

$$p = \frac{h}{\lambda} = \left(\frac{h}{2\pi} \right) k$$

$$k = 2\pi/\lambda$$

$$E = p^2/2m$$

$$= \frac{\left(\frac{h}{2\pi} \right)^2 k^2}{2m}$$

$$= \frac{h^2 k^2}{8\pi^2 m}$$

$$k_n = \frac{n\pi}{l}$$

$$= \frac{h^2 n^2 \pi^2}{8\pi^2 m l^2} = \frac{h^2}{8m l^2} \cdot n^2$$

$$f = \frac{c}{\lambda} = \frac{c}{2\pi}$$

$$df = \frac{c}{2\pi} dk$$

$$dn = \frac{k dk}{2\pi^2} \quad L_x L_y L_z$$

$$= \frac{4\pi^2 f^2}{c^2 2\pi^2} dk \quad \text{per unit volume}$$

$$= \frac{4f^2}{c^2} \frac{2\pi}{c} df$$

$$= \frac{4\pi f^2}{c^3} df$$

each state if filled has energy of $E = hf$. dn states in df , would constitute energy dE of $dE = \frac{4\pi h f^3}{c^3} df$

But each state is not necessarily filled. The probability of a state of energy E being filled given by the Bose-Einstein probability function,

$$p(E) = \frac{1}{\left(\exp \frac{hf}{kT} - 1\right)}$$

But because each state can be filled twice over, in effect, due to two possible orthogonal polarisation, hence

$$dE = \frac{8\pi hf^3}{c^3} \frac{dF}{\left(\exp \frac{hf}{kT} - 1\right)}$$

Electromagnetic Waves

The laws of electrostatics, magnetostatics and electromagnetism can be written in mathematical form as

$$\oint_S \underline{D} \cdot d\underline{s} = \int_V \rho \, dv \quad (1)$$

$$\oint_S \underline{B} \cdot d\underline{s} = 0 \quad (2)$$

$$\oint_L \underline{E} \cdot d\underline{l} = - \int_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{s} \quad (3)$$

$$\oint_L \underline{H} \cdot d\underline{l} = \int_S \left(\underline{J} + \frac{\partial \underline{D}}{\partial t} \right) \cdot d\underline{s} \quad (4)$$

where $\underline{J} = \sigma \underline{E}$

$\underline{D} = \epsilon \epsilon_0 \underline{E}$

$\underline{B} = \mu \mu_0 \underline{H}$

These equations are often called collectively Maxwell's Equations. They can be rewritten in differential form, by making use of two vector theorems;

$$\text{Gauss' Theorem} \quad \int_S \underline{F} \cdot d\underline{s} = \int_V \text{div } \underline{F} \, dv$$

$$\text{Stokes Theorem} \quad \int_L \underline{F} \cdot d\underline{l} = \int_S \text{curl } \underline{F} \cdot d\underline{s}$$

$$\text{Hence} \quad \int_V \text{div } \underline{D} \, dv = \int_V \rho \, dv$$

$$\text{so that} \quad \text{div } \underline{D} = \rho \quad (5)$$

$$\text{similarly} \quad \text{div } \underline{B} = 0 \quad (6)$$

$$\text{curl } \underline{E} = - \frac{\partial \underline{B}}{\partial t} \quad (7)$$

$$\text{curl } \underline{H} = (\underline{J} + \frac{\partial \underline{D}}{\partial t}) \quad (8)$$

If we restrict attention to non-conducting media, $\sigma = J = 0$; also exclude free charges, $\rho = 0$, and restrict to non-magnetic media ($\mu = 1$), then

$$\text{div } \underline{D} = 0 \quad (\nabla \cdot \underline{D} = 0) \quad (9)$$

$$\text{div } \underline{B} = 0 \quad (\nabla \cdot \underline{B} = 0) \quad (10)$$

$$\text{curl } \underline{E} = - \frac{\partial \underline{B}}{\partial t} = - \mu_0 \frac{\partial \underline{H}}{\partial t} \quad (\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t}) \quad (11)$$

$$\text{curl } \underline{H} = \frac{\partial \underline{D}}{\partial t} \quad (\nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t}) \quad (12)$$

We can easily combine eqs 11, 12 to eliminate $E(D)$ or $H(B)$. Taking curl of 11 gives

$$\begin{aligned} \text{curl curl } \underline{E} &= - \frac{\partial}{\partial t} \text{curl } \underline{B} = - \mu_0 \frac{\partial}{\partial t} \{ \text{curl } \underline{H} \} \\ &= - \mu_0 \frac{\partial^2}{\partial t^2} \underline{D} \end{aligned}$$

Using the vector identity

$$\text{curl curl } \underline{F} = \text{grad div } \underline{F} - \nabla^2 \underline{F} \quad (\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$$

then

$$\text{grad div } \underline{E} - \nabla^2 \underline{E} = - \mu_0 \frac{\partial^2}{\partial t^2} \underline{D}$$

Since $\underline{D} = \epsilon \epsilon_0 \underline{E}$, we can simplify this expression provided we impose some restrictions on ϵ .

1. If ϵ is a position independent scalar quantity, then

$$\text{div } \underline{D} = \text{div } \epsilon \epsilon_0 \underline{E} = \epsilon \epsilon_0 \text{div } \underline{E} = 0$$

$$\therefore \text{div } \underline{E} = 0, \text{ and } \text{grad div } \underline{E} = 0$$

leading to

$$\nabla^2 \underline{E} - \mu_0 \frac{\partial^2}{\partial t^2} \underline{D} = 0 \quad (13)$$

This restricts us strictly to uniform media which are isotropic. It is sometimes necessary to consider propagation in anisotropic media, where ϵ is a tensor, when this assumption is not valid (crystalline materials often have this property). If ϵ is isotropic but varies with position (inhomogeneous), then provided the variation is slow (ϵ effectively constant over distances comparable to a wavelength) the medium can be divided into uniform elements, the simple solutions in the elements being linked by suitable boundary conditions.

2. ϵ is not time dependent.

With this restriction,

$$\frac{\partial^2 \underline{D}}{\partial t^2} = \epsilon \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$

and eqn. 13 becomes

$$\nabla^2 \underline{E} - (\mu_0 \epsilon_0) \epsilon \frac{\partial^2 \underline{E}}{\partial t^2} = 0 \quad (14)$$

Similarly we could deduce, by taking the curl of (12) etc,

$$\nabla^2 \underline{H} - (\mu_0 \epsilon_0) \epsilon \frac{\partial^2 \underline{H}}{\partial t^2} = 0 \quad (15)$$

The time dependence of ϵ can be considered in two categories; if ϵ changes little during an oscillation of the E-M wave (whose existence we shall shortly justify) (i.e. 10^{-15} s for optical waves) then the equations can be solved assuming ϵ is constant, the time dependence being introduced subsequently: if the change in ϵ takes

place in a short time interval, then, at least for optical waves, the time scale is such that essentially only another optical field could be responsible. This leads to a range of phenomena, called collectively 'non-linear optics' including second harmonic generation, electro-optic effect, parametric amplification, self-focusing.

