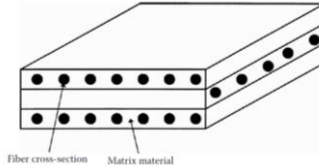


Macromechanical analysis of a lamina

An alternative to the Micromechanical approach is to use the Macromechanical approach to understanding composite materials. Macromechanical approaches can be applied to either a Lamina or a Laminate. A Lamina is 1 ply and is typically ~0.125mm thick. A Laminate is constructed by stacking a number of Laminae.



The above shows a Laminate made up of 3 Laminae. In order to understand the Laminate, we must first understand the Lamina.

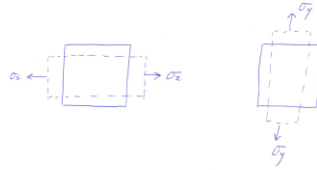
As a reminder, consider the following definitions when applied to materials:

Homogeneous:	uniform composition throughout.
Isotropic:	identical values of a property in all directions.
Anisotropic:	different properties in different directions.
Orthotropic:	properties that differ along three mutually-orthogonal axes

The Lamina is not a homogeneous material, however macromechanical analysis is based on average properties based on the individual mechanical properties of the fiber and the matrix as discussed in the micromechanical section. However, even with the homogenization of a Lamina it behaves different from a homogeneous isotropic material.

Stiffness and Compliance Matrices

Consider a Linear Elastic material stressed in x and y;



The Strain in x is given by

$$\varepsilon_x = \frac{1}{E} \sigma_x - \frac{\nu}{E} \sigma_y$$

Similarly,

$$\varepsilon_y = \frac{1}{E} \sigma_y - \frac{\nu}{E} \sigma_x$$

If we considered this in 3 dimensions we would have the following

$$\varepsilon_x = \frac{1}{E} \sigma_x - \frac{\nu}{E} \sigma_y - \frac{\nu}{E} \sigma_z \quad [1]$$

$$\varepsilon_y = -\frac{\nu}{E} \sigma_x + \frac{1}{E} \sigma_y - \frac{\nu}{E} \sigma_z \quad [2]$$

$$\varepsilon_z = -\frac{\nu}{E} \sigma_x - \frac{\nu}{E} \sigma_y + \frac{1}{E} \sigma_z \quad [3]$$

If this material were also subjected to shear, since $G = \frac{\tau}{\gamma}$, then we would also have

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \quad [4]$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \quad [5]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad [6]$$

If we write equations 1-6 in matrix form we thus get;

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}, \quad [7]$$

Equation 7 is called the ‘compliance matrix’, [S] of an isotropic material. If this were to be inverted we would get the ‘stiffness matrix’, [C]. Assuming that the composite material is linear and elastic IS acceptable, however, assuming that the material is isotropic IS NOT acceptable. Therefore, the constants relating stress and strain are more than those seen in equation 7. In the most generalized form for an isotropic material we would see the below (using 1,2,3 instead of x,y,z)

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}. \quad [8]$$

Thus for an Isotropic material

$$S_{11} = \frac{1}{E} = S_{22} = S_{33}$$

$$S_{12} = -\frac{\nu}{E} = S_{13} = S_{21} = S_{23} = S_{31} = S_{32}$$

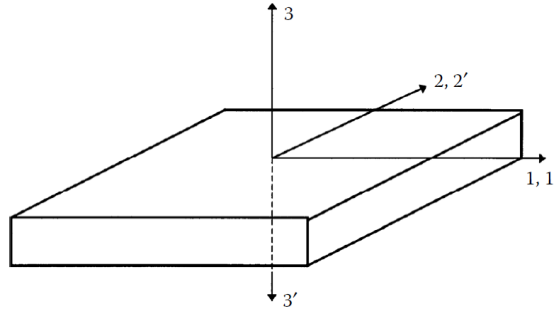
$$S_{44} = \frac{1}{G} = S_{55} = S_{66},$$

The compliance matrix [8] appears to have 36 unknowns, however, due to the fact that this matrix is symmetrical, $S_{61}=S_{16}$ etc so this matrix reduces to the following with only 21 unknowns.

