \text{ year:} & 5\% \text{ of } £250 = \frac{5}{100} \times 250 = £12.50 \\ \text{Balance after 1}^{\text{st}} \text{ year:} & 250 + 12.50 = £262.50 \end{array}$$

$$\begin{array}{ll} \text{Interest given at end of 2}^{\text{nd}} \text{ year:} & 5\% \text{ of } £262.50 = \frac{5}{100} \times 262.50 = £13.13 \\ \text{Balance after 2}^{\text{nd}} \text{ year:} & £262.50 + £13.13 = £275.63 \end{array}$$

12. Ratio

12.1. Introduction

Ratios are basically just fractions on the side and can be treated in much the same way. If the ratio of boys to girls in a class is 3 : 4 then this means that for every 3 boys there are 4 girls.

Alternatively by dividing both sides of the ratio by 3 we obtain:

$$1 : \frac{4}{3}$$

This means that for every boy then will be $\frac{4}{3}$ of a girl.

So if there were 12 boys there would be

$$12 \times \frac{4}{3} = 16$$

girls.

If there were 20 girls there would be

$$\begin{aligned} 20 \div \frac{4}{3} &= 20 \times \frac{3}{4} \\ &= 15 \end{aligned}$$

boys.

Ratios are usually easier to deal with when they are in the form 1 : n and they can always be got into this form by dividing both sides by the smallest side.

Exercise

1. A drawing of a house is drawn to a scale 1:100.
 - a. If the height of a wall on the drawing is 2.8 cm what will it be in the house?
 - b. If the length of a room in the house is 450 cm what will it be on the drawing?

12.2. Splitting into Ratios

Imagine you were asked to divide £9100 into the ratio 2:4:7. How would you go about doing this? The method is as follows:

- Work out the total number of parts: $2 + 4 + 7 = 13$ parts
- Work out the value of one part: $9100 \div 13 = £700$
- Now find the values of 2, 4 and 7 parts respectively:
 $2 \times 700 = £1400$ $4 \times 700 = £2800$ $7 \times 700 = £4900$

As a check we can add up the parts to check we have the right amount:
 $1400 + 2800 + 4900 = £9100$

Example

If I divide £100 in the ratio 2:3:5 how much is each part worth?

Answer

Total number of parts: $2 + 3 + 5 = 10$

Value of each part: $100 \div 10 = £10$

Hence

2 parts are worth $\rightarrow 2 \times 10 = £20$

3 parts are worth $\rightarrow 3 \times 10 = £30$

5 parts are worth $\rightarrow 5 \times 10 = £50$

Example

A sum of money is split between three people A, B, and C in the ratio 2:3:4.

Assuming that A gets £8 work out how much B and C receive and the total amount of money.

Answer

This is slightly different because we don't know the total amount of money. First we work out the value of each part:

Person A has received 2 parts so each part is worth $8 \div 2 = £4$

Person B receives: $3 \times 4 = £12$

Person C receives: $4 \times 4 = £16$

Hence total amount is $8 + 12 + 16 = £36$

13. Square Roots and Cube Roots

13.1. Square Roots

When we **square** a number we multiply it by itself. So

$$\begin{aligned}3^2 &= 3 \times 3 = 9 \\4^2 &= 4 \times 4 = 16 \\x^2 &= x \times x\end{aligned}$$

To find the **square root** of 9 you want to know what number times itself gives 9, the answer is 3 because $3 \times 3 = 9$. We write

$$\sqrt{9} = 3$$

Example

What is $\sqrt{49}$? Since $7 \times 7 = 49$, we have $\sqrt{49} = 7$.

You can also work out square roots using the $\sqrt{}$ button on your calculator. Depending on your calculator you may have to press the number before or after the number you want the square root of.

Example

Using a calculator find $\sqrt{36}$

13.2. Cube Roots

When we **cube** a number we multiply it by itself 3 times. So

$$\begin{aligned}2^3 &= 2 \times 2 \times 2 = 8 \\3^3 &= 3 \times 3 \times 3 = 27 \\x^3 &= x \times x \times x\end{aligned}$$

To find the **cube root** of 27 you want to know what number times itself three times gives 27, the answer is 3 because $3 \times 3 \times 3 = 27$. We write

$$\sqrt[3]{27} = 3$$

Example

What is $\sqrt[3]{64}$? Since $4 \times 4 \times 4 = 64$ we have $\sqrt[3]{64} = 4$

You can find out cube roots on your calculator by using the $\sqrt[3]{}$ button.

Exercise

Using your calculator work out $\sqrt[3]{125}$

14. Indices

We use indices as a shorthand method

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^7$$

3 is called the **BASE** and 7 is called the **INDEX**. We say "3 to the power 7".

Examples

$$\begin{array}{lll} 6 \times 6 & = & 6^2 \quad (6 \text{ squared}) \\ 7 \times 7 \times 7 & = & 7^3 \quad (7 \text{ cubed}) \\ 8 \times 8 \times 8 \times 8 & = & 8^4 \quad (8 \text{ to the power } 4) \\ 9 \times 9 \times 9 \times 9 \times 9 & = & 9^5 \quad (9 \text{ to the power } 5) \end{array}$$

You can work out indices on your calculator using the X^Y or Y^X button. To work out 8^4 you would press:

$$8 \quad X^Y \quad 4 \quad =$$

14.1. Rules for Working with Indices

- When multiplying numbers **with the same base** you add the indices:

$$3^2 \times 3^3 = 3^{2+3} = 3^5 \qquad 5^2 \times 5^7 = 5^9$$

- When dividing numbers **with the same base** you subtract the indices:

$$3^3 \div 3^2 = 3^{3-2} = 3^1 \qquad 5^8 \div 5^{11} = 5^{8-11} = 5^{-3}$$

- When raising one power to another you multiply the powers:

$$(3^2)^4 = 3^{2 \times 4} = 3^8 \qquad (6^3)^9 = 6^{3 \times 9} = 6^{27}$$

- Any number to the power 1 is just itself:

$$33^1 = 33 \qquad 99^1 = 99$$

- Any number to the power 0 is 1:

$$100^0 = 1 \qquad 9561^0 = 1$$

(Apart from 0 itself!)
 $0^0 = ?$

- Fractional Powers:

The power $\frac{1}{2}$ means square root: $25^{1/2} = \sqrt{25} = 5$

The power $\frac{1}{3}$ means cube root: $64^{1/3} = \sqrt[3]{64} = 4$

- Negative Powers:

$$5^{-6} = \frac{1}{5^6} \qquad 6^{-2} = \frac{1}{6^2} \qquad 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$$

Example

Write 2^{-2} as a decimal.

Answer: $2^{-2} = \frac{1}{2^2} = \frac{1}{4} = 0.25$

Example

Simplify $7^3 \times 3^2 \times 7^5$

Answer: $7^{3+5} \times 3^2 = 7^8 \times 3^2$

15. Substitution

Prior to doing substitution we need to understand some of the notation used in algebra:

15.1. Notation

- $4x$ means $4 \times x$ [The multiplication sign is often left out]
- $5y$ means $5 \times y$
- xyz means $x \times y \times z$ [The multiplication sign is often left out]
- xy^2 means $x \times y \times y$ [It is only the y that is squared]
- $(xy)^2$ means $x \times y \times x \times y$ [Both the x and y are squared]
- $x(y-z)^2$ means $x \times (y-z) \times (y-z)$ [Only the brackets get squared]

15.2. Substitution

Imagine you were given the formula

$$X = 3A + 2B$$

and asked to find the value of X when $A = 5$ and $B = 7$. All you do is write down the formula again but this time you substitute numbers for letters on the right hand side. So we would have

$$\begin{aligned} X &= (3 \times 5) + (2 \times 7) \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

Make sure you do it in stages as we have done above. **DO NOT TRY TO DO IT ALL IN ONE GO!!!**

Example

If $C = \frac{5}{9}(F - 32)$ find the value of C when $F = 59$.

$$\begin{aligned} \text{Ans } C &= \frac{5}{9}(59 - 32) \\ &= \frac{5}{9}(27) \\ &= 15 \end{aligned} \quad \text{[Remember we do brackets first]}$$

16. Simplifying Expressions

Consider the expression

$$5x - 7y - y^2 - x + 3y^2$$

We say

$5x$	is an x term
$-7y$	is a y term
$-y^2$	is a y^2 term
$-x$	is an x term
$3y^2$	is a y^2 term

The idea here is to first **collect like terms together**, so we would write:

$$5x - 7y - y^2 - x + 3y^2 = \underbrace{(5x - x)}_{x \text{ terms}} \underbrace{(-7y)}_{y \text{ terms}} \underbrace{(-y^2 + 3y^2)}_{y^2 \text{ terms}}$$

Next we **combine like terms**:

$$5x - x = 4x$$

$$-7y = -7y$$

$$-y^2 + 3y^2 = 2y^2$$

So we have

$$5x - 7y - y^2 - x + 3y^2 = 4x - 7y + 2y^2$$

WARNING: Don't be confused into thinking that y and y^2 are both y terms!!! They are different and hence cannot be combined.

Example

Simplify $7xy + 9yz - 3xy - 3yz + 7xy - 2yz$

Firstly we **collect the like terms together**:

$$\begin{aligned} 7xy + 9yz - 3xy - 3yz + 7xy - 2yz &= \boxed{7xy - 3xy + 7xy} + \boxed{9yz - 3yz - 2yz} \\ &\quad \text{(Collect like terms together)} \\ &= 11xy + 4yz \quad \text{(Combine like terms)} \end{aligned}$$

17. Expanding Brackets

17.1. Multiplying Out Brackets

Recall that when letters are multiplied together then we just write them next to each other:

$$x \times y = xy$$

$$x \times y \times z = xyz$$

Also remember from our work on indices that:

$$x \times x = x^2$$

$$x \times x \times x = x^3$$

Finally don't forget the important effect of brackets:

$$xy^2 = x \times y \times y$$

$$(xy)^2 = x \times x \times y \times y$$

When expanding brackets the term outside the bracket multiplies each separate term inside the bracket:

$$3(5x + 2) = 3 \times 5x + 3 \times 2 = 15x + 6$$

$$-4(4x^2 + y) = -4 \times 4x^2 + -4 \times y = -16x^2 - 4y$$

$$4x(3y + 2x) = 4x \times 3y + 4x \times 2x = 12xy + 8x^2$$

To expand a bracket that just has a minus sign outside the bracket you simply change the signs of each of the terms inside the bracket:

$$-(x + y) = -x - y$$

$$-(-x - y) = x + y$$

$$-(x - y) = -x + y$$

$$-(-x + y) = x - y$$

Example

Expand and simplify $7x + 8y + 3(2x + 4y)$

Ans

Firstly we expand the bracket:

$$\begin{aligned} 3(2x + 4y) &= 3 \times 2x + 3 \times 4y \\ &= 6x + 12y \end{aligned}$$

Hence we have

$$\begin{aligned} 7x + 8y + 3(2x + 4y) &= 7x + 8y + 6x + 12y \\ &= 7x + 6x + 8y + 12y \\ &= 13x + 20y \end{aligned}$$

[Group together like terms]
[Combine like terms]