Thus for simple linear systems, we are justified in using the separated field equations (14), (15). They are obviously in the form of wave equations, so that the basic laws of electromagnetism lead directly to the notion of electromagnetic waves. In principle, all problems in electro-magnetic theory (subject to the restrictions outlined above) can be treated by solving these equations with the appropriate boundary conditions. In practice, such a general approach is often very difficult to carry through, and is frequently unnecessary. Image forming systems, and most diffraction problems, can be treated adequately in a less rigorous way, while optical propagation on a thin glass fibre requires analysis based on the wave equations.

In order to gain some feeling for the characteristics of E-M waves, we shall treat a very simple system - namely an infinite uniform space supporting a plane wave.

Plane Waves

Look for solutions representing plane waves propagating in the z-direction. Since the waves extend infinite distances in the x-y plane, we must have

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial y} = 0 ; \quad (16)$$

$$\text{and } \frac{\partial H}{\partial x} = \frac{\partial H}{\partial y} = 0. \quad (17)$$

Considering one scalar component only of the vector equation (14), say E_x , we have

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0 \epsilon_0 \epsilon \frac{\partial^2 E_x}{\partial t^2} = 0$$

For any one harmonic component of the solution, put

$$E_x(z, t) = E_x(z) \exp(j\omega t),$$

and thus

$$\frac{\partial^2 E_x}{\partial z^2} + (\mu_0 \epsilon_0) \epsilon \omega^2 E_x = 0,$$

giving

$$E_x = E_{x+} \exp(-j\beta z) + E_{x-} \exp(+j\beta z),$$

$$\text{where } \beta^2 = (\mu_0 \epsilon_0) \epsilon \omega^2 \quad (18)$$

Hence the full solution is

$$E_x = E_{x+} \exp j(\omega t - \beta z) + E_{x-} \exp j(\omega t + \beta z);$$

representing waves travelling in the +z and -z directions. We can find similar expressions for E_y , H_x , H_y .

$$\text{Since } \operatorname{div} E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0,$$

$$\text{then using (16) we have } \frac{\partial E_z}{\partial z} = 0;$$

thus E_z is either zero or a constant, and the wave is transverse. (Similarly for H_z).

$$\text{The phase velocity, } v_{ph} = \frac{\omega}{\beta}$$

$$\text{and from (18) } v_{ph} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \cdot \frac{1}{\sqrt{\epsilon}}.$$

In a vacuum, or approx in air, $\epsilon = 1$

$$\begin{aligned} \therefore (v_{ph})_{vac} = C &= \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{(8.85 \cdot 10^{-12} \cdot 4\pi \cdot 10^{-7})^{\frac{1}{2}}} \\ &= 3 \times 10^8 \text{ m/s}. \end{aligned}$$

In other media $\epsilon > 1$, $\therefore v_{ph} < C$

Defining $\frac{C}{v_{ph}} = n$ (refractive index)

then $n = \sqrt{\epsilon}$, $n^2 = \epsilon$ (this is the permittivity at optical frequency which is likely to be lower than that at low frequency).

From eqn 7

$$\begin{aligned} \text{curl } \underline{E} &= \underline{i} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \underline{j} \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) + \underline{k} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \\ &= -\mu_0 \left\{ \underline{i} \frac{\partial H_x}{\partial t} + \underline{j} \frac{\partial H_y}{\partial t} + \underline{k} \frac{\partial H_z}{\partial t} \right\}. \end{aligned}$$

Since $\frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y} = 0$, then $\frac{\partial H_z}{\partial t} = 0$, by comparing k components.

$\therefore H_z$ (and similarly E_z) are zero or independent of t , as well as independent of x, y, z .

Comparing i th and j th components, we have

$$\begin{aligned} \frac{\partial E_y}{\partial z} &= +\mu_0 \frac{\partial H_x}{\partial t} \\ \text{and } \frac{\partial E_x}{\partial z} &= -\mu_0 \frac{\partial H_y}{\partial t} \end{aligned}$$

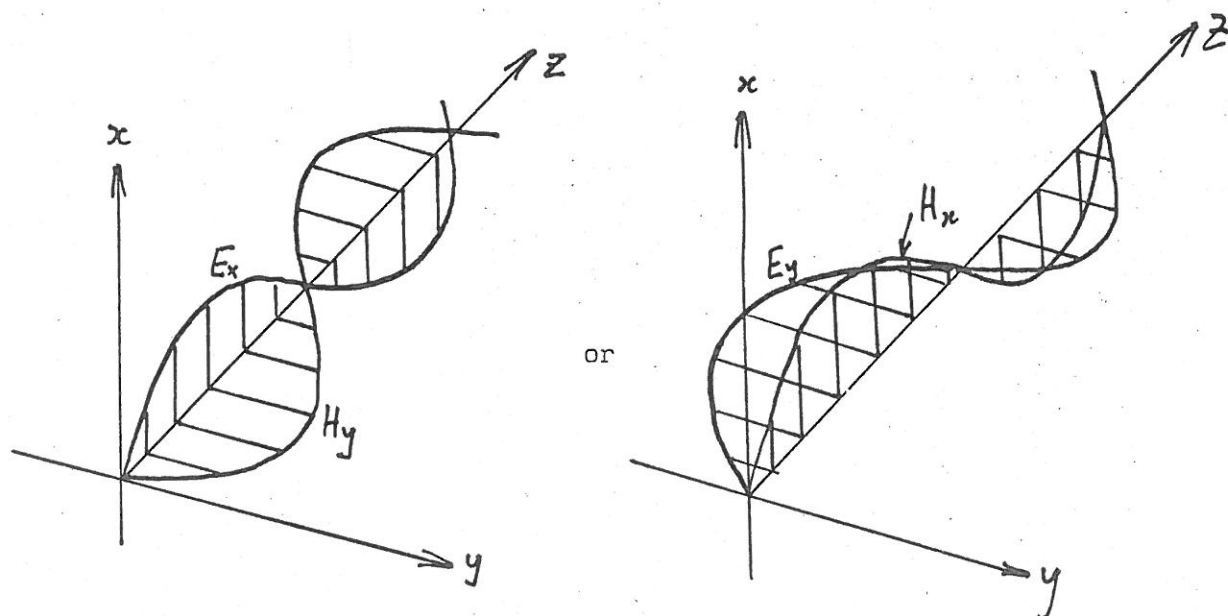
Using $E_x = E_{x+} \exp j(\omega t - \beta z)$ etc, (i.e. forward going waves)

$$E_{y+} (-j\beta) = +\mu_0 j\omega H_{x+}$$

$$E_{x+} (-j\beta) = -j\mu_0 \omega H_{y+}$$

or $H_{x+} = -\left(\frac{\beta}{\mu_0 \omega}\right) E_{y+}$

$$H_{y+} = +\left(\frac{\beta}{\mu_0 \omega}\right) E_{x+}$$



As $\frac{\beta}{\omega} = (\mu_0 \epsilon_0)^{\frac{1}{2}} \epsilon^{\frac{1}{2}}$ (from 18)

then $-\frac{H_{x+}}{E_{y+}} = +\frac{H_{y+}}{E_{x+}} = (\frac{\epsilon_0}{\mu_0})^{\frac{1}{2}} \epsilon^{\frac{1}{2}}$

for vacuum $-\frac{H_{x+}}{E_{y+}} = \frac{H_{y+}}{E_{x+}} = (\frac{\epsilon_0}{\mu_0})^{\frac{1}{2}} = \frac{1}{377} \quad (\frac{A/m}{V/m})$

$\therefore \frac{E}{H} \text{ (plane waves)} = 377 \, \Omega$ (called impedance of free space).