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} \quad [9]$$

Monoclinic Composites

If the material has one plane of symmetry (i.e. it is MONOCLINIC) such as in the following figure in which the structure is symmetrical about the 3 plane then you can simplify the compliance matrix;



The **direct strains** are not related to the shear stresses in the 23 or the 31 planes

- ϵ_1 is not related to τ_{23} , therefore $S_{14}=0$
- ϵ_1 is not related to τ_{31} , therefore $S_{15}=0$
- ϵ_2 is not related to τ_{23} , therefore $S_{24}=0$
- ϵ_2 is not related to τ_{31} , therefore $S_{25}=0$
- ϵ_3 is not related to τ_{23} , therefore $S_{34}=0$
- ϵ_3 is not related to τ_{31} , therefore $S_{35}=0$

The direct stresses are not related to the **shear strains in the 23 plane**

- γ_{23} is not related to σ_1 , therefore $S_{41}=0$
- γ_{23} is not related to σ_2 , therefore $S_{42}=0$
- γ_{23} is not related to σ_3 , therefore $S_{43}=0$

The **shear strain in the 23 plane**, is not related to the shear stress in the 12 plane

- γ_{23} is not related to τ_{12} , therefore $S_{46}=0$

Similarly, for the **shear strains in the 31 plane**

- γ_{31} is not related to σ_1 , therefore $S_{51}=0$
- γ_{31} is not related to σ_2 , therefore $S_{52}=0$
- γ_{31} is not related to σ_3 , therefore $S_{53}=0$
- γ_{31} is not related to τ_{12} , therefore $S_{56}=0$

Finally, the **shear strain in the 12 plane** is not related to the shear stress in the 23 plane or the 31 planes

- γ_{12} is not related to τ_{23} , therefore $S_{64}=0$
- γ_{12} is not related to τ_{31} , therefore $S_{65}=0$

The direction perpendicular to the plane of symmetry is the **principal direction**, and the 21 unknowns has now reduced to 13 unknowns, thus the compliance matrix is now;

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \quad [10]$$

Feldspar is an example of a monoclinic material.

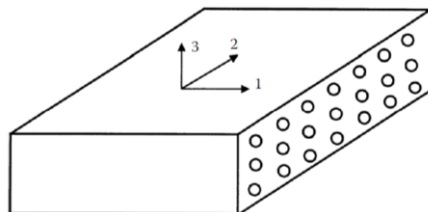
Orthotropic Composites

If the material has three mutually perpendicular planes of symmetry, i.e. it is Orthotropic then

- ε_1 is not related to τ_{12} , therefore $S_{16}=0$
- ε_2 is not related to τ_{12} , therefore $S_{26}=0$
- ε_3 is not related to τ_{12} , therefore $S_{36}=0$
- γ_{23} is not related to τ_{31} , therefore $S_{45}=0$
- γ_{31} is not related to τ_{23} , therefore $S_{45}=0$
- γ_{12} is not related to σ_1 , therefore $S_{16}=0$
- γ_{12} is not related to σ_2 , therefore $S_{26}=0$
- γ_{13} is not related to σ_3 , therefore $S_{36}=0$

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \quad [11]$$

Note that the number of unknowns has now reduced to 9. An example of an orthotropic material would be a single lamina of continuous fiber composite arranged in a rectangular array;



A unidirectional 2D lamina

Since $\sigma_3 = 0$, $\tau_{23} = 0$ and $\tau_{31} = 0$ then the problem of understanding the material behavior reduces to a 2D problem;

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{bmatrix}$$

Since γ_{23} and γ_{31} are clearly zero, these can be omitted thus

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 \\ S_{12} & S_{22} & S_{23} & 0 \\ S_{13} & S_{23} & S_{33} & 0 \\ 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ \tau_{12} \end{bmatrix}$$

If it is assumed that ε_3 is zero, this can also be omitted, i.e.

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Where [S] is the reduced compliance matrix and

$$S_{11} = \frac{1}{E_1}; S_{22} = \frac{1}{E_2}; S_{12} = -\frac{\nu_{12}}{E_1} \text{ and } S_{66} = \frac{1}{G_{12}}$$

Examples

1)

- a) Write the compliance matrix in terms of the 9 engineering constants for an orthotropic material.
- b) Hence find the compliance matrix for a Graphite/Epoxy lamina whose material properties are as follows;

$$\begin{aligned}E_1 &= 181 \text{ GPa} \\E_2 &= 10.3 \text{ GPa} \\E_3 &= 10.3 \text{ GPa} \\\nu_{12} &= 0.28 \\\nu_{23} &= 0.60 \\\nu_{13} &= 0.27 \\G_{12} &= 7.17 \text{ GPa} \\G_{23} &= 3.0 \text{ GPa} \\G_{31} &= 7.0 \text{ GPa}\end{aligned}$$

2) For a Graphite/Epoxy unidirectional lamina, find

- a) The reduced compliance matrix
- b) Strains in the 1-2 co-ordinate axis if the following stresses are applied.
 $\sigma_1 = 2 \text{ MPa}$, $\sigma_2 = -3 \text{ MPa}$, $\tau_{12} = 4 \text{ MPa}$.

The material properties are as follows;

$$\begin{aligned}E_1 &= 181 \text{ GPa} \\E_2 &= 10.3 \text{ GPa} \\\nu_{12} &= 0.28 \\G_{12} &= 7.17 \text{ GPa}\end{aligned}$$

Notes adapted from/further reading:
Mechanics of composite materials, 2nd edition, Autar K Kaw