17.2. Brackets Multiplied by Brackets

To expand an expression like $(2x + 1)(x + 2)$ you need to multiply each term in the first bracket by each term in the second:

$$\begin{aligned}(2x + 1)(x + 2) &= 2x \times x + 2x \times 2 + 1 \times x + 1 \times 2 \\ &= 2x^2 + 4x + x + 2 \\ &= 2x^2 + 5x + 2\end{aligned}$$

Example

Expand and simplify $(3x + 7)(5x - 2)$

$$\begin{aligned}(3x + 7)(5x - 2) &= 3x \times 5x + 3x \times -2 + 7 \times 5x + 7 \times -2 \\ &= 15x^2 - 6x + 35x - 14 \\ &= 15x^2 + 29x - 14\end{aligned}$$

17.3. Squared Brackets

To expand something like $(5x + 3)^2$ you need to remember that

$$(5x + 3)^2 = (5x + 3)(5x + 3)$$

Then we can use the same method as above.

$$\begin{aligned}(5x + 3)^2 &= (5x + 3)(5x + 3) \\ &= 5x \times 5x + 5x \times 3 + 3 \times 5x + 3 \times 3 \\ &= 25x^2 + 15x + 15x + 9 \\ &= 25x^2 + 30x + 9\end{aligned}$$

Example

Expand and simplify $(x - 4)^2$

Ans

$$\begin{aligned}(x - 4)^2 &= (x - 4)(x - 4) \\ &= x \times x + x \times -4 + -4 \times x + -4 \times -4 \\ &= x^2 - 4x - 4x + 16 \\ &= x^2 - 8x + 16\end{aligned}$$

18. Factorising

This is the exact opposite of multiplying out brackets. Now instead of getting rid of brackets we are interested in putting them back in. The idea is to find common factors amongst the terms and bring them out. This is best learnt by example:

Example

Factorise $9x + 12x^2y$

- What is the highest common factor of 9 and 12?
3 can be brought out as a common factor.
- What is the biggest power of x that can be brought out from each term?
We can bring an x out of each term but we can't bring x^2 out of each term. So x can be brought out as a common factor.
- Since y doesn't occur in each term it won't be able to come out as a common factor.

Hence we can bring out $3x$ as a common factor. We then need to work out what will be left once we've taken out the $3x$:

$$9x + 12x^2y = 3x(3 + 4xy)$$

We need 3
to get $9x$

We need $4xy$ to
get $12x^2y$

Example

Factorise $15x^4y + 20x^2y^3z - 35x^3yz^2$

- What is the highest common factor of 15, 20 and 35?
5 can be brought out as a common factor.
- What is the biggest power of x that can be brought out of each term?
 x^2 can be brought out as a common factor.
- What is the biggest power of y that can be brought out of each term?
 y can be brought out as a common factor.
- Since z doesn't occur in each term it won't be able to come out as a common factor.

Hence we can bring out $5x^2y$ as a common factor. Then work out what is left:

$$15x^4y + 20x^2y^3z - 35x^3yz^2 = 5x^2y(3x^2 + 4y^2z - 7xz^2)$$

We need $3x^2$
to get $15x^4y$

We need $4y^2z$
to get $20x^2y^3z$

We need $7xz^2$
to get $35x^3yz^2$

19. Solving Linear Equations

An **equation** is a statement containing an “=” sign, such as

$$x + 4 = 7 \qquad 12x + 3 = 2x$$

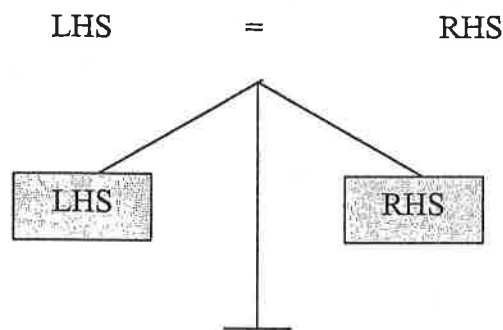
In the above equations x is the unknown. A **linear equation** is one that has no squares or higher power of the unknown.

A **solution** of an equation is a number which, when put instead of the letter, makes the equation true. For example,

$$x + 4 = 7 \quad \text{has the solution} \quad x = 3 \quad \text{because } 3 + 4 = 7$$

19.1. Keeping the Balance

You should think of an equation like a set of scales in balance.



The equation, like scales, will still balance if

whatever you do to one side you also do to the other

The balance will be unchanged if you

- I. add the same quantity to each side
- II. subtract the same quantity from each side
- III. multiply each side by the same quantity
- IV. divide each side by the same quantity

In order to solve an equation the idea is to get all the “letter terms” on one side and all the “number terms” on the other. To obtain this you may need to get rid of unwanted terms from each side of the equation.

The inverse of + is -. So to get rid of a ‘+ term’ use the matching ‘- term’. For example,

$$\text{to get rid of '+2' use '-2' because } +2 - 2 = 0$$

The inverse of - is +. So to get rid of a ‘- term’ use the matching ‘+ term’. For example,

$$\text{to get rid of '-3' use '+3' because } -3 + 3 = 0$$

Once you have solved the equation always **check** your answer by substitution.

Example

Solve $x + 3 = 7$

Ans

To get rid of $+ 3$ we use $- 3$. So we subtract 3 from **both sides** of the equation:

$$\begin{array}{rcl} x + 3 - 3 & = & 7 - 3 \\ x & = & 4 \end{array}$$

Let's check that this answer is correct by using substitution:

$$\begin{array}{rcl} \text{LHS: } x + 3 & = & 4 + 3 \\ & = & 7 \end{array} \qquad \text{RHS: } 7$$

Since LHS = RHS our answer is correct.

Example

Solve $x - 4 = 10$

Ans

To get rid of $- 4$ we use $+ 4$. So we add 4 to **both sides** of the equation:

$$\begin{array}{rcl} x - 4 + 4 & = & 10 + 4 \\ x & = & 14 \end{array}$$

Let's check that this answer is correct by using substitution:

$$\begin{array}{rcl} \text{LHS: } x - 4 & = & 14 - 4 \\ & = & 10 \end{array} \qquad \text{RHS: } 10$$

Since LHS = RHS our answer is correct.

19.2. Inverses of $\times$ and $\div$

Recall that x means $1 \times x$. Sometimes after rearranging your equation the letter term may have a coefficient that is not 1. For example,

$3x$ has a coefficient 3; it means $3 \times x$

$\frac{x}{2}$ has a coefficient $\frac{1}{2}$; it means $\frac{x}{2}$

We will want to make the coefficient equal to 1. The inverses of $\times$ and $\div$ will help you do this:

The inverse of $\times$ is $\div$. So to undo a multiplication use a matching division. For example:

If you have a coefficient 3 then you need to divide by 3 since $3 \div 3 = 1$

The inverse of $\div$ is $\times$. So to undo a division use a matching multiplication. for example:

If you have a coefficient $\frac{1}{2}$ then you need to multiply by 2 since $\frac{1}{2} \times 2 = 1$

Example

Solve $4x = 8$

Ans

To undo the $\times 4$, we use $\div 4$. So we must divide **both sides** by 4.

$$\frac{4x}{4} = \frac{8}{4}$$

Now since $4 \div 4 = 1$ we have

$$x = 2$$

Check:

$$\begin{array}{lcl} \text{LHS: } 4x & = & 4 \times 2 \\ & = & 8 \end{array} \qquad \text{RHS: } 8$$

Since LHS = RHS our answer is correct.

Example

Solve $\frac{x}{7} = 3$

Ans

To undo $\div 7$, we use $\times 7$. So we must multiply **both sides** by 7.

$$\frac{x}{7} \times 7 = 3 \times 7$$

Now since $\frac{1}{7} \times 7 = 1$ we have

$$x = 21$$

Check:

$$\begin{array}{lcl} \text{LHS: } \frac{x}{7} & = & \frac{21}{7} \\ & = & 3 \end{array} \qquad \text{RHS: } 3$$

Since LHS = RHS our answer is correct.

19.3. More Complicated Examples

Example

Solve $2x + 5 = 11$

Ans

To get rid of $+5$ we use -5 , so we subtract 5 from both sides:

$$\begin{aligned} 2x + 5 - 5 &= 11 - 5 \\ 2x &= 6 \end{aligned}$$

To undo the $\times 2$ we use $\div 2$, so we divide both sides by 2:

$$\begin{aligned} \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

Check:

$$\begin{aligned} \text{LHS: } 2x + 5 &= 2 \times 3 + 5 & \text{RHS: } 11 \\ &= 6 + 5 \\ &= 11 \end{aligned}$$

Since $\text{LHS} = \text{RHS}$ our answer is correct.

Example

Solve $3x + 4 = 7 - 2x$

Ans

To get rid of $-2x$ we use $+2x$, so we add $2x$ to **both sides**

$$\begin{aligned} 3x + 4 + 2x &= 7 - 2x + 2x \\ 5x + 4 &= 7 \end{aligned}$$

To get rid of $+4$ we use -4 , so we subtract 4 from **both sides**

$$\begin{aligned} 5x + 4 - 4 &= 7 - 4 \\ 5x &= 3 \end{aligned}$$

To undo the $\times 5$ we use $\div 5$, so we divide **both sides** by 5

$$\begin{aligned} \frac{5x}{5} &= \frac{3}{5} \\ x &= \frac{3}{5} = 0.6 \end{aligned}$$

Check:

$$\text{LHS: } 3x + 4 = 3 \times 0.6 + 4 = 5.8$$

$$\text{RHS: } 7 - 2x = 7 - 2 \times 0.6 = 5.8$$

Since $\text{LHS} = \text{RHS}$ our answer is correct.

19.4. Equations with Brackets

If an equation involves brackets **always** multiply out the brackets first before you attempt to solve the equation.

Example

Solve $2(x + 5) = 5(x - 7)$

Ans

Firstly we expand the brackets, this gives

$$2x + 10 = 5x - 35$$

Subtract $2x$ from both sides:

$$\begin{array}{rcl} 2x + 10 - 2x & = & 5x - 35 - 2x \\ 10 & = & 3x - 35 \end{array}$$

Add 35 to both sides:

$$\begin{array}{rcl} 10 + 35 & = & 3x - 35 + 35 \\ 45 & = & 3x \end{array}$$

Divide both sides by 3:

$$\begin{array}{rcl} \frac{45}{3} & = & \frac{3x}{3} \\ 15 & = & x \end{array}$$

Check:

$$\begin{array}{rclclcl} \text{LHS: } 2(x + 5) & = & 2(15 + 5) & \text{RHS: } 5(x - 7) & = & 5(15 - 7) \\ & = & 2 \times 20 & & = & 5 \times 8 \\ & = & 40 & & = & 40 \end{array}$$

Since LHS = RHS, our answer is correct.

19.5. Equations with Fractions

If an equation involves a fraction it is easier to get rid of the fraction first before attempting to solve the equation.