Power Flow in Plane Waves

Since E is measured in V/m and H in A/m , the units of $E H$ are W/m^2 and we might expect this to represent the power flowing through unit area. However, for each polarization component E and H are perpendicular. Hence we need to express the product of $\underline{E} \times \underline{H}$. Further, we need a factor of $\frac{1}{2}$ for the time average and since we are using the exponential form, one of them should be the complex conjugate. Thus we may expect the power/unit area S to be

$$S = \frac{1}{2} \int_{\text{unit area}} \underline{E} \times \underline{H}^* \cdot d\mathbf{s}$$

(this can be justified in a more rigorous way, of course).

For plane waves

$$\underline{E} \times \underline{H}^* = \underline{i} \{ \underset{\text{(zero)}}{E_y H_z - E_z H_y} \} + \underline{j} \{ \underset{\text{(zero)}}{E_z H_x - E_x H_z} \} + \underline{k} \{ E_x H_y - E_y H_x \}$$

∴ Power flows in direction of propagation only (note $E_y H_x$ is -ve).

Reflection and Refraction

Although strictly ideal plane waves would not seem likely to occur in real systems, this limiting case can be extremely helpful. Consider the interaction of a plane wave with a plane dielectric interface. We can expect that the interaction will give rise in general to a reflected and a transmitted wave - just as in the case of waves on a string meeting a discontinuity. As in that problem, we have to consider boundary conditions, (for string, equal displacement and velocity on either side of the discontinuity).

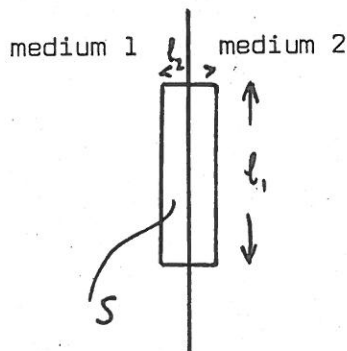
For electro-magnetic fields, the boundary conditions are the continuity of the tangential components of the electric and magnetic fields.

From eqns (3) and (4)

$$\oint \underline{E} \cdot d\underline{l} = \int_S -\mu_0 \frac{\partial H}{\partial t} \cdot d\underline{s}$$

$$\oint \underline{H} \cdot d\underline{l} = \int_S \epsilon \frac{dE}{dt} \cdot d\underline{s} \quad \text{(linear approx).}$$

For an element of the boundary,



$$l_2 \ll l_1$$

$$l_1 \ll \lambda$$

(Hence essentially uniform E, H .)

Due to refractive index n, n' , cannot have B.C.'s satisfied by perfect transmission of E, H . (at least H must change)

- 9 -

$$\frac{E_r}{E_i} = \frac{n - n'}{n + n'}$$

As $\ell_2 \rightarrow 0$, $S \rightarrow 0$. $\therefore$ R.H.S. (above) $\rightarrow 0$.

$$\therefore \oint \underline{E} \cdot d\ell = \oint \underline{H} \cdot d\ell = 0.$$

$$(E_{t_1} - E_{t_2}) \ell_1 = 0; \quad E_{t_1} = E_{t_2} \quad (\text{tangential components})$$

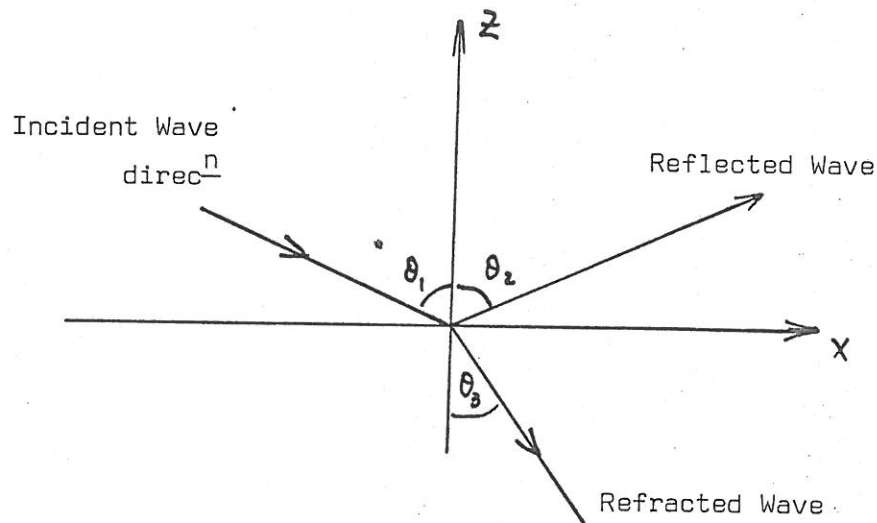
Similarly, $H_{t_1} = H_{t_2}$ (tangential components).

phase shift on reflection: either 0 or π

eg. air to glass: 0.2 (amplitude reflection) = 0.46 (intensity reflection)

Consider a monochromatic, plane wave approaching a plane dielectric surface, lying in the x-y plane, the incident wave propagating in the x-z plane.

i.e.

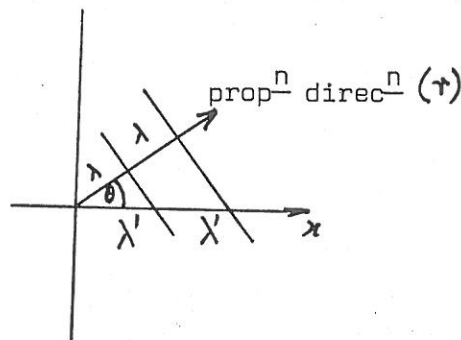


We can write, for the y components of the electric vectors,

$$E_{1y} = A_1 \exp j (\omega_1 t - k_1 (\ell_1 x + m_1 y + n_1 z)).$$

ℓ_1, m_1, n_1 , are direction cosines;

i.e.



$$\lambda' \cos \theta = \lambda$$

$$\frac{2\pi}{\lambda} \cdot r \equiv \frac{2\pi}{\lambda'} \cdot x = \frac{2\pi}{\lambda} \cos \theta \cdot x$$

$$= k \cos \theta \cdot x = k \ell x.$$

In 3 dimensions we need 3 direction cosines.

$$\text{Similarly } E_{2y} = A_{2y} \exp j (\omega_2 t - k_2 (\ell_2 x + m_2 y + n_2 z)).$$

$$E_{3y} = A_{3y} \exp j (\omega_3 t - k_3 (\ell_3 x + m_3 y + n_3 z)).$$

The boundary condition requires that

$$E_{1y} + E_{2y} = E_{3y}.$$

This can only be satisfied for all t if $\omega_1 = \omega_2 = \omega_3 (= \omega)$. It follows that $k_1 = k_2 = k$, (same medium)

$$\text{but } \frac{v}{v'} = \frac{n'}{n} = \frac{k'}{k} \quad \therefore k' = \left(\frac{n'}{n}\right) k.$$

With incident wave in x - z plane,

$$\ell_1 = \sin \theta_1, \quad m_1 = 0, \quad n_1 = -\cos \theta_1 \quad (-ve \text{ because } z \text{ away}).$$

The boundary condition can only be satisfied for all y if $m_2 = m_3 = 0$. ($\therefore$ Reflected and refracted waves lie in the incident plane.)

