Differential Equations

Ordinary Differential Equations

Dr Owen Williams – Swansea Metropolitan University:

1.1 Introduction

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics, and other disciplines.

Differential equations arise in many areas of science and technology: whenever a deterministic relationship involving some continuously varying quantities (modelled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time varies. Newton's Laws allow one to relate the position, velocity, acceleration and various forces acting on the body and state this relation as a differential equation for the unknown position of the body as a function of time. In some cases, this differential equation (called an equation of motion) may be solved explicitly.

An example of modelling a real world problem using differential equations is determination of the velocity of a ball falling through the air, considering only gravity and air resistance. The ball's acceleration towards the ground is the acceleration due to gravity minus the deceleration due to air resistance. Gravity is constant but air resistance may be modelled as proportional to the ball's velocity. This means the ball's acceleration, which is the derivative of its velocity, depends on the velocity. Finding the velocity as a function of time involves solving a differential equation.

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions, which is the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

There are very few methods of explicitly solving nonlinear differential equations; those that are known typically depend on the equation having particular symmetries. Nonlinear differential equations can exhibit very complicated behaviour over extended time intervals, characteristic of chaos. Linear differential equations frequently appear as approximations to nonlinear equations. These approximations are only valid under restricted conditions. For example, the harmonic oscillator equation is an approximation to the nonlinear pendulum equation that is valid for small amplitude oscillations.

1.2 Basic Examples

Generally,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

is called a 'second order differential equation' (ODE2), where *a*, *b* and *c* are constants and *f(x)* is a function of *x*. Also *y* is a function of *x*.

Normally we are dealing with the case where *f(x)*=0

Then the equation becomes the 'homogeneous second order differential equation'.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

If we need to solve an equation of this type, how do we go about it?

We need to find *y*, a function of *x*, which satisfies the above equation. Note that the solution needs to be a function of *x*.

Lets try the function $y = e^{mx}$. *But WHY?*

Because if you differentiate e^{mx} you end up with another e^{mx} term. Thus,

$$\frac{dy}{dx} = me^{mx}$$

and differentiating again gives us:

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

Substituting these into

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

gives:

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

or, factorising out e^{mx}

$$e^{mx}(am^2 + bm + c) = 0$$

So, now we know that either $e^{mx} = 0$, or $am^2+bm+c = 0$?

For any real vales of mx, $e^{mx} \neq 0$. (One of the special properties of the exponential function is that is can never equal zero). Therefore we must have:

$$(am^2 + bm + c) = 0$$

which, of course, is a quadratic equation that we call the **characteristic equation**. Since every quadratic equation is solvable, $y = e^{mx}$ is a solution of the original equation provided that m satisfies

$$(am^2 + bm + c) = 0$$

We can get the solutions to the above equation by using the quadratic equation formula or by factorising the quadratic where appropriate.

So how many different types of 'roots' are there to a quadratic equation?

Three!

1. Real and Different, m_1 *and* m_2
2. Real and Equal, $m_1 = m_2 = m$
3. Complex, $m = \alpha \pm i\beta$

For the case of real and different roots, we have the general solution:

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

If the roots are real and equal we get:

$$y = (A + Bx)e^{mx}$$

and if the roots are complex, we have:

$$y = e^{\alpha x}[A\cos(\beta x) + B\sin(\beta x)]$$

Example 1

Solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$$

We need to find *y* which satisfies the above equation. What is the **characteristic equation**?

$$m^2 + 2m - 3 = 0$$
$$Factorise....$$
$$(m+3)(m-1)$$
$$Solve....$$
$$m_1 = -3, m_2 = 1$$

Hence the solution to the characteristic equation is two real and different roots, $m_1=-3$ and $m_2=1$. Substituting these into the general solution

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

we get:

$$y = Ae^{-3x} + Be^{x}$$

This is the general solution to the above example.

Example 2

Solve

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

We need to find *y* which satisfies the above equation. What is the **characteristic equation**?

$$m^2 + 2m + 1 = 0$$
$$Factorise....$$
$$(m+1)^2$$
$$Solve....$$
$$m_1 = m_2 = -1$$

Hence the solution to the characteristic equation is two real and different roots, $m_1=-3$ and $m_2=1$. Substituting these into the general solution

$$y = (A + Bx)e^{mx}$$

we get:

$$y = (A + Bx)e^{-x}$$

This is the general solution to the above example.

<u>Example 3</u>

Solve

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

We need to find *y* which satisfies the above equation. What is the **characteristic equation**?

$$m^2 + m + 1 = 0$$

To solve this quadratic equation we need to use the formula:

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Putting *a = 1, b = 1 and c = 1* into the equation gives:

$$m = \frac{-1 \pm \sqrt{1^2 - 4}}{2}$$

$$m = \frac{-1 \pm \sqrt{-3}}{2}$$

$$m = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$$

$$m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

As we can see, this is a '*complex*' root.