Example

Solve $\frac{2x-1}{3} = 7$

Ans

Multiply each side by the denominator 3:

$$3 \times \frac{(2x-1)}{3} = 3 \times 7$$

$$2x - 1 = 21$$

Add 1 to each side

$$\begin{array}{rcl} 2x - 1 + 1 & = & 21 + 1 \\ 2x & = & 22 \end{array}$$

Divide both sides by 2

$$\begin{array}{rcl} \frac{2x}{2} & = & \frac{22}{2} \\ x & = & 11 \end{array}$$

As always you should check this answer, we leave this as an exercise.

Solution of Quadratic Equations by Formula.

Ex. use the formula:-

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to solve the equation:-

$$x^2 - 6x - 2 = 0$$

here $a = 1, b = -6, c = -2$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-2)}}{2(1)}$$

$$x = \frac{6 \pm \sqrt{36 + 8}}{2}$$

$$x = \frac{6 \pm \sqrt{44}}{2}$$

$$x = \frac{6 \pm \sqrt{4 \times 11}}{2}$$

$$x = \frac{6 \pm 2\sqrt{11}}{2}$$

$$x = 3 \pm \sqrt{11}$$

either
 $x = 3 - \sqrt{11}$

OR
 $x = 3 + \sqrt{11}$

$x = -0.32 \text{ (2dp's)}$

$x = 6.32 \text{ (2dp's)}$

20. Solving Quadratic Equations

20.1. What if $a \times b = 0$?

An important fact that you should know is: "If two terms multiply together to give zero then at least one of them must be zero", i.e.

If $a \times b = 0$ then either $a = 0$ OR $b = 0$

This can be useful in solving some types of equations.

OR $a = 0$ and $b = 0$

Example

Solve the equation

$$(x + 3)(x - 7) = 0$$

From above we know that either

$$x + 3 = 0$$

OR

$$x - 7 = 0$$

$$x = -3$$

$$x = 7$$

Example

Solve the equation

$$(x - 5)(x + 11) = 0$$

From above we know that either

$$x - 5 = 0$$

OR

$$x + 11 = 0$$

$$x = 5$$

$$x = -11$$

20.2. Solving Quadratics

A quadratic equation is one that looks like

$$ax^2 + bx + c = 0$$

In the quadratic equations you will consider we will always have $a = 1$, namely

$$x^2 + bx + c = 0$$

Examples of such equations include:

$$x^2 - 2x + 1$$

$$x^2 + 5x - 3$$

$$x^2 - 11x + 22$$

To solve a quadratic equation you need to factorise it using the method below:

- First write the equation in the form $x^2 + bx + c = 0$
- Our aim is to write the equation in the form $(x \quad)(x \quad) = 0$
- To achieve this we need to find two numbers that multiply to give c but also add to give b .
- If c is positive then both numbers must have the same sign, but if c is negative the numbers will have different signs.
- Once the equation has been factorised it can then be easily solved.

Example 1

Solve the equation

$$x^2 + 7x + 10 = 0$$

Ans

Firstly we must factorise the quadratic:

$$(x \quad)(x \quad)$$

We want to find two numbers that multiply to give 10 and add to give 7. Since 10 is positive the numbers must have the same sign and since they add to give 7 they must both be positive. Let's try and find them:

$1 \times 10 = 10$	but	$1 + 10 = 11$	[No good]
$2 \times 5 = 10$	and	$2 + 5 = 7$	[Success!!!]

Hence

$$x^2 + 7x + 10 = (x + 5)(x + 2)$$

Thus

$$(x + 5)(x + 2) = 0$$

So we have

$$\begin{aligned} x + 5 &= 0 \\ x &= -5 \end{aligned}$$

OR

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

Example 2

Solve the equation

$$x^2 - 5x + 6 = 0$$

Ans

Firstly we must factorise the quadratic:

$$(x \quad)(x \quad)$$

We want to find two numbers that multiply to give 6 and add to give -5. Since 6 is positive the numbers must have the same sign and since they add to give -5 they must both be negative. Let's try and find them:

$-1 \times -6 = 6$	but	$-1 + -6 = -7$	[No good]
$-2 \times -3 = 6$	and	$-2 + -3 = -5$	[Success!!!]

Hence

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

Thus

$$(x - 2)(x - 3) = 0$$

So we have

$$\begin{aligned} x - 2 &= 0 \\ x &= 2 \end{aligned}$$

OR

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

Example 3

Solve the equation

$$x^2 + x - 12 = 0$$

Ans

Firstly we must factorise the quadratic:

$$(x \quad)(x \quad)$$

We want to find two numbers that multiply to give -12 and add to give 1. Since -12 is negative the numbers must have different signs and since they add to give 1 the bigger number must be positive. Let's try and find them:

$-1 \times 12 = -12$	and	$-1 + 12 = 11$	[No good]
$-2 \times 6 = -12$	and	$-2 + 6 = 4$	[No good]
$-3 \times 4 = -12$	and	$-3 + 4 = 1$	[Success!!!]

Hence

$$x^2 + x - 12 = (x - 3)(x + 4)$$

Thus

$$(x - 3)(x + 4) = 0$$

So we have

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

OR

$$\begin{aligned} x + 4 &= 0 \\ x &= -4 \end{aligned}$$

Example 4

Solve the equation

$$x^2 - 4x - 12 = 0$$

Ans

Firstly we must factorise the quadratic:

$$(x \quad)(x \quad)$$

We want to find two numbers that multiply to give -12 and add to give -4. Since -12 is negative the numbers must have different signs and since they add to give -4 the bigger number must be negative. Let's try and find them:

$1 \times -12 = -12$	and	$1 - 12 = -11$	[No good]
$2 \times -6 = -12$	and	$2 - 6 = -4$	[Success!!!]

Hence

$$x^2 - 4x - 12 = (x + 2)(x - 6)$$

Thus

$$(x + 2)(x - 6) = 0$$

So we have

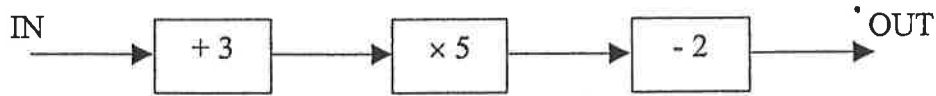
$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

OR

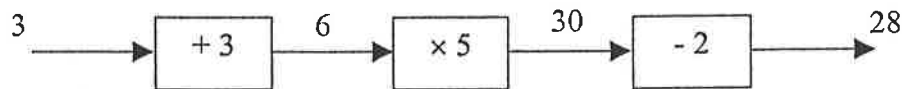
$$\begin{aligned} x - 6 &= 0 \\ x &= 6 \end{aligned}$$

21. Function Machines

An example of a function machine is



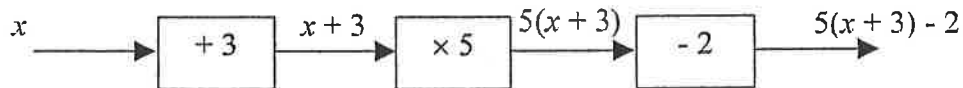
The secret of a function machine is to write down what is coming out from each box. If we put 3 in the above machine we'd have



Put 3 in the 1 st box	$3 + 3 = 6$	6 comes out.
Put 6 in the 2 nd box	$6 \times 5 = 30$	30 comes out.
Put 30 in the 3 rd box	$30 - 2 = 28$	28 comes out.

Example

What comes out if you put x in?



Put x in the 1 st box	$x + 3$	$x + 3$ comes out.
Put $x + 3$ in the 2 nd box	$5 \times (x + 3)$	$5(x + 3)$ comes out.
Put $5(x + 3)$ in the 3 rd box	$5(x + 3) - 2$	$5(x + 3) - 2$ comes out.

What comes out can be simplified as follows:

$$\begin{aligned}
 5(x + 3) - 2 &= 5x + 15 - 2 \\
 &= 5x + 13
 \end{aligned}$$

Sometimes you may be told what comes out and asked to find what was originally put in the machine. To do this you simply use the machine from right to left and do the opposite operations.

Example

For the above machine what would you need to put in to get 48 out.

Working from right to left and doing the opposite operations gives.

48 goes in.	$48 + 2 = 50$	50 comes out.
50 goes in.	$50 \div 5 = 10$	10 comes out.
10 goes in.	$10 - 3 = 7$	7 comes out.

So you need to put 7 in.

22. Rearranging Formulae

Consider the formula

$$3z - x = 5y + 2p$$

We could re-arrange this formula so that we had

$$x = \text{"something"}$$

where there is no x on the right hand side. We say we have made x the **subject** of the formula.

Alternatively we could rearrange to obtain

$$y = \text{"something"}$$

where there is no y on the right hand side. This time we've made y the **subject** of the formula.

You could also make x or p the subject of the formula.

To re-arrange formulae we use the same methods that we used when we learnt to solve equations.

Example

Make p the subject of the formula: $\frac{p}{p+y} = 4$

Firstly we multiply both sides by $(p+y)$ to get rid of the fraction:

$$\begin{aligned}\frac{p}{p+y} \times (p+y) &= 4 \times (p+y) \\ p &= 4p + 4y\end{aligned}$$

To get rid of $+4p$ we use $-4p$

$$\begin{aligned}p - 4p &= 4p + 4y - 4p \\ -3p &= 4y\end{aligned}$$

To undo $\times -3$ we use $\div -3$ which gives

$$\begin{aligned}\frac{-3p}{-3} &= \frac{4y}{-3} \\ p &= -\frac{4}{3}y\end{aligned}$$

Example

Make y the subject of the formula: $z = \sqrt{2x - \frac{x^2}{2y+1}}$

Firstly we must get rid of the square root, to do this we square both sides:

$$z^2 = 2x - \frac{x^2}{2y+1}$$

To get rid of the fraction we must multiply both sides by $(2y+1)$

$$z^2 \times (2y+1) = \left(2x - \frac{x^2}{2y+1}\right) \times (2y+1)$$

$$z^2(2y+1) = 2x \times (2y+1) - \frac{x^2}{2y+1} \times (2y+1)$$

$$z^2(2y+1) = 2x(2y+1) - x^2$$

Now we need to expand the brackets

$$2yz^2 + z^2 = 4xy + 2x - x^2$$

Let's get all terms involving y on the LHS and the rest on the RHS. To get rid of $+4xy$ we use $-4xy$

$$2yz^2 + z^2 - 4xy = 4xy + 2x - x^2 - 4xy$$

$$2yz^2 - 4xy + z^2 = 2x - x^2$$

To get rid of $+z^2$ we use $-z^2$

$$2yz^2 - 4xy + z^2 - z^2 = 2x - x^2 - z^2$$

$$2yz^2 - 4xy = 2x - x^2 - z^2$$

Now let's factorise out the y term on the LHS:

$$y(2z^2 - 4x) = 2x - x^2 - z^2$$

To undo the $\times (2z^2 - 4x)$ we use $\div (2z^2 - 4x)$, this gives

$$\frac{y(2z^2 - 4x)}{(2z^2 - 4x)} = \frac{2x - x^2 - z^2}{(2z^2 - 4x)}$$

$$y = \frac{2x - x^2 - z^2}{(2z^2 - 4x)}$$

Note: The question in your GCSE examination will probably be easier than the above example.