The directions indicated earlier now lie in one plane only, and hence

$$\ell_2 = \sin \theta_2, \quad \ell_3 = \sin \theta_3.$$

$$n_2 = \cos \theta_2, \quad n_3 = -\cos \theta_3.$$

For the boundary condition to be independent of x ,

$$A'_1 \exp(-j k \sin \theta_1) x + A'_2 \exp(-j k \sin \theta_2) x = A'_3 \exp(-j k' \sin \theta_3) x.$$

$\therefore$ we require

$$(A'_1 + A'_2) \exp(-j k \sin \theta) x \quad \text{for left-hand side}$$

$$\text{i.e. } \sin \theta_1 = \sin \theta_2 \quad (= \sin \theta).$$

$$\therefore \theta_1 = \theta_2 \quad (\text{law of reflection}).$$

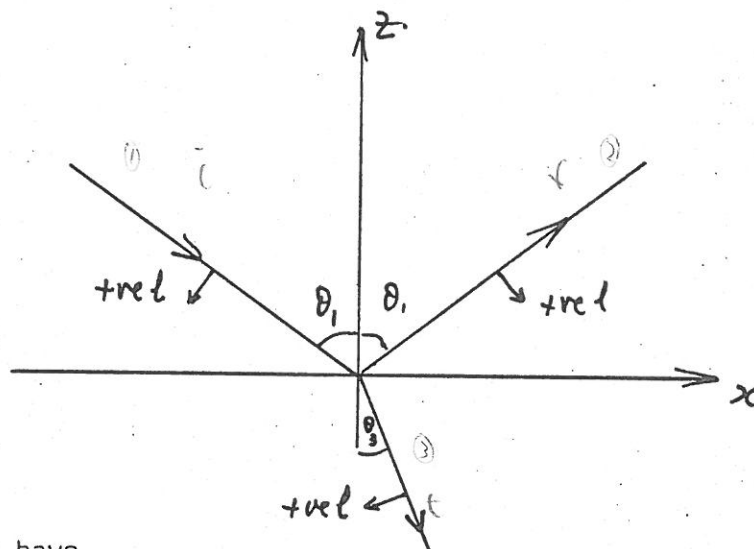
and also $k \sin \theta_1 = k' \sin \theta_3$.

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{v} f = \frac{2\pi}{c} f n$$

$n \sin \theta_1 = n' \sin \theta_3$ (Snell's law of refraction).

To deduce these results we used only one component of the particular boundary conditions specifically. We need all of them to deduce amplitudes, before and after reflection, refraction.

We have to take account of the fact that the incident electric and magnetic fields are vectors, and can be directed anywhere in a plane perpendicular to the wave direction. Since we are interested in tangential fields at the interface, one of the components we will want is simply the y component, which lies perpendicular to the plane of incidence (i.e. $E_y = E_r$, $H_y = H_r$). The other components, at right angles to the y-direction, we will call E_z , H_z (parallel). The E_z , H_z positive directions are taken to be;



Thus we have

$$\begin{aligned} E_{1r} = E_{1y} &= A_{1y} \exp j (\omega t - k (\ell_1 x + m_1 y + n_1 z)) \\ &= A_{1y} \exp j (\omega t - k (\sin \theta_1 x - \cos \theta_1 z)) \end{aligned}$$

$$\text{Similarly } E_{2r} = A_{2y} \exp j (\omega t - k (\sin \theta_1 x + \cos \theta_1 z))$$

$$E_{3r} = A_{3y} \exp j (\omega t - k' (\sin \theta_3 x - \cos \theta_3 z)) ,$$

and corresponding expressions for H_r , E_ℓ , H_ℓ .

We now apply the boundary conditions in the plane $z = 0$.
Continuity of E_r gives

$$E_{1r} + E_{2r} = E_{3r} \quad (19)$$

Continuity of H_ℓ gives

$$-H_{1\ell} \cos \theta_1 + H_{2\ell} \cos \theta_1 = -H_{3\ell} \cos \theta_3. \quad (20)$$

$$\text{Since } H_\ell = -E_r \epsilon^{\frac{1}{2}} \left(\frac{\epsilon_0}{\mu_0} \right)^{\frac{1}{2}} = -n E_r \left(\frac{\epsilon_0}{\mu_0} \right)^{\frac{1}{2}}.$$

(note sign)

eqn 20 becomes

$$n E_{1r} \cos \theta_1 - n E_{2r} \cos \theta_1 = n' E_{3r} \cos \theta_3. \quad (21)$$

Continuity of E_ℓ gives

$$(E_{1\ell} - E_{2\ell}) \cos \theta_1 = E_{3\ell} \cos \theta_3 \quad (22)$$

and continuity of H_r ,

$$H_{1r} + H_{2r} = H_{3r}$$

$$\text{and since } H_r = +n E_\ell \left(\frac{\epsilon_0}{\mu_0} \right)^{\frac{1}{2}}.$$

$$n E_{1\ell} + n E_{2\ell} = n' E_{3\ell}. \quad (23)$$

Since we are concerned with the plane $z = 0$, and also that
 $k \sin \theta_1 = k' \sin \theta_3$, these conditions can be written as

$$A_{1r} + A_{2r} = A_{3r}$$

$$(A_{1r} - A_{2r}) n \cos \theta_1 = A_{3r} n' \cos \theta_3$$

$$(A_{1\ell} - A_{2\ell}) \cos \theta_1 = A_{3\ell} \cos \theta_3$$

$$(A_{1\ell} + A_{2\ell}) n = A_{3\ell} n'$$

Fresnel's Equations

By judicious simplification, we can deduce

$$A_{2\ell} = A_{1\ell} \frac{\tan (\theta_1 - \theta_3)}{\tan (\theta_1 + \theta_3)} \quad (24)$$

$$A_{2r} = - A_{1r} \frac{\sin (\theta_1 - \theta_3)}{\sin (\theta_1 + \theta_3)} \quad (25)$$

$$A_{3\ell} = A_{1\ell} \frac{2 \sin \theta_3 \cos \theta_1}{\sin (\theta_1 + \theta_3) \cos (\theta_1 - \theta_3)} \quad (26)$$

$$A_{3r} = A_{1r} \frac{2 \cos \theta_1 \sin \theta_3}{\sin (\theta_1 + \theta_3)} \quad (27)$$

These expressions predict one property which is important in the design of lasers, amongst other things. Consider eqn 24. When $\theta_1 + \theta_3 = \pi/2$, $A_{2\ell} = 0$ ($A_{2r} \neq 0$). Thus at an angle of incidence that satisfies this condition, the reflected wave is plane polarized (perpendicular to the plane of incidence).