Equating this with $m = \alpha + i\beta$ we have $\alpha = -\frac{1}{2}$ *and* $\beta = \frac{\sqrt{3}}{2}$

Substituting these into the general solution

$$y = e^{\alpha x}[A\cos(\beta x) + B\sin(\beta x)]$$

we get:

$$y = e^{\frac{-x}{2}}[A\cos(\frac{\sqrt{3}}{2}x) + B\sin(\frac{\sqrt{3}}{2}x)]$$

This is the general solution to the above example.

1.3 Basic Examples in MATLAB

Let's consider a parachutist who falls from a plane. The plane is at 1000ft and when the parachutist jumps out we assume his initial downward velocity to by zero. (we ignore the speed of the plane as we are only interested in the vertical fall rate).

We assume that:

m= Parachutists mass (kg)

g = Acceleration due to gravity (ms^{-2})

v = Velocity (ms^{-1})

h = Height

The initial velocity is assumed to be zero, and the governing equation is:

$$ma = -mg + \frac{4}{15}V^2$$

Which is simply :

$$m\frac{d^2h}{dt^2} = -mg + \frac{4}{15}V^2$$

To be able to solve an equation of this type we need it in the form:

$$\frac{d^2h}{dt^2} = f\left(t, h, \frac{dh}{dt}\right)$$

So in the case of our parachutist, our equation becomes:

$$\frac{d^2h}{dt^2} = -g + \frac{4}{15}\frac{V^2}{m}$$

Now, MATLAB only solves systems of first order differential equations, and clearly this is a second order differential equation. We need to convert it to a first order equation:

To achieve this we introduce a variable *z*, which is the first derivative of h (the velocity).

So now we have:

$$V = \frac{dh}{dt} = z \qquad \text{So, we have}$$

$$\frac{dz}{dt} = \frac{d^2h}{dt^2} = -g + \frac{4}{15}\frac{z^2}{m}$$

And the initial conditions we know are:

$$h(0) = 1000\,ft$$
$$z(0) = 0ms^{-1}$$

In MATLAB we will address this problem using the ode45 command.

Ode45 is one of the many differential equation solvers in MATLAB and is generally the most used. It required some parameters before we can use it.

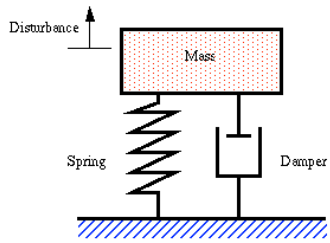
We must provide:

1. A function that defines the functions derived above (the first derivatives of *h* and *z*)
2. The initial values of both *h* and *z*.
3. The time range over which the solution should be sovled.

Now in MATLAB, in a new m-file, we need to do the following:

```
function ODE1
timerange=[0 10]
initialvalues=[304.8 0]
[t,y]=ode45(@fall,timerange,initialvalues)
plot(t,y(:,1))
ylabel('height (m)')
xlabel('time (s)')

function rk=fall(t,y)
mass=[75]
g=[9.81]
rk=[y(2); -g+4/14*y(2).^2/mass]
```

1.4 Another Mathematical Example:



The motion of a damped spring-mass system is described by the equation:

$$\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 10x = 0$$

Where x is the initial displacement and t is the time. Determine the particular solution for this differential equation with the initial conditions, when t = 0, x = 0 and $\frac{dx}{dt}$=3.

SOLUTION

The characteristic equation is

$m^2 + 7m + 10 = 0$

$(m+5)(m+2) = 0$

$m_1 = -5$ and $m_2 = -2$

We can see that we have 'real and different' roots, so we put these into:

$$x = Ae^{m_1 t} + Be^{m_2 t}$$

which gives

$$x = Ae^{-5t} + Be^{-2t}$$

We now substitute the given initial conditions t=0 and x=0 into the above equation giving

$$0 = Ae^0 + Be^0$$

$$A + B = 0$$

(because $e^0 = 1$)

Using the other given initial condition t=0 and $\frac{dx}{dt}$=3, we will need to differentiate:

$$x = Ae^{-5t} + Be^{-2t}$$

Which becomes

$$\frac{dx}{dt} = -5Ae^{-5t} + -2Be^{-2t}$$

$$3 = -5Ae^{0} + -2Be^{0}$$

Thus we now have two simultaneous equations:

A + B = 0
and
-5A – 2B = 3

From the first equation we have:

A = -B

And putting this in the second we get:

-5(-B) – 2B = 3
5B – 2B = 3
3B = 3
B = 1

Therefore we have B = 1 and A = -1

Substituting A = -1 and B = 1 into:

$$x = Ae^{-5t} + Be^{-2t}$$

gives

$$x = -e^{-5t} + e^{-2t}$$

Or more simply

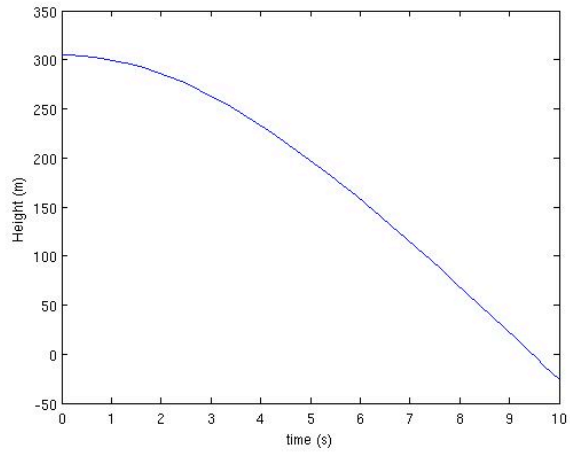
$$x = e^{-2t} - e^{-5t}$$

This is a particular solution since we have found particular values of A and B.

And in MATLAB?

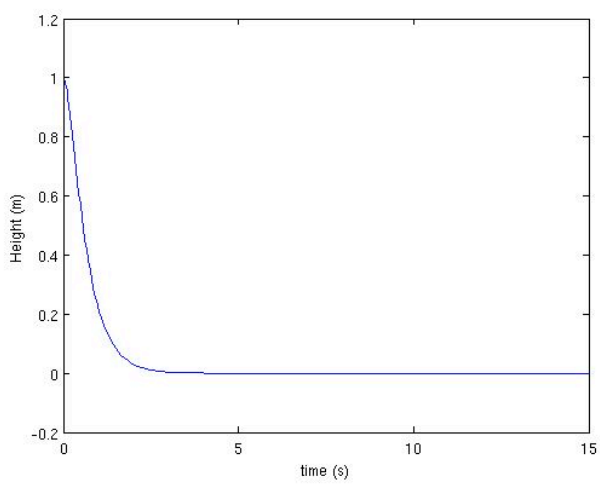
```
function ODE1
timerange=[0 15]
initialvalues=[0 3]
[t,y]=ode45(@smd,timerange,initialvalues)
plot(t,y(:,1)

function Spg=smd(t,y)
mass=[10]
g=[9.81]
Spg=[y(2); -7*y(2)-10*y(1)]
```

Solutions:



Our parachutist can be seen to fall from the plane from rest, accelerate due to gravity until he reached his terminal velocity, then come to rest very terminally after about 9 seconds.

It does not stop on reaching zero as we have not set it up to do this, remember MATLAB will do exactly what you ask it…. Even if it is a bit silly…



In the case of our mass-spring-damper system, we can clearly see that the system is 'over-damped' as there is no oscillation of the system, it just slowly comes to rest.

Compare this solution with the one we calculated by hand?

*ezplot (exp(-2*t)-exp(-5*t))*

Homework

1. A body of mass *m* is attached to a spring with a spring stiffness *k*. The displacement, *x*, is related by:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Show that $x = A\cos(\omega t) + B\sin(\omega t)$ where A, B are constants and $\omega = \sqrt{\frac{k}{m}}$

2. The figure above shows a simple mass *m* attached to a spring of stiffness *k* and a viscous damper with a damping coefficient *c*. The displacement, *x*, is given by

$$\frac{d^2x}{dt^2} + 2\zeta\omega\frac{dx}{dt} + \omega^2 x = 0$$

where $\omega = \sqrt{\frac{k}{m}}$ and the viscous damping factor $\zeta = \frac{c}{2m\omega}$

Show that the solution of this differential equation for ξ > 1 is given by:

$$x = Ae^{r_1 t} + Be^{r_2 t}$$

where

$$r_1 = \omega(-\zeta + \sqrt{\zeta^2 - 1})$$
$$r_2 = -\omega(\zeta + \sqrt{\zeta^2 - 1})$$

3. The equation of motion for a vertical spring-mass system is given by:

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0$$

Where ζ is the damping factor, ω is the angular frequency and *x* is the displacement of the mass from its equilibrium position. When *t* = 0, *x* = 5 and $\dot{x}$ = 0. Find an expression for *x* in terms of t with ζ = 1

4. The equation of motion for a spring-mass system is given by $\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0$, where ζ is the damping factor, ω is the angular frequency, and *x* is the displacement of the mass from its equilibrium position. For ζ < 1 and the initial conditions, when *t* = 0, *x* = 0 and $\dot{x}$ = βω where β = $\sqrt{1-\zeta^2}$, show that $x = e^{-\zeta\omega t}\sin(\beta\omega t)$.