23. Inequalities

23.1. Different Types

There are 4 types of inequality signs you need to know:

$>$	means greater than	$<$	means less than
$\geq$	means greater than or equal to	$\leq$	means less than or equal to

An **integer** is just a word to mean a whole number. You will need to know this.

Example

If x is an integer (whole number) and $3 < x \leq 7$ what values can x take?

We know x must be bigger than 3 so the smallest value x can be is 4. It must also be less than or equal to 7, so the biggest x can be is 7. Hence x can be 4, 5, 6 or 7.

Sometimes you will see

$$x > 5 \quad \text{written as} \quad 5 < x$$

Don't be confused by this, both are telling you that x is bigger than 5.

23.2. Solving Inequalities

When solving inequalities you can just treat the inequality as if it were an "=" sign.

However there is one important difference:

Whenever you multiply or divide by a negative number the inequality sign must be flipped!!!!!!

Example

Solve $3x < x + 4$

To get rid of $+x$ we use $-x$

$$3x - x < x + 4 - x$$

$$2x < 4$$

$$\frac{2x}{2} < \frac{4}{2}$$

To undo $\times 2$ we use $\div 2$

$$x < 2$$

Example

Solve $5x \leq 6x + 2$

To get rid of $+6x$ we use $-6x$

$$5x - 6x \leq 6x + 2 - 6x$$

$$-x \leq 2$$

To get rid of the $-$ in front of x we must divide both sides by -1 . Remembering that the inequality sign flips we obtain

$$\begin{aligned} \frac{-x}{-1} &\geq \frac{2}{-1} \\ x &\geq -2 \end{aligned}$$

24. Sequences

24.1. Types of Sequences

A sequence is just a list of numbers. In your exam you will be shown some numbers in a sequence and asked to write down the next two or three terms in the sequence. There are several different types of sequences but they are easy to spot if you always write down the difference in the gaps between each pair of numbers.

Example - Common Difference

Write down the next 2 terms in the sequence 7 11 15 19 23

Begin by writing down the difference in the gaps between each pair of numbers

7 11 15 19 23
 +4 +4 +4 +4

From this we can see that the next two terms are

$$23 + 4 = 27 \quad \text{and} \quad 27 + 4 = 31$$

Example – Increasing Difference

Write down the next 2 terms in the sequence 8 11 15 20 26

Begin by writing down the difference in the gaps between each pair of numbers

8 11 15 20 26
 +3 +4 +5 +6

From this we can see that the next two terms are

$$26 + 7 = 33 \quad \text{and} \quad 33 + 8 = 41$$

Example – Decreasing Difference

Write down the next 2 terms in the sequence 53 43 34 26 19

Begin by writing down the difference in the gaps between each pair of numbers

53 43 34 26 19
 -10 -9 -8 -7

From this we can see that the next two terms are

$$19 - 6 = 13 \quad \text{and} \quad 13 - 5 = 8$$

Example – Multiplying Factor

Write down the next 2 terms in the sequence 5 10 20 40 80

Begin by writing down the difference in the gaps between each pair of numbers

5 10 20 40 80
 $\times 2$ $\times 2$ $\times 2$ $\times 2$

From this we can see that the next two terms are

$$80 \times 2 = 160 \quad \text{and} \quad 160 \times 2 = 320$$

Write down the next 2 terms in the sequence

1 1 2 3 5 8

$$\begin{array}{ccccccc} 1 & & 1 & & 2 & & 3 & & 5 & & 8 \\ & +1 & & +1 & & +1 & & +2 & & +3 & \end{array}$$
$$8 + 5 = 13 \qquad \text{and} \qquad 13 + 8 = 21$$

For the sequence type “Common Difference” you will be expected to be able to write down a formula for the n^{th} term of the sequence. This is easy if you follow the method in the example below:

Find the n^{th} term for the sequence 7 11 15 19 23 ...

When $n = 1$ we have $4 \times 1 = 4$ but the 1st term should be 7.

When $n = 2$ we have $4 \times 2 = 8$ but the 2^{nd} term should be 11.

When $n = 3$ we have $4 \times 3 = 12$ but the 3rd term should be 15.

$$4n + 3$$

Find the n^{th} term for the sequence 1 4 7 10 13 ...

When $n = 1$ we have $3 \times 1 = 3$ but the 1st term should be 1.

When $n = 2$ we have $3 \times 2 = 6$ but the 2^{nd} term should be 4.

When $n = 3$ we have $3 \times 3 = 9$ but the 3rd term should be 7.

$$3n - 2$$

Find the n^{th} term for the sequence 1 4 9 16 25 ...

When $n = 1$ we have $1^2 = 1$

When $n = 3$ we have $3^2 = 9$

When $n = 2$ we have $2^2 = 4$

When $n = 4$ we have $4^2 = 16$

So in general the n^{th} term will be n^2

25. Standard Index Form

25.1. Definition

Index form (or standard index form) is a method scientists use to write very big or small numbers. Examples of numbers in standard form are

36 000	In standard form this is	3.6×10^4
505, 000	In standard form this is	5.05×10^5
0.00365	In standard form this is	3.65×10^{-3}
0.0000000078	In standard form this is	7.8×10^{-9}

Numbers in standard form must always be of the form
 $x \times 10^n$

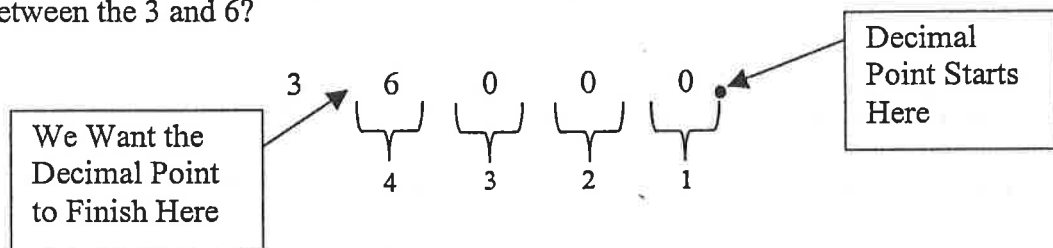
where x is a number between 1 and 10 which cannot be 10, i.e. $1 \leq x < 10$ and n tells us the number of places the decimal point moves. If the number is big then n is positive but if it is small n is negative.

25.2. Method for Writing Numbers in Standard Form

Example

Write 36000 in standard form.

The front number must be between 1 and 10, so in this case it will be 3.6.
Consider 36000. How many places do we have to move the decimal point so that it lies between the 3 and 6?



Hence we have to move the decimal point 4 places, and since the number is big n will be positive. So we have

$$36000 = 3.6 \times 10^4$$

Example

Write 0.000000543 in standard form.

The front number in this case will be 5.43.

Now how many places do we have to move the decimal point so that it moves from its starting point in 0.000000543 to between the 5 and 4?

Clearly it must move 7 places, but because the number is small this time n will be negative. Hence

$$0.000000543 = 5.43 \times 10^{-7}$$

25.3. Multiplying and Dividing Number in Standard Form

To do this you must remember the rules about multiply and dividing indices.

Example

Calculate the product of 3×10^2 and 2×10^5 giving your answer in standard form

Ans:

$$\begin{aligned} 3 \times 10^2 \times 2 \times 10^5 &= 3 \times 2 \times 10^2 \times 10^5 \\ &= 6 \times 10^{2+5} \\ &= 6 \times 10^7 \end{aligned}$$

Example

Work out $\frac{6 \times 10^{11}}{2 \times 10^{-4}}$

Ans:

$$\begin{aligned} \frac{6 \times 10^{11}}{2 \times 10^{-4}} &= \frac{6}{2} \times \frac{10^{11}}{10^{-4}} \\ &= 3 \times 10^{11-(-4)} \\ &= 3 \times 10^{15} \end{aligned}$$

Example

Calculate the product of 5×10^3 and 7×10^2 giving your answer in standard form.

Ans:

$$\begin{aligned} 5 \times 10^3 \times 7 \times 10^2 &= 5 \times 7 \times 10^3 \times 10^2 \\ &= 35 \times 10^{3+2} \\ &= 35 \times 10^5 \end{aligned}$$

However this answer is not in standard form. To achieve this we first write 35 in standard form, $35 = 3.5 \times 10^1$

$$\begin{aligned} \text{This then gives } 35 \times 10^5 &= 3.5 \times 10^1 \times 10^5 \\ &= 3.5 \times 10^{1+5} \\ &= 3.5 \times 10^6 \end{aligned}$$

25.4. Using a Calculator

To do standard form on a calculator you use the key **EXP** or **EE**.

To enter the number 3.5×10^6 you press

3 . 5 **EXP** 6

To enter the number 1.56×10^{-7} you press

1 . 5 6 **EXP** - 7

26. Trial and Improvement

26.1. Introduction

One day Jack and Jill decide to play a number game instead of going to fetch water. Jack thinks of a number x between 1 and 50 and Jill tries to guess what x is.

Jill	:	25?	
Jack	:	Too low	
Jill	:	I now know $x > 25$.	Try 35.
Jack	:	Too high	
Jill	:	I now know $25 < x < 35$.	Try 27
Jack	:	Too low	
Jill	:	I now know $27 < x < 35$.	Try 29
Jack	:	Too high	
Jill	:	I now know $27 < x < 29$	Try 28
Jack	:	Correct!!	

Having had so much fun playing this game they decide to play again. This time Jill picks a number between 1 and 10 and Jack tries to guess it:

Jack	:	5?	
Jill	:	Too low	
Jack	:	I now know $x > 5$.	Try 7
Jill	:	Too high	
Jack	:	I now know $5 < x < 7$.	Try 6
Jill	:	Too low	

At this point Jack looks somewhat confused but then realises that Jill didn't say she had chosen an integer (a whole number):

Jack	:	I know $5 < x < 7$	Try 6.5
Jill	:	Too low	
Jack	:	I know $6.5 < x < 7$	Try 6.7
Jill	:	Too high	
Jack	:	I know $6.5 < x < 6.7$	Try 6.6
Jill	:	Too low	
Jack	:	I know $6.6 < x < 6.7$	Try 6.65
Jill	:	Too low	

Jack now knows $6.65 < x < 6.7$ and proudly proclaims "The number you have chosen is 6.7 to 1 decimal place." Notice that when Jack knew $6.6 < x < 6.7$ he was unable to give x to 1 decimal place because he didn't know whether it was closer to 6.6 or 6.7. However by trying 6.65 he discovered that $6.65 < x < 6.7$. Now he knows x is closer to 6.7 than 6.6 so he can say $x = 6.7$ to 1 decimal place.

Example

Find a solution of the equation

$$x^2 + x - 5 = 0$$

to 1 decimal place by using trial and improvement.