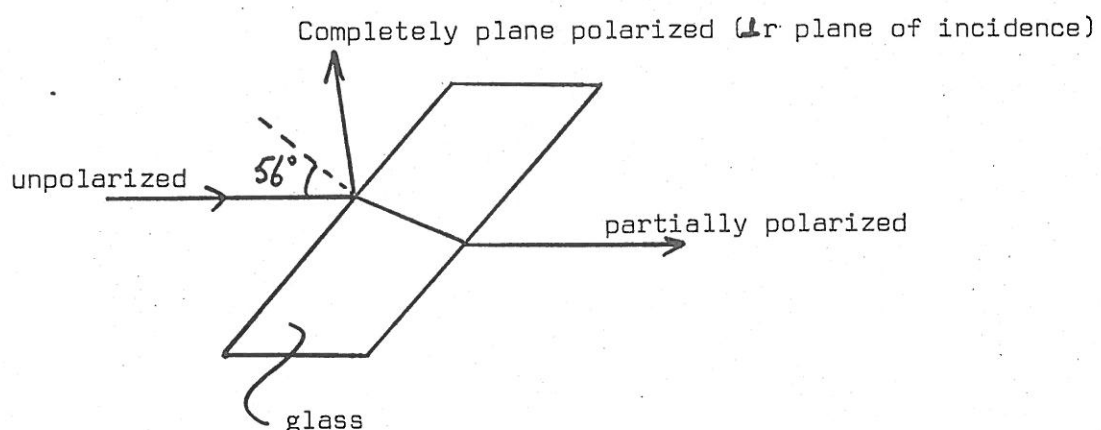
$$\text{Since } \frac{\sin \theta_1}{\sin \theta_3} = \frac{n'}{n}$$

then at this critical angle

$$\frac{\sin \theta_1}{\sin (\pi/2 - \theta_1)} = \tan \theta_1 = \frac{n'}{n}$$

This angle of incidence is called Brewster's angle. For a typical glass, $n' \approx 1.5$, $n = 1$

giving $\theta_1 \approx 56^\circ$.



Some useful results can also be obtained for normal incidence.

Thus $\theta_1 = 0$, hence $\theta_3 = 0$ and eqns 24, 25, 26, 27 become

$$A_{2_l} = A_{1_l} \left(\frac{n' - n}{n' + n} \right)$$

$$A_{2_r} = - A_{1_r} \left(\frac{n' - n}{n' + n} \right)$$

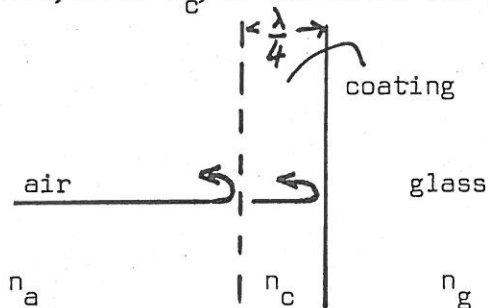
$$A_{3_l} = A_{1_l} \left(\frac{2n}{n' + n} \right)$$

$$A_{3_r} = A_{1_r} \left(\frac{2n}{n' + n} \right)$$

Note, at normal incidence there is nothing to distinguish l and r components, The results for A_{2_l} and A_{2_r} are in fact identical, the sign change arises simply from the definition of +ve l .

For $n' > n$, A_{2_r} has opposite sign to A_{1_r} , hence there has been a π phase shift. For $n' < n$, no phase shift.

This result is made use of in the blooming of lenses, also in the manufacture of laser mirrors. If a plane wave approaches a plane glass-air interface normally, some reflection will occur. If a thin layer of dielectric, index n_c , is coated on the glass, i.e.



then we can arrange that the reflection from the air-coating interface destructively interferes with that from the coating-glass interface.

If $n_a < n_c < n_g$, then phase shifts of π at each reflection, and thickness $d = \lambda/4$ required for destructive interferences.

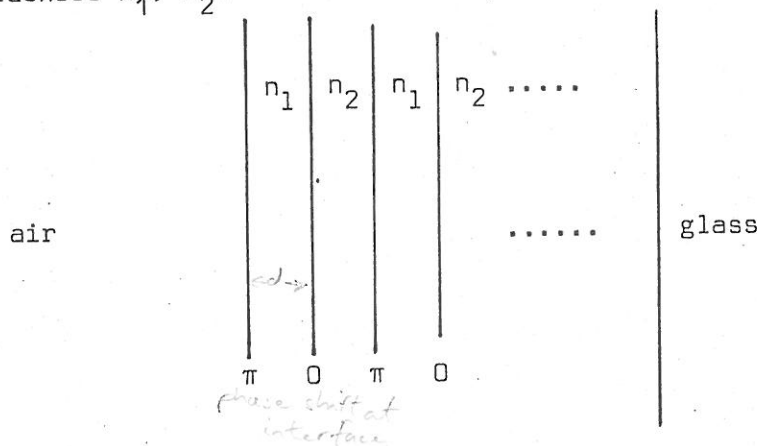
Also if

$$\frac{n_c - n_a}{n_c + n_a} = \frac{n_g - n_c}{n_g + n_c},$$

i.e. $n_c = \sqrt{n_a n_g}$,

then interference is complete and no net reflection occurs. (Notice, the required thickness is wavelength dependent.)

For laser mirrors, put on many coatings of two materials with incidences n_1, n_2 .



With $n_1, n_2 > n_a$ and $n_1 > n_2$, phase shifts alternately π and 0 . Hence for $d = \lambda/4$, constructive interference occurs, and with sufficient layers, very high reflectance is obtained.

Loughurst : Geometrical & Physical Optics

Total internal reflection, angle greater than critical angle : get total reflection, but also get a small field inside the glass

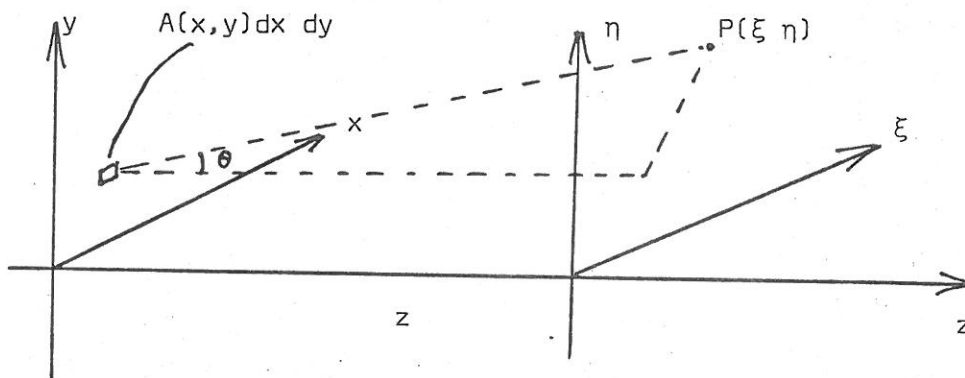
field is exponentially decaying $\left(e^{-\kappa' z \left(\frac{1}{n_1} \right)^2 \sin^2 \theta_1} - 1 \right)^{1/2}$

May use "frustrated int. refl." for measuring small distances.