Let's play the same game as Jack and Jill and try to guess the answer. We'll then test our guess to see whether it is too big or too small. Letting g represent our guess we proceed as follows:

g	$g^2 + g - 5 = 0$	Outcome	Conclusion
0	$0^2 + 0 - 5 = -5$		
1	$1^2 + 1 - 5 = -3$		
2	$2^2 + 2 - 5 = 1$	2 is too big	$1 < x < 2$
1.5	$1.5^2 + 1.5 - 5 = -1.25$	1.5 is too small	$1.5 < x < 2$
1.7	$1.7^2 + 1.7 - 5 = -0.41$	1.7 is too small	$1.7 < x < 2$
1.8	$1.8^2 + 1.8 - 5 = 0.04$	1.8 is too big	$1.7 < x < 1.8$
1.75	$1.75^2 + 1.75 - 5 = -0.1875$	1.75 is too small	$1.75 < x < 1.8$

Pay particular attention to the last two lines. When we knew $1.7 < x < 1.8$ we were unable to give x to 1 d.p. because we did not know whether x was closer to 1.7 or 1.8. To resolve this problem we tried 1.75 and found that $0.75 < x < 1.8$. Hence x is closer to 1.8 than 1.7 so we can safely say $x = 1.7$ (to 1 d.p.)

Example

Calculate $\sqrt{13}$ to 2 d.p. without the use of the $\sqrt{\quad}$ key on your calculator.

Solution

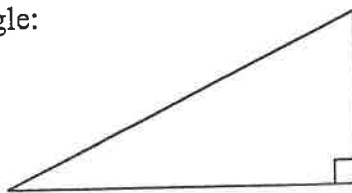
We want to find x such that $x = \sqrt{13}$. This is the same as trying to find a solution of the equation $x^2 = 13$ [We have squared both sides]. We proceed in the same manner as before:

g	g^2	Outcome	Conclusion
3	$3^2 = 9$	3 is too small	$x > 3$
4	$4^2 = 16$	4 is too big	$3 < x < 4$
3.5	$3.5^2 = 12.25$	3.5 is too small	$3.5 < x < 4$
3.7	$3.7^2 = 13.69$	3.7 is too big	$3.5 < x < 3.7$
3.6	$3.6^2 = 12.96$	3.6 is too small	$3.6 < x < 3.7$
3.65	$3.65^2 = 13.3225$	3.65 is too big	$3.6 < x < 3.65$
3.61	$3.61^2 = 13.0321$	3.61 is too big	$3.60 < x < 3.61$
3.605	$3.605^2 = 12.996$	3.605 is too small	$3.605 < x < 3.61$

Hence $x = 3.61$ to 2 d.p.

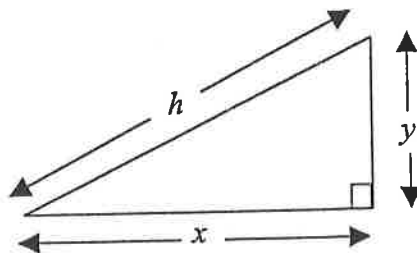
45. Pythagoras

Consider a right-angled triangle:



The longest side in the triangle is the one opposite the right angle. We call this side the **HYPOTENUSE**.

Pythagoras' Theorem is an equation that relates the lengths of the sides of the triangle. Labelling the sides as



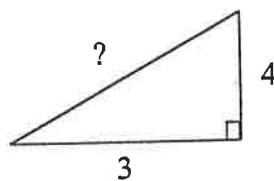
Pythagoras' Theorem states:

$$h^2 = x^2 + y^2$$

Notice that the HYPOTENUSE is on its own on the left – this is **ALWAYS** the case.

Example

Find the unknown length

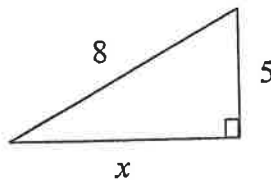


By Pythagoras

$$\begin{aligned} h^2 &= 3^2 + 4^2 \\ &= 9 + 16 \\ &= 25 \\ \therefore h &= \sqrt{25} \\ &= 5 \end{aligned}$$

Example

Find the value of x



By Pythagoras

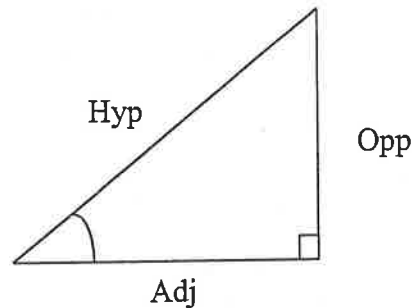
$$\begin{aligned} 8^2 &= x^2 + 5^2 \\ 64 &= x^2 + 25 \\ x^2 &= 64 - 25 \\ &= 39 \\ \therefore x &= \sqrt{39} = 6.24 \text{ (2 d.p.)} \end{aligned}$$

46. Trigonometry

46.1. Labelling Triangle Sides

Before we can do any trigonometry we need to know how to label the sides of a right angle triangle. **The way we label the sides depends on the angle we are considering.**

Consider the triangle below and the angle A.

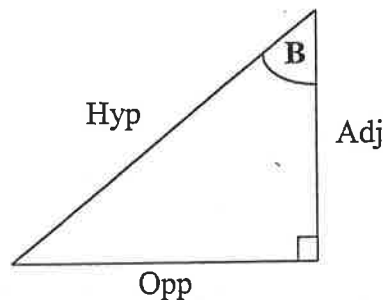


We know the side opposite the right angle is called **the hypotenuse**.

The side opposite the angle A is called **the opposite**.

The remaining side is called **the adjacent** (because it is adjacent to A).

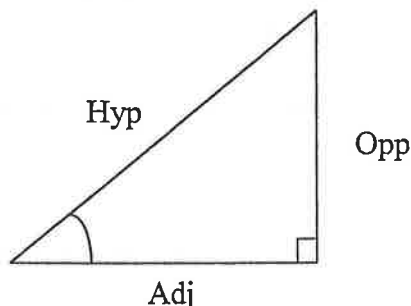
Note that if we were considering the angle B as shown below then the sides are labelled differently.



Remember: The labels for the sides depend on the angle you are considering.

46.2. Finding Lengths and Angles

Trigonometry shows us how the lengths of the sides of a right angled triangle are related to the angle we are considering. Consider again the angle A.



Trigonometry tells us that

$$\sin A = \frac{Opp}{Hyp}$$

$$\cos A = \frac{Adj}{Hyp}$$

$$\tan A = \frac{Opp}{Adj}$$

To remember these three relationships just remember the name

SOH CAH TOA

then from this we obtain

Sin is Opposite over Hypotenuse

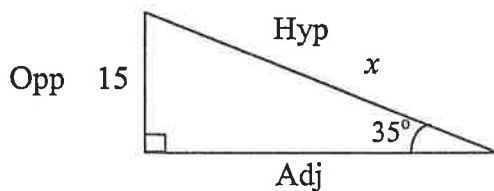
Cos is Adjacent over Hypotenuse

Tan is Opposite over Adjacent

We use these trigonometric ratios to find the lengths of sides or sizes of angles in **right angle triangles!!!!**

Example 1

Find x in the triangle shown



1. Label the sides!!
2. We know the opposite and want the hypotenuse. What relates these two? The answer is **Sin**.
3. Now we use the trigonometric ratio

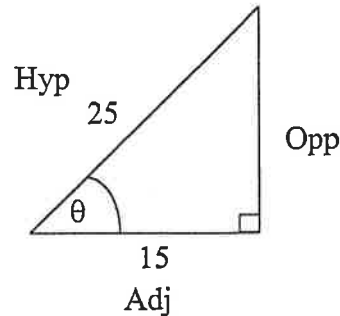
$$\begin{aligned}\sin 35 &= \frac{Opp}{Hyp} \\ &= \frac{15}{x} \\ x \times \sin 35 &= 15 \\ x &= \frac{15}{\sin 35} = 26.2\end{aligned}$$

On your calculator press: 15 ÷ 35 sin =
if it is a logical calculator press: 15 ÷ sin 35 =

In this next example we show how to work out an unknown angle when two lengths are known.

Example 2

Find the angle of θ in this triangle:



1. Label the sides Hyp, Adj and Opp.
2. We know the hypotenuse and the adjacent. Which trigonometric ratio links these two? The answer is **Cos**.
3. Now do the calculation:

$$\begin{aligned}\cos \theta &= \frac{\text{adj}}{\text{hyp}} \\ &= \frac{15}{25} \\ &= 0.6\end{aligned}$$

To work out the value of θ we need to work out the **INVERSE COS** of 0.6. To do this we use the **INV** key on the calculator.

$$\begin{aligned}\theta &= \text{INV COS}(0.6) \\ &= 53.1\end{aligned}$$

To do this calculation on your calculator you will need to do one of the following:

PRESS 0.6 INV COS =

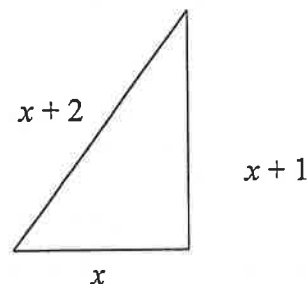
PRESS INV COS 0.6 =

PRESS 2nd F COS 0.6 =

65.3. Quadratic Equations

Example 1

A right-angled triangle has the dimensions, measured in centimetres, shown in the diagram.



- a) By using Pythagoras, show that $x^2 - 2x - 3 = 0$.
- b) Solve the equation in a)
- c) Write down the lengths of the sides of the triangle.

a) By Pythagoras:

$$x^2 + (x + 1)^2 = (x + 2)^2$$

Expanding the brackets on either side gives:

$$x^2 + x^2 + 2x + 1 = x^2 + 4x + 4$$

$$\therefore 2x^2 + 2x + 1 = x^2 + 4x + 4$$

$$2x^2 + 2x + 1 - x^2 - 4x - 4 = 0$$

$$\therefore x^2 - 2x - 3 = 0$$

b) To solve this we factorise the expression on the left hand side:

$$(x - 3)(x + 1) = 0$$

$$\begin{array}{lcl} \text{Either} & x - 3 = 0 & \rightarrow x = 3 \\ \text{or} & x + 1 = 0 & \rightarrow x = -1 \end{array}$$

- c) In our triangle x must be 3 cm because you can't have negative lengths!!!!
So the lengths of the triangles are:

(x)	3 cm
$(x + 1)$	4 cm
$(x + 2)$	5 cm

But in the question we are told that the area is the same as the perimeter. This means that

$$\begin{aligned}
 12x - 48 &= 2x + 16 \\
 12x &= 2x + 16 + 48 \\
 12x - 2x &= 64 \\
 10x &= 64 \\
 \therefore x &= 64 \div 10 \\
 &= 6.4 \text{ cm}
 \end{aligned}$$

$$\begin{aligned}
 \text{So } A &= 12 \times 6.4 - 48 \\
 &= 28.8 \text{ cm}^2
 \end{aligned}$$

65.2. Simultaneous Equations

Example 1

Three mugs and two plates cost £7.20, but four mugs and one plate cost £7.90. Let x represent the cost of a mug and y the cost of a plate. Write down two equations involving x and y . Solve these equations and calculate the cost of seven mugs and 6 plates.