Huygens, Fresnel, Scalar Theory of Diffraction

Huygens' principle (1678) - that every point on a wave front acts as a secondary source of waves - should be familiar. The principle was developed by Fresnel (1815) - see Longhurst Chapter 10, Second Edition. For any wave emanating from a point we may expect the phase factor, $\exp j(\omega t - kr)$; also the amplitude should fall away as A_0/r (energy as $1/r^2$). In addition to these obvious points, Fresnel found it necessary to include the factor $\frac{1}{j\lambda} f(\theta)$ where $f(\theta)$ is slowly varying for small θ , and is zero for $\theta = \pi$ (i.e. no backward waves). The justification is essentially to make the theory self-consistent. The principle in this form works well enough for most diffraction problems, and it is an enormously useful technique.

For example, if we know the amplitude distribution in a plane of some complex wave, we can simply write down the amplitude distribution in another distant plane.



$$A(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{j\lambda r} A(x, y) \exp j(\omega t - kr_{A \rightarrow P}) dx dy$$

We have assumed that z is large enough, that θ is always small, and $f(\theta) \rightarrow 1$.

$$r_{A \rightarrow P} = [z^2 + (\xi - x)^2 + (\eta - y)^2]^{\frac{1}{2}}$$

For small angles $z \gg (\xi - x), (\eta - y)$, hence using the binomial theorem,

$$r_{A \rightarrow P} = z + \frac{(\xi^2 + \eta^2)}{2z} + \frac{(x^2 + y^2)}{2z} - \frac{(\xi x + \eta y)}{z}$$

learn
derivation
- provide
exam
question

more formally $[\exp(j\omega t - kz)]$ with time dependence

$$\therefore A(\xi, \eta) = \frac{1}{j\lambda z} \exp - jkz \cdot \exp - jk \frac{(\xi^2 + \eta^2)}{2z}$$

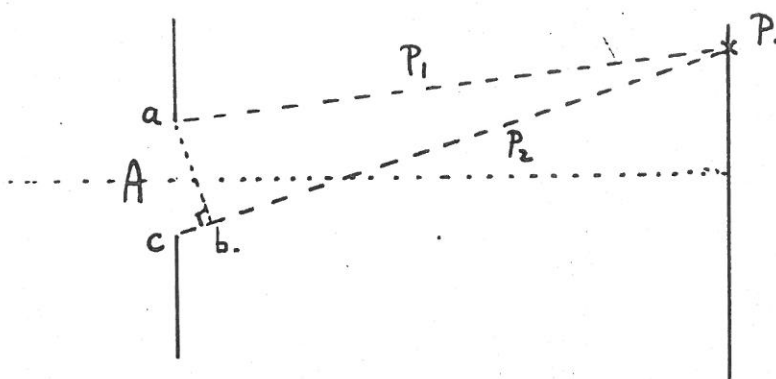
(28)

$$\lim_{z \rightarrow \infty} \iint_{-\infty}^{\infty} A(x, y) \exp - jk \frac{(x^2 + y^2)}{2z} \times \exp jk \frac{(\xi x + \eta y)}{z} dx dy$$

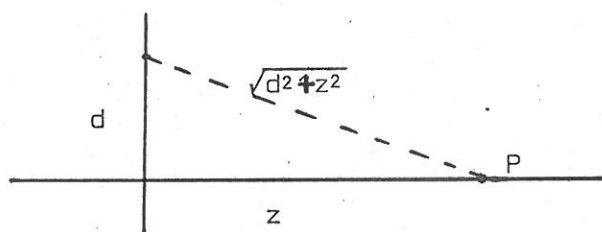
// times

(Note in first term on R.H.S., r replaced by z , since $A(\xi, \eta)$ insensitive to this multiplying factor and $r \approx z$).

We can often simplify this expression further. Consider diffraction through an aperture in a screen e.g.



In order to determine the amplitude at P , we need to consider the phases at P of the sources distributed across A . Clearly if z is large, the two limiting paths P_1, P_2 are virtually parallel and the line ab is $\perp$ to both aP and cP . This simplifies the analysis considerably. If we take P to lie on the axis, then, for an aperture width $2d$,



we require $[\sqrt{d^2 + z^2} - z] < \lambda$ for the 'far field' approximation to be strictly valid.

$$\text{Thus } (z + \frac{d^2}{2z}) - z < \lambda$$

$$\frac{kd^2}{2z} < 2\pi.$$

Generally, if the factor $\frac{kd^2}{2z} < 1$ radn, then the far-field is taken to be satisfied. As numerical examples;

fairly wide
(π maybe better)

for $d = 1 \text{ cm}; \lambda = 5.10^{-7} \text{ m}$ (green) then $z_c > 500 \text{ m}$
 $d = 1 \text{ mm} \quad \lambda = 5.10^{-7} \text{ m} \quad \text{then } z_c > 6 \text{ m}$

When $z < z_c$, we are concerned with near field or Fresnel diffraction; when $z > z_c$ we have far-field or Fraunhofer diffraction.

Fraunhofer Diffraction

Amplitude distribution reduces to

$$A(\xi, \eta) = \frac{1}{j\lambda z} \exp(-jkz) \exp(-jk \frac{(\xi^2 + \eta^2)}{2z}) \iint_{-\infty}^{\infty} A(x, y) \exp(jk \frac{(\xi x + \eta y)}{z}) dy dx$$

if we write $\omega_x = -\frac{2\pi}{\lambda} \frac{\xi}{z}$; $\omega_y = -\frac{2\pi}{\lambda} \frac{\eta}{z}$, (spatial frequency)

then

$$A(\xi, \eta) = \{ \} \iint_{-\infty}^{\infty} A(x, y) \exp(-j(\omega_x x + \omega_y y)) dx dy$$

and the integral is the Fourier Transform of the aperture field distribution, $A(x, y)$.
2 dimensions

Fresnel Diffraction

As an example of Fresnel Diffraction, consider the following. Suppose that in some plane (x, y) we have a Gaussian amplitude distribution

$$A(x, y) = A_0 \exp\left(-\frac{x^2 + y^2}{a^2}\right)$$



Since $A(x, y)$ is real, the phase distribution is uniform.

The amplitude distribution in a plane distance z away is

$$A(\xi, \eta) = \{ \} \int_{-\infty}^{\infty} A \exp\left(-\frac{x^2 + y^2}{a^2}\right) \exp\left(-\frac{jk(x^2 + y^2)}{2z}\right) \exp\left(jk \frac{(x\xi + y\eta)}{z}\right) dy dx$$

Since the variables are separable we need consider only part of the integral

Put $I = \int_{-\infty}^{\infty} \exp(-x^2) \cdot \left[\frac{jk}{2z} + \frac{1}{a^2} \right] \exp\left(j \frac{k\xi x}{z}\right) dx$

with $\left(\frac{jk}{2z} + \frac{1}{a^2}\right) = b$

$$\frac{jk\xi}{z} = c_1$$

$$\frac{jk\eta}{z} = c_2$$

$$I = \int_{-\infty}^{\infty} \exp(-bx^2 + cx) dx = \sqrt{\frac{\pi}{b}} \exp\left(\frac{c^2}{4b}\right)$$

substitution used for integration was $p = \sqrt{b}x - \frac{c}{\sqrt{b}}$

Hence,

$$A(\xi, \eta) = \left\{ \right\} A_0 \frac{2\pi z a^2}{jka^2 + 2z} \exp - \frac{k^2 a^2 (\xi^2 + \eta^2)}{2z(jka^2 + 2z)}$$

$$= \left[\frac{A_0}{j\lambda z} \exp - jkz \frac{2\pi z a^2}{(jka^2 + 2z)} \right] \exp - \frac{k}{2z} (\xi^2 + \eta^2) \left(j + \frac{ka^2}{jka^2 + 2z} \right)$$

When we note that, for any particular output plane, z is a constant, we see that the square bracket is constant. $\therefore$ The information about the amplitude distribution in the ξ, η plane is contained in the exponential term only.

Rearranging exponential gives

$$A(\xi, \eta) = [] \exp - \frac{k}{2z} (\xi^2 + \eta^2) \left[j \frac{4z^2}{4z^2 + k^2 a^4} + \frac{2kza^2}{4z^2 + k^2 a^4} \right] \quad (29)$$

The real part of the exponential describes the distribution of $|A|$, and the imaginary part the phase distribution.

For real part, putting $\xi^2 + \eta^2 = r^2$

$$|A| \propto \exp - \frac{k^2 a^2}{4z^2 + k^2 a^4} \cdot r^2 \quad = \exp - \frac{r^2}{\omega^2} \quad \leftarrow \text{see below}$$

proportional to

The shape of the beam remains Gaussian. Defining beam width, ω as the $1/e$ radius.

then $\omega^2 = \frac{4z^2 + k^2 a^4}{k^2 a^2}$

for $4z^2 \ll k^2 a^4$

then $\omega^2 = a^2 = \omega_0^2$ (const, independent of z)

for $4z^2 \gg k^2 a^4$

$$\omega^2 = \frac{4z^2}{k^2 a^2} \quad \omega = \frac{2z}{ka}$$

and $\frac{\omega}{z} = \frac{2}{k\omega_0} = \text{const.}$

$$\omega = \frac{2z}{k\omega_0}$$

If we put $z = z_0$ at the transition from one regime to the other, then

$$4z_0^2 = k^2 a^4$$

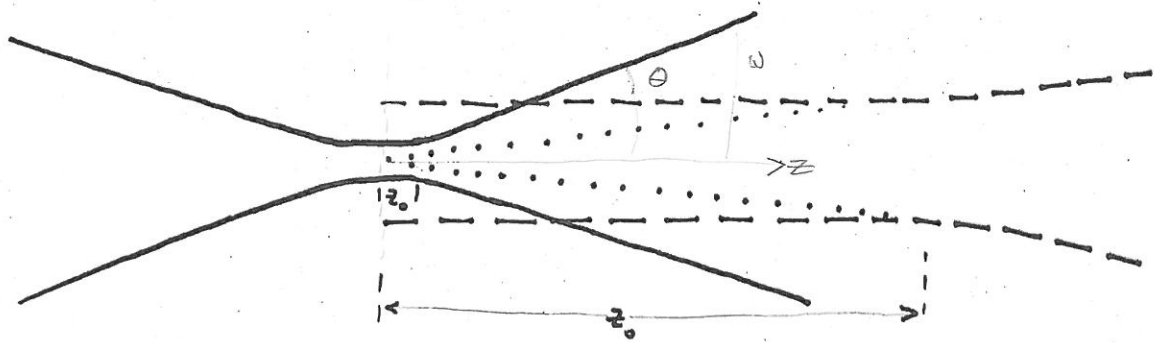
$$z_0 = \frac{ka^2}{2} = \frac{k\omega_0^2}{2}$$

i.e. $z \ll \frac{ka^2}{2}$

i.e. for a certain distance, the field is the same shape as the input (in amplitude)

i.e. propagates out in a uniform cone, appearing to have come from a pt. at the origin.

The Gaussian beam, therefore, has the following shape



$$4z_0^2 = k^2 a^4$$

i.e. z_0 = distance where change from constant width to constant divergence

$$\therefore z_0 = \frac{k a^2}{2}$$

$$\theta = \frac{\omega}{z} = \frac{2}{k a} = \frac{\lambda}{\pi a} = \left(\frac{1}{\pi}\right) \frac{\lambda}{a}$$

$$= \frac{\omega_0}{z_0}$$

Thus, small divergence gives large spot diameter and long focal region; or large aperture (ω_0) gives small divergence ($\frac{2}{k \omega_0}$)

Numerical example:

Typical laser; $\lambda = 5000 \text{ \AA} = 0.5 \cdot 10^{-6} \text{ m}$

$$\omega_0 \approx 0.3 \text{ mm}$$

$$\therefore z_0 \approx 20 \text{ cms}$$

$$\theta = 1.5 \text{ m rdns}$$

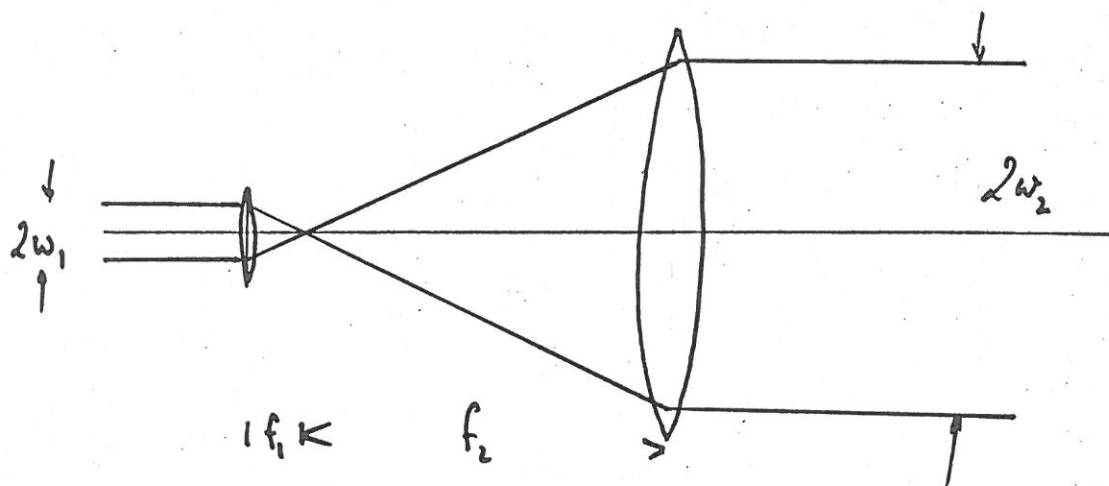
Microwave dish; $\lambda = 1 \text{ cm}$

$$\omega_0 \approx 10 \text{ cm}$$

$$\therefore z_0 = 3 \text{ m}$$

$$\theta = 30 \text{ m rdns}$$

The divergence of a laser beam can be easily reduced by using a telescope to increase the beam width.