Solution

We are told

$$3 \text{ mugs and } 2 \text{ plates cost } £7.20, \text{ i.e. } 3x + 2y = 7.20 \quad \text{--- (1)}$$

$$4 \text{ mugs and } 1 \text{ plate cost } £7.90, \text{ i.e. } 4x + y = 7.90 \quad \text{--- (2)}$$

Multiplying equation (2) by 2 gives the following:

$$3x + 2y = 7.20 \quad \text{--- (3)}$$

$$8x + 2y = 15.8 \quad \text{--- (4)}$$

(3) - (4) gives,

$$\begin{aligned}
 3x - 8x &= 7.20 - 15.80 \\
 -5x &= -8.6 \\
 x &= 1.72
 \end{aligned}$$

Substituting into (1) gives,

$$\begin{aligned}
 3 \times 1.72 + 2y &= 7.20 \\
 5.16 + 2y &= 7.20 \\
 2y &= 2.04 \\
 y &= 1.02
 \end{aligned}$$

Hence Price of a mug = £1.72 Price of a plate = £1.02

$$\begin{aligned}
 \text{Thus } 7x + 6y &= 7 \times 1.72 + 6 \times 1.02 \\
 &= 12.04 + 6.12 \\
 &= £18.16
 \end{aligned}$$

65. Using Equations

We have already seen how to solve linear, simultaneous and quadratic equations. In this section we will see how to answer questions when the equations are not given.

65.1. Linear Equations

Example 1

Sue has x pounds in her account.

- a) She withdraws 50 pounds. Write down an expression to show how much she now has.

$$\begin{aligned}\text{Amount in account} &= \text{Original amount} - 50 \\ &= x - 50\end{aligned}$$

- b) The next week Sue deposits twice the amount she originally had in the account. Write down an expression to show how much she now has.

$$\begin{aligned}\text{Amount in account} &= \text{Current Amount} + 2 \times \text{Original Amount} \\ &= x - 50 + 2 \times x \\ &= 3x - 50\end{aligned}$$

- c) Given that Sue originally had 140 pounds in the bank, how much does she now have?

We are told that $x = 140$, so using the formula in b) we have

$$\begin{aligned}\text{Amount in account} &= 3 \times 140 - 50 \\ &= 420 - 50 \\ &= \text{£}370\end{aligned}$$

Example 2

The length of a rectangle is 12cm and its width is $(x - 4)$ cm. If its perimeter is numerically the same as its area, calculate the value of x and hence its area.

Solution

Let A denote the area of the rectangle and p denote its perimeter. Then

$$\begin{aligned}A &= \text{Length} \times \text{Breadth} \\ &= 12 \times (x - 4) \\ A &= 12x - 48\end{aligned}$$

also

$$\begin{aligned}p &= 12 + (x - 4) + 12 + (x - 4) \\ &= 2x + 12 - 4 + 12 - 4 \\ p &= 2x + 16\end{aligned}$$

Example

Solve the simultaneous equations:

$$2x + 4y = 2 \quad \text{--- (1)}$$

$$6x + 5y = 13 \quad \text{--- (2)}$$

Let us try to eliminate x from these equations. First we need to make each equation have the same number of x 's. To do this we multiply the first equation by 3 so that there will be 6 x 's in it.

$$3 \times (1) \rightarrow \begin{array}{rcl} 3 \times (2x + 4y) & = & 3 \times 2 \\ 6x + 12y & = & 6 \end{array}$$

“Whatever you do to one side you must do to the other”

So now we have the equations

$$6x + 12y = 6 \quad \text{--- (3)}$$

$$6x + 5y = 13 \quad \text{--- (4)}$$

Now we subtract equation (4) from (3)

$$\begin{array}{rcl} (3) - (4) \rightarrow & 6x + 12y - (6x + 5y) & = 6 - 13 \\ & \therefore 12y - 5y & = -7 \\ & 7y & = -7 \\ & \therefore y & = \frac{-7}{7} = -1 \end{array}$$

To find the value of x we substitute this value of y into equation (1):

$$\begin{array}{rcl} 2x + 4 \times (-1) & = & 2 \\ 2x - 4 & = & 2 \\ 2x & = & 6 \\ \therefore x & = & \frac{6}{2} = 3 \end{array}$$

So our solution is $x = 3, y = -1$

64. Simultaneous Equations

We have already seen how to solve equations involving only one variable.

Examples

1. $x = 2x + 1$
2. $3x - 5 = 2x$

In our work on graphs we also saw how to solve a pair of equations involving two variables.

Example

Solve the simultaneous equations

$$\begin{aligned}x + y &= 5 \\x - y &= 1\end{aligned}$$

The way we did this earlier was to draw a graph of each and see where they intersect. Now we will learn how to solve them algebraically. The idea is that we try to **ELIMINATE** one of the variables:

$$\begin{array}{rcl}x + y &= & 5 \quad \text{--- (1)} \\x - y &= & 1 \quad \text{--- (2)}\end{array}$$

Let's try to eliminate the x from these equations. The way we do this is to subtract equation (2) from equation (1):

$$\begin{array}{rcl}(1) - (2) \rightarrow & x + y - (x - y) &= 5 - 1 \\& x - x + y + y &= 6 \\& 2y &= 6 \\& y &= 3\end{array}$$

Now that we know the value of y we can substitute this into equation (1) or (2) to find out the value of x .

$$\begin{array}{rcl}\text{In (1):} & x + 3 &= 5 \\& x &= 5 - 3 \\& x &= 2\end{array}$$

So our solution is $x = 2, y = 3$

This example was easy because we already had the same number of x 's in each equation to start with so we could easily eliminate x . We could have just as easily eliminated y by adding the equations.

The next example shows what we do when we don't have the same number of x 's or y 's:

Base units.

All quantities can be described by using 7 base units.

The Seven Base units

Mass	-	kilogram	-	kg
length	-	metre	-	m
time	-	second	-	s
Current	-	Ampere	-	A
Temperature	-	Kelvin	-	K
Amount of Substance	-	mole	-	mol.
Luminous Intensity	-	Candela	-	Cd.

All other units can be derived from these seven base units.

Derived units.

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} \Rightarrow \text{units } \frac{\text{m}}{\text{s}} \text{ or } \text{ms}^{-1}$$

$$\text{Velocity} = \frac{\text{Displacement}}{\text{Time}} \Rightarrow \text{units } \frac{\text{m}}{\text{s}} \text{ or } \text{ms}^{-1}$$

$$\begin{aligned}\text{Acceleration} &= \text{Rate of change of velocity} \\ &\Rightarrow \frac{\text{ms}^{-1}}{\text{s}} \\ &\Rightarrow \text{ms}^{-2}\end{aligned}$$

$$\begin{aligned}\text{Force} &= \text{mass} \times \text{acceleration} \\ &\Rightarrow \text{units } \text{kg ms}^{-2}\end{aligned}$$

$$\begin{aligned}\text{Work} &= \text{Force} \times \text{distance} \\ &\Rightarrow \text{units of } \text{kgms}^{-2} \cdot \text{m} \\ &\Rightarrow \text{kgm}^2\text{s}^{-2}\end{aligned}$$

$$\begin{aligned}\text{charge } Q &= It \\ &\Rightarrow \text{units of AS}\end{aligned}$$

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

$$\Rightarrow \text{units } \frac{\text{kgms}^{-2}}{\text{m}^2}$$

$$\Rightarrow \text{units } \text{kgm}^{-1}\text{s}^{-2}$$

$$\text{If } F = \frac{G m_1 m_2}{r^2}$$

Find the base units of ' G ' if m_1, m_2 are masses, ' r ' the distance between the masses and ' F ' is the gravitational force of attraction.

Rearranging the formula $\Rightarrow$

$$\frac{F r^2}{m_1 m_2} = G$$

$$\Rightarrow \frac{\text{kgms}^{-2} \cdot \text{m}^2}{\text{kg}^2} = G$$

$$\Rightarrow \text{kg}^{-1}\text{m}^3\text{s}^{-2} = G$$

Ex: From the Ideal gas equation:-

$$\frac{PV}{T} = R$$

Find Base units of Molar gas Constant:

$$P = \frac{\text{Force}}{\text{Area}} = \frac{\text{kgms}^{-2}}{\text{m}^2} = \text{kgm}^{-1}\text{s}^{-2}$$

$$V - \text{volume} = \text{m}^3$$

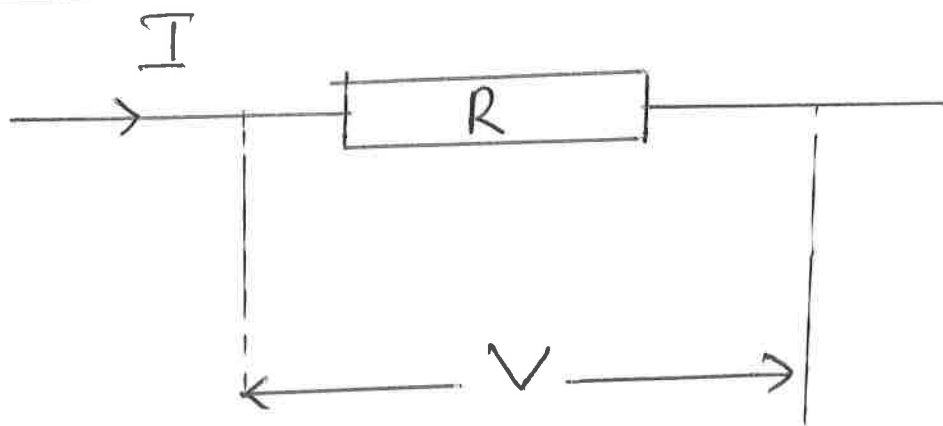
$$T - \text{temperature} = \text{K}$$

$$\therefore R = \frac{\text{kgm}^{-1}\text{s}^{-2} \cdot \text{m}^3}{\text{K}}$$

$$\underline{R = \text{kgm}^2\text{s}^{-2}\text{K}^{-1}}$$

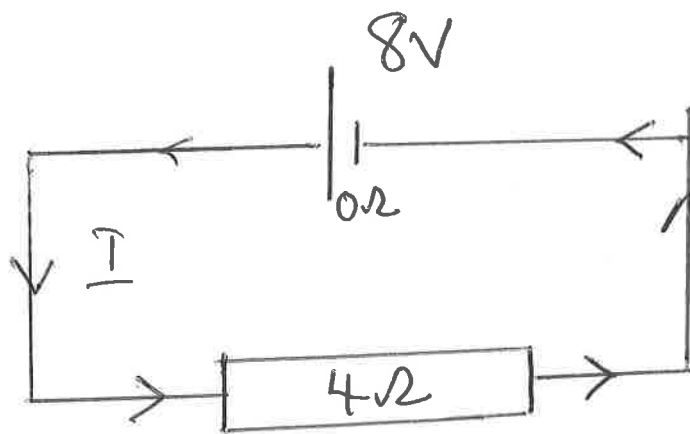
Simple Electrical Circuits.

Ohm's Law



$$I = \frac{V}{R}$$

Ex.

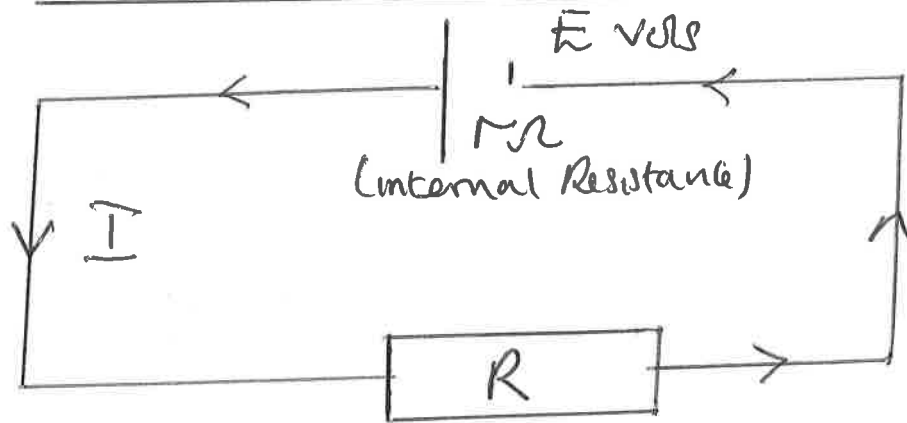


$$I = \frac{V}{R}$$

$$\Rightarrow I = \frac{8}{4}$$

$$\underline{I = 2A}$$

The Circuit Equation:



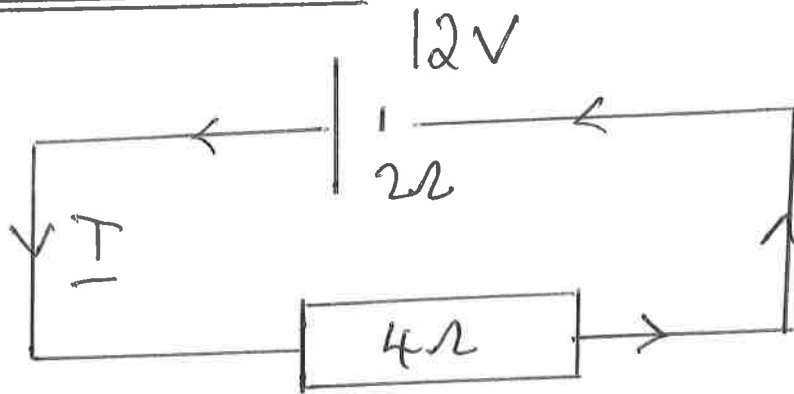
from Kirchhoff's Law

$$\sum_1 E = \sum_1 V \quad (\text{Conservation of Energy})$$

$$E = IR + Ir$$

$$E = I(R + r)$$

Example:



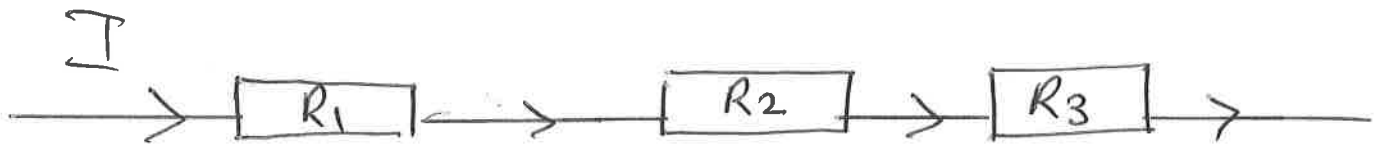
$$E = I(R + r)$$

$$12 = I(4 + 2)$$

$$12 = 6I$$

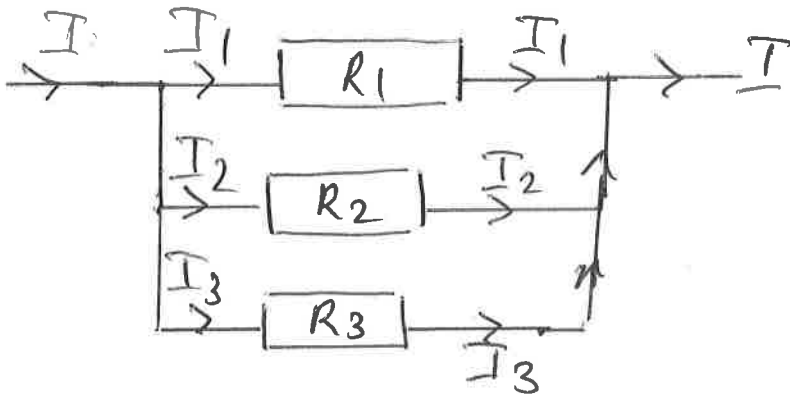
$$\underline{2A = I}$$

Resistors in Series



$$\text{Total Resistance } R = R_1 + R_2 + R_3 \quad (\text{etc.})$$

Resistors in Parallel



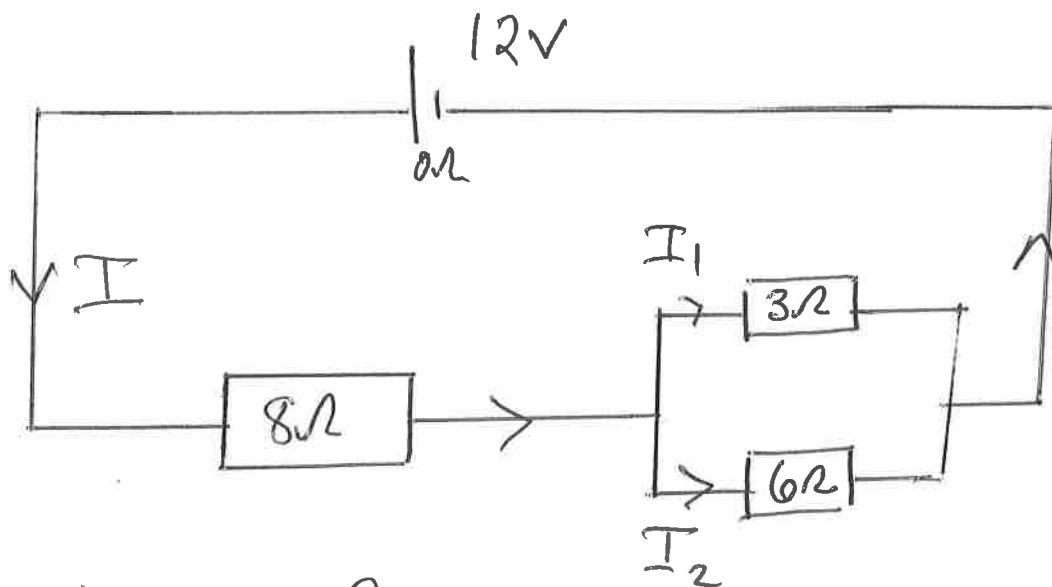
$$\frac{1}{R_{\text{Total}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \quad (\text{etc.})$$

Examples



$$\begin{aligned} R &= R_1 + R_2 \\ &= 8\Omega + 4\Omega \\ &= 12\Omega \end{aligned}$$

Ex.



Find (i) total Resistance

(ii) Current I

(iii) I_1, I_2

(iv) Power supplied by the Cell.

For Resistors in Parallel,

$$\frac{1}{R} = \frac{1}{3} + \frac{1}{6}$$

$$= \frac{2}{6} + \frac{1}{6}$$

$$\frac{1}{R} = \frac{3}{6}$$

$$\frac{1}{R} = \frac{1}{2}$$

$$\Rightarrow \underline{R = 2\Omega}$$

For Resistors in Series

$$R = R_1 + R_2 + R_3$$

$$R = 8 + 2 + 0$$

$$\underline{R = 10\Omega}$$

$$\therefore \text{as } R_{\text{total}} = 10 \Omega$$

from Ohm's Law

$$\underline{I} = \frac{V}{R}$$

$$\underline{I} = \frac{12}{10}$$

$$\underline{I = 1.2 \text{ A}}$$

The Potential Drop across the Parallel Resistors.

$$V = I \cdot R$$

$$V = (1.2)(2)$$

$$V = 2.4 \text{ volts.}$$

(iii)

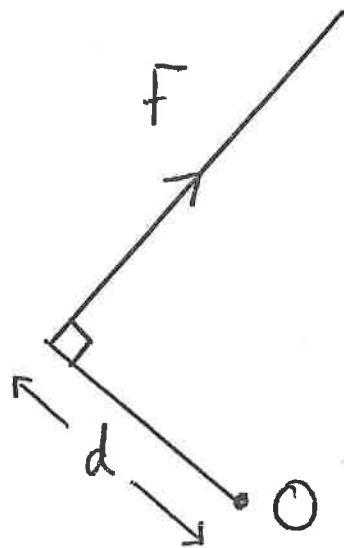
$$\therefore \underline{I_1} = \frac{2.4}{3} = 0.8 \text{ A}$$

$$\underline{I_2} = \frac{2.4}{6} = 0.4 \text{ A}$$

$$\begin{aligned} \text{(iv) Power} &= VI \\ &= 12 \times 1.2 \text{ Watts,} \\ &= \underline{14.4 \text{ Watts.}} \end{aligned}$$

Moments.

The Moment of a force about an axis is defined as the product of the magnitude of the force and the perpendicular distance of the line of action of the force from the axis. It measures the turning effect of the force. i.e.



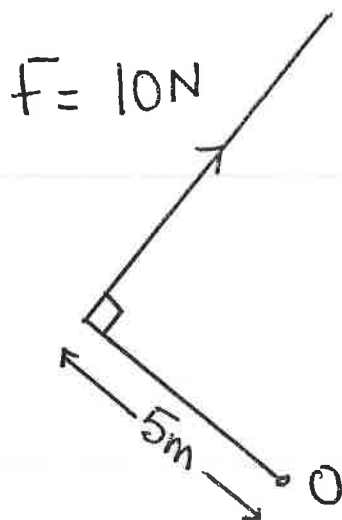
$$\begin{aligned} \text{The moment of } F \text{ about } O &= \text{Force} \times \text{br distance from } O \\ &= \underline{F \times d} \end{aligned}$$

Units are Nm.

This force causes a rotation in a clockwise sense about 'O', and hence is known as a clockwise Moment.

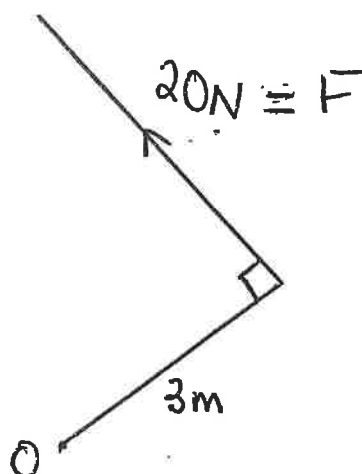
EXAMPLES

(i)



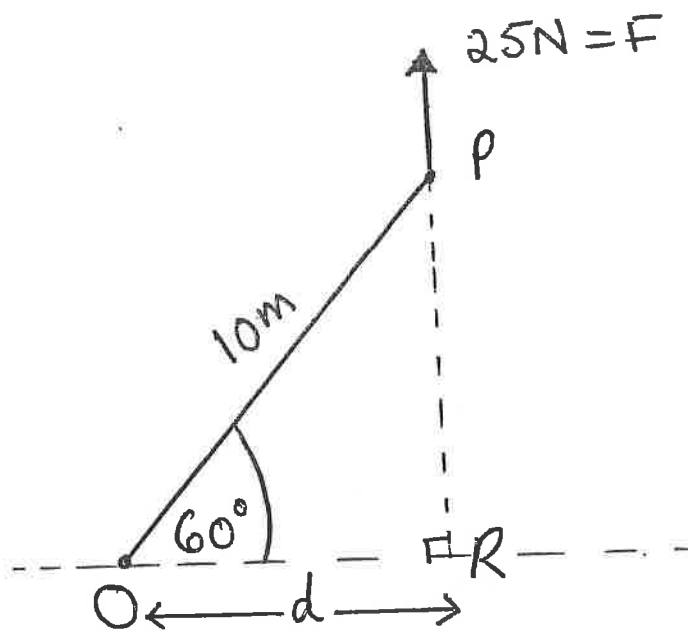
Moment of F about O
 $= 10 \times 5$
 $= 50\text{Nm}$
(Clockwise)

(ii)



Moment of F about O
 $= 20 \times 3$
 $= 60\text{Nm}$
Anticlockwise.