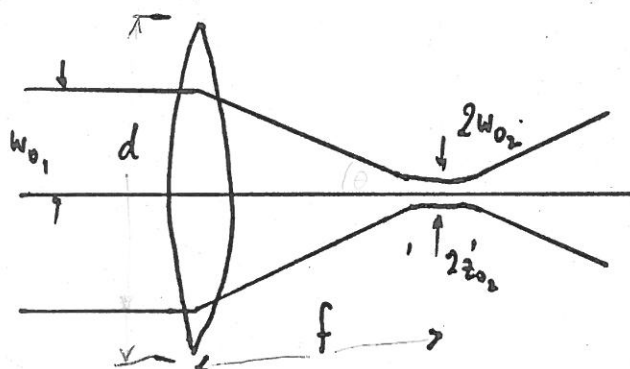


$$\omega_2 = \left(\frac{f_2}{f_1} \right) \omega_1$$

$$z_{o2} = \frac{\pi}{\lambda} \left(\frac{f_2}{f_1} \right)^2 \omega_1^2$$

for $f_2/f_1 = 100$, $\omega_2 = 3 \text{ cms.}$ $z_{o2} = 2 \text{ kms}$
 $\omega_2 = 0.015 \text{ m rdns}$

Another useful application of this result is in spatial filtering. An ideal Gaussian beam can be focussed to the smallest possible spot size.



$$\theta \approx \frac{w_{01}}{f} \approx \frac{d}{4f}$$

Also $\theta = \frac{w_{02}}{z_{02}}$

$$\therefore w_{02} = \frac{d}{4f} \cdot z_{02} = \frac{d}{4f} \cdot \frac{k w_0^2}{2}$$

$$\therefore w_{02} = \frac{8f}{d} \frac{1}{k} = \frac{4f}{\pi d} \cdot \lambda$$

The larger f/d , the larger w_{02} and hence z_{02} and consequently the depth of field that remains in focus. The smallest spot size is limited by $f/d \approx 1$, hence $w_{02} \approx \lambda$. If the beam is not quite Gaussian, the deviations will cause the image spot to broaden. If a pin-hole of suitable size is placed at the focal

*d = diameter of lens
must make d larger than the 1/2 radius to avoid diffraction effects.
to take d = 1/2
1/2 radius = d/4*

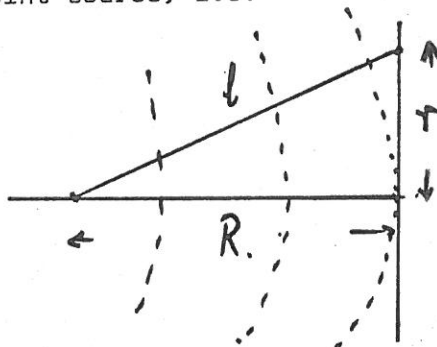
point, the non-Gaussian elements of the beam can be removed.

Returning to eq. (29), the imaginary part of the exponential describes the phase distribution in the output plane.

i.e. Phase distribution corresponds to $\exp - j \frac{2zk}{4z^2 + k^2 a^2} \cdot r^2$.

$$= \exp - j \frac{k^2 z}{2(z^2 + z_0^2)} \cdot r^2. \quad (30)$$

If we consider the phase distribution across a plane distance R from a point source, i.e.



$$l^2 = R^2 + r^2 = R^2 \left(1 + \frac{r^2}{R^2} \right)$$

$$l = R + \frac{r^2}{2R} \quad (\text{for } R \gg r)$$

$$\begin{aligned} \text{Phase distribution, } \phi(r) &= \exp - jkl \\ &= \underbrace{\exp - jkR}_{\substack{\text{independent} \\ \text{of } r, \text{ hence} \\ \text{affects absolute phase only.}}} \cdot \exp - \frac{jk}{2R} r^2. \end{aligned}$$

By comparison with (30), we see that the phase fronts of the Gaussian beam are spherical, with radius of curvature, R, given by

$$R = \frac{z^2 + z_0^2}{z}$$

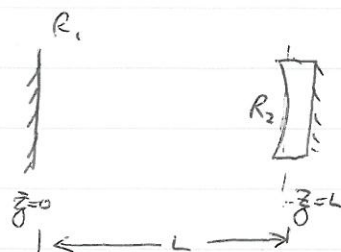
Thus at $z = 0$, $R \rightarrow \infty$ (plane phase fronts, as expected due to symmetry)

at $z = z_0$, $R = 2z_0$

$z \gg z_0$, $R \approx z$.

The shapes and curvatures of these phase fronts have great importance in the design of optical cavities for lasers, as we shall see.

Lasers: Configuration of laser mirrors



Plano-Concave

$$R_1 = \infty$$

$$R_2 = L$$

$$R_2 = \frac{z^2 + z_0^2}{z^2} = L + \frac{z_0^2}{L}$$

$$z_0 = [L(R_2 - L)]^{1/2}$$

$$\therefore \boxed{R_2 > L}$$

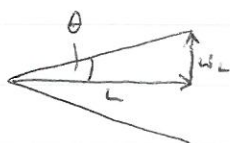
$$\omega_0^2 = \frac{2}{k} z_0$$

$$\omega_0 = \left(\frac{\lambda z_0}{\pi} \right)^{1/2}$$

$$\theta = \frac{\omega_0}{z_0}$$

$$\omega_L = L\theta$$

$$(L > z_0)$$



If $L = 0.5\text{m}$ (typical He-Ne laser)

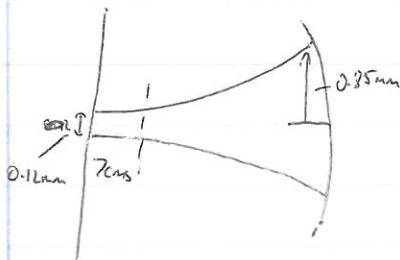
$$\lambda = 633\text{nm}$$

(see table overleaf)

$$L = 0.5\text{m}, \lambda = 633\text{nm}$$

R_2	Z_0	W_0	θ	W_L	
0.51	7cm	0.12mm	1.7mrad	0.85mm	①
0.6	22cm	0.21mm	1mrad	0.5mm	②
1.00	50cm	0.32mm	0.6mrad	0.32mm	③

①



③



Want

- ① Low beam divergence
- ② Optimum "filling factor" for fitting the beam to make the best use of the amplification gap (fixed by other effect)

Can also use 2' concave mirrors



$$z=0$$