(iii)



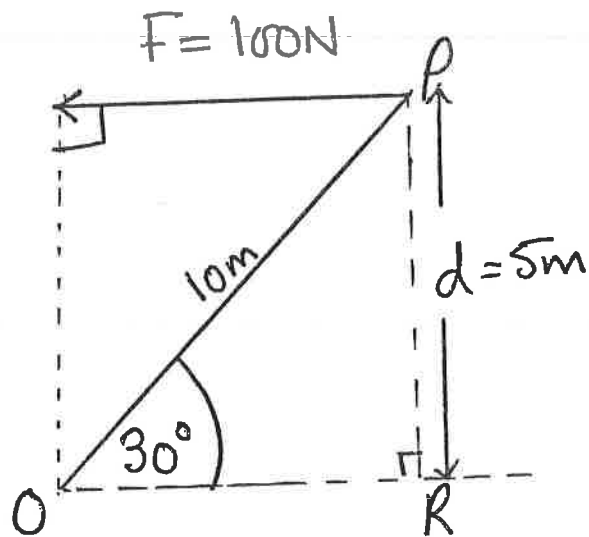
From triangle POR, $\cos 60^\circ = \frac{d}{10}$
 $\Rightarrow 10 \cos 60^\circ = d$
 $10 \cdot \frac{1}{2} = d$
 $5\text{m} = d$

Thus moment of
25N force
about O

$$\begin{aligned} &= F \times d \\ &= 25 \times 5 \\ &= 125\text{Nm} \end{aligned}$$

In an anticlockwise
sense.

EXAMPLE.



$$\text{from } \triangle OPR, \sin 30^\circ = \frac{d}{10}$$

$$10 \sin 30^\circ = d$$

$$10 \cdot \frac{1}{2} = d$$

$$\underline{5\text{m} = d.}$$

$$\begin{aligned} \text{Moment of } F \text{ about } O &= F \times d \\ &= 100 \times 5 \\ &= \underline{500\text{Nm}} \quad \curvearrowright \end{aligned}$$

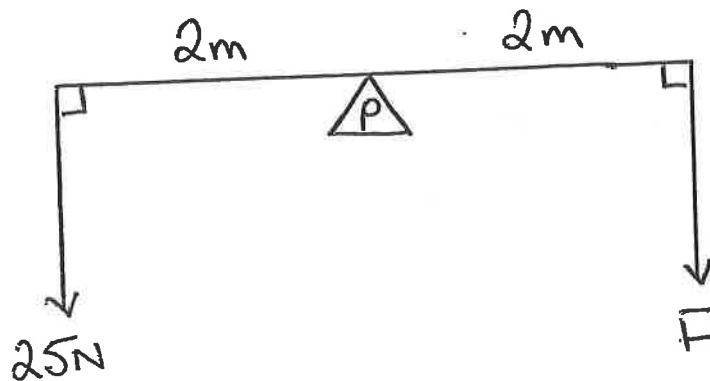
The Equilibrium of a Rigid body.

For an object to remain in equilibrium two conditions must be satisfied.

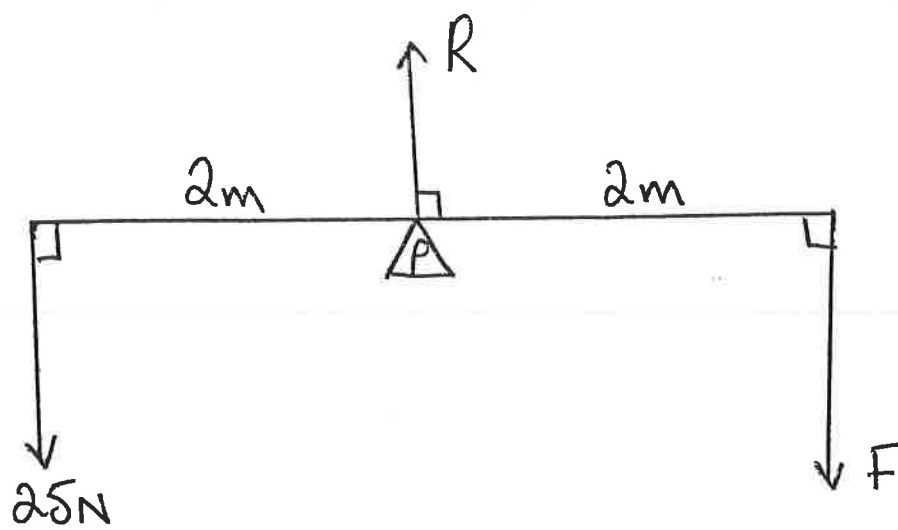
- (i) The vertical and horizontal forces must balance (or sum to zero)
- (ii) The sum of the clockwise moments must equal the sum of the Anti-clockwise moments, otherwise there would be rotation.
ie $\sum CM = \sum ACM.$

This is known as the PRINCIPLE OF MOMENTS

EXAMPLE:



- Find (i) force 'F' to keep rod (see-saw) balanced
(ii) The Normal reaction at pivot 'P'



(i) Taking Moments about P

$$\sum_1^1 CM = F \times 2$$

$$\sum_1^1 ACM = 25 \times 2 \quad \text{Nm}$$

In Equilibrium: $\sum_1^1 CM = \sum_1^1 ACM$

$$F \times 2 = 25 \times 2$$

$$\Rightarrow \underline{F = 25\text{N}}$$

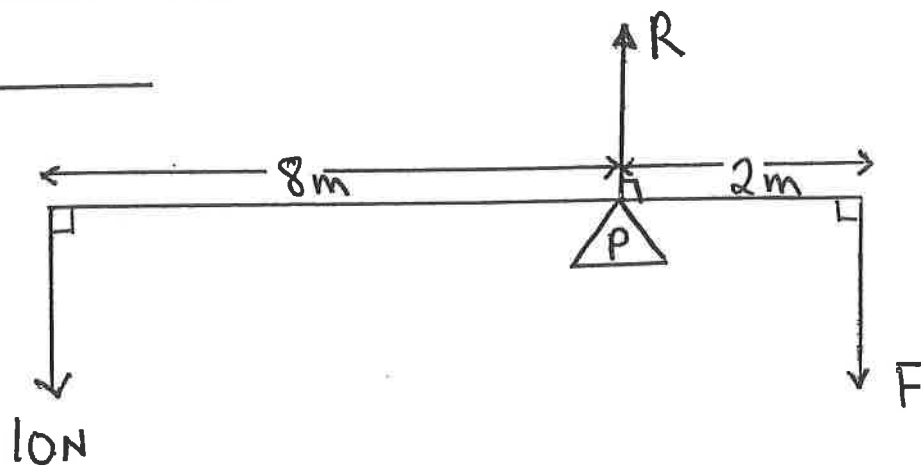
(ii) Resolve Vertically

$$R = 25 + F$$

$$R = 25 + 25$$

$$\underline{R = 50\text{N}}$$

EXAMPLE:



- (i) Find 'F' necessary to preserve Equilibrium
(ii) Find the reaction at the Pivot P.

Take Moments about P.

$$\sum CM = F \times 2 \quad \text{Nm}$$

$$\begin{aligned} \sum ACM &= 10 \times 8 \\ &= 80 \text{ Nm} \end{aligned}$$

In Equilibrium $\sum CM = \sum ACM$

$$F \times 2 = 80$$

$$\Rightarrow \underline{F = 40 \text{ N}}$$

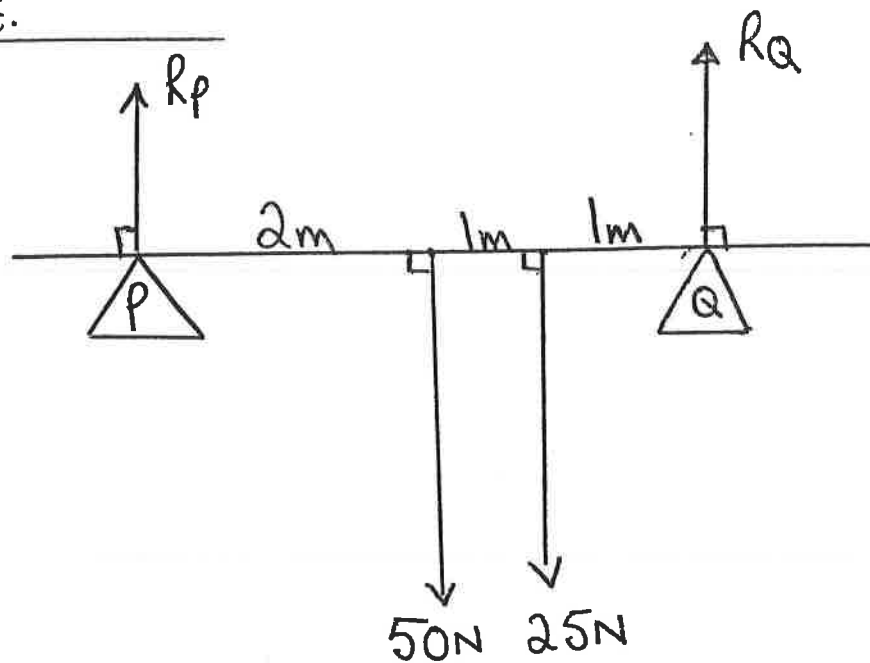
Resolve Vertically ↑ +ve

$$R = 10 + F$$

$$R = 10 + 40$$

$$\underline{R = 50 \text{ N}}$$

EXAMPLE.



Find the reactions at P and Q, if in Equilibrium

Resolve Vertically

$$R_P + R_Q = 50 + 25$$

$$R_P + R_Q = 75 \quad \text{--- (1)}$$

Take Moments about P

$$\sum_1 CM = (50 \times 2) + (25 \times 3)$$

$$= 100 + 75$$

$$\sum_1 CM = 175 \text{ Nm}$$

$$\sum_1 ACM = R_Q \times 4$$

$$\text{In Equilibrium} \quad \sum_1 CM = \sum_1 ACM$$

$$\Rightarrow \quad 175 = 4R_Q$$

$$\frac{175}{4} = R_Q$$

$$\underline{43\frac{3}{4} \text{ N} = R_Q}$$

∴ Substituting $R_Q = 43.75\text{N}$

Into ① $\Rightarrow$

$$R_P + 43.75 = 75$$

$$R_P = 75 - 43.75$$

$$\underline{R_P = 31.25\text{N}}$$

