



EXAMINATION SOLUTIONS

(to accompany the examination paper sent to the external examiner
but NOT to be seen by students)

Programme:	BEng Motorsport Engineering and Design	Level:	6
Module:	Advanced Motorsport Vehicle Dynamics M6X00415		
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Q 1	Question	Marks
	a) <i>Explain with respect to optimising tyre slip angles, why a car may utilise Pro or Anti Ackerman steering geometry. Use sketches of tyre slip angle versus lateral force curves at different vertical loads to support your explanation.</i> [3 marks] Explain how ackerman selection is function of tyre curves to keep slip angles for two differently loaded tyres at peaks of slip angle curves. Correct diagrams showing relative pro and anti ackerman cases as below. b) <i>A vehicles suspension may be modelled as a combination of spring mass damper systems. A quarter car system may be modelled as shown below. Write down an equation in matrix form which describes the motions of the two masses.</i> [5 marks] $m_s \ddot{x} + c_s(\dot{x}_s - \dot{x}_u) + k_s(x_s - x_u) = 0$ $m_u \ddot{x} + c_u(\dot{x}_u - \dot{y}) - c_s(\dot{x}_s - \dot{x}_u) + k_u(x_u - y) - k_s(x_s - x_u) = 0$ $m_s \ddot{x}_s + c_s \dot{x}_s - c_s \dot{x}_u + k_s x_s - k_s x_u = 0$ $m_u \ddot{x} + c_u \dot{x}_u - c_u \dot{y} - c_s \dot{x}_s + c_s \dot{x}_u + k_u x_u - k_u y - k_s x_s + k_s x_u = 0$ $m_s \ddot{x}_s + c_s \dot{x}_s - c_s \dot{x}_u + k_s x_s - k_s x_u = 0$ $m_u \ddot{x}_u + c_u \dot{x}_u - c_s \dot{x}_s + c_s \dot{x}_u + k_u x_u - k_s x_s + k_s x_u = c_u \dot{y} + k_u y$ $\begin{bmatrix} m_s & 0 \\ 0 & m_u \end{bmatrix} \begin{bmatrix} \ddot{x}_s \\ \ddot{x}_u \end{bmatrix} + \begin{bmatrix} c_s & -c_s \\ -c_s & c_s + c_u \end{bmatrix} \begin{bmatrix} \dot{x}_s \\ \dot{x}_u \end{bmatrix} + \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} = \begin{bmatrix} 0 \\ c_u \dot{y} + k_u y \end{bmatrix}$	1 2 1 1 1 2

- c) The motion of a certain spring mass damper (SMD) system may be described by the following equation:

$$15 \frac{d^2 x}{dt^2} + 3 \frac{dx}{dt} + 4x = 0$$

- i) Determine the particular solution for this differential equation with the initial conditions, given the characteristic equations below, when $t=0$, $x = 0$ and $\frac{dx}{dt} = 3$. State whether this SMD system under damped, over damped or critically damped.

[6 marks]

Case	Roots	Characteristic Equations
Overdamped	r_1 and r_2	$x = Ae^{r_1 t} + Be^{r_2 t}$
Critically Damped	$r_1 = r_2 = r$	$x = (A + Bt)e^{rt}$
Underdamped	$r = \alpha \pm i\beta$	$x = e^{\alpha t} [A \cos(\beta t) + B \sin(\beta t)]$

$$15 \frac{d^2 x}{dt^2} + 3 \frac{dx}{dt} + 4x = 0$$

$$15r^2 e^{rt} + 3r e^{rt} + 4e^{rt} = 0$$

$$e^{rt} (15r^2 + 3r + 4) = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 240}}{30}$$

$$r = -\frac{3}{30} \pm \frac{\sqrt{-231}}{30} \quad r = -\frac{3}{30} \pm i \frac{\sqrt{231}}{30}$$

$$\alpha = -0.1 \quad \beta = \frac{\sqrt{231}}{30}$$

$$x = e^{\alpha t} [A \cos(\beta t) + B \sin(\beta t)]$$

$$x = e^{-0.1t} \left[A \cos\left(\frac{\sqrt{231}}{30} t\right) + B \sin\left(\frac{\sqrt{231}}{30} t\right) \right]$$

$$\text{If } t = 0, x = 0 \text{ and } \frac{dx}{dt} = 3$$

$$0 = 1 [A + 0] \quad A = 0$$

$$x = e^{-0.1t} \left[B \sin\left(\frac{\sqrt{231}}{30} t\right) \right]$$

$$\frac{dx}{dt} = -0.1 e^{-0.1t} \left[B \sin\left(\frac{\sqrt{231}}{30} t\right) \right] + e^{-0.1t} \left[\frac{\sqrt{231}}{30} B \cos\left(\frac{\sqrt{231}}{30} t\right) \right]$$

$$3 = -0.1 [0] + 1 \left[\frac{\sqrt{231}}{30} B(1) \right]$$

$$B = \frac{90}{\sqrt{231}}$$

$$x = e^{-0.1t} \left[\frac{90}{\sqrt{231}} \sin\left(\frac{\sqrt{231}}{30} t\right) \right]$$

Case is underdamped

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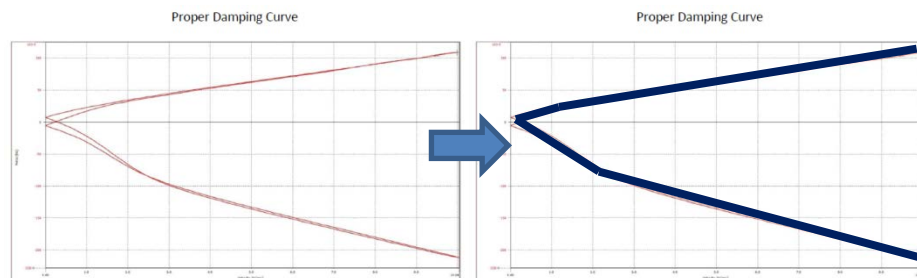
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- ii) *Race vehicle suspension systems do not have a single damping coefficient. Draw typical damper curves and explain how these could be modelled for use in the analysis above.*

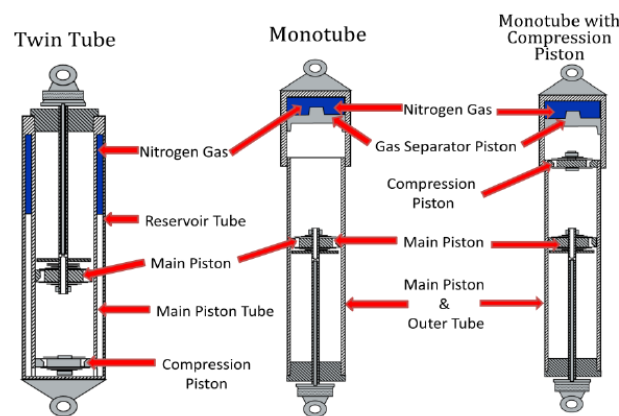
[2 marks]

Model may approximate curves distinct sections, low speed and high speed, as linear curvefits .



- d) *Outline the 3 main damper constructions available and give advantages for each construction.*

[4 marks]



	Advantages
Twin Tube	- o Inexpensive - o Compression Piston. This means lower compression pressures and typically lower gas pressure can be used.
Mono Tube with compression piston	- o Oil is separated from the gas. - o Higher quality construction than most Twin Tube shocks
Mono Tube	- o Oil is separated from the gas. - o Compression Piston. This means lower compression pressures and typically lower gas pressure can be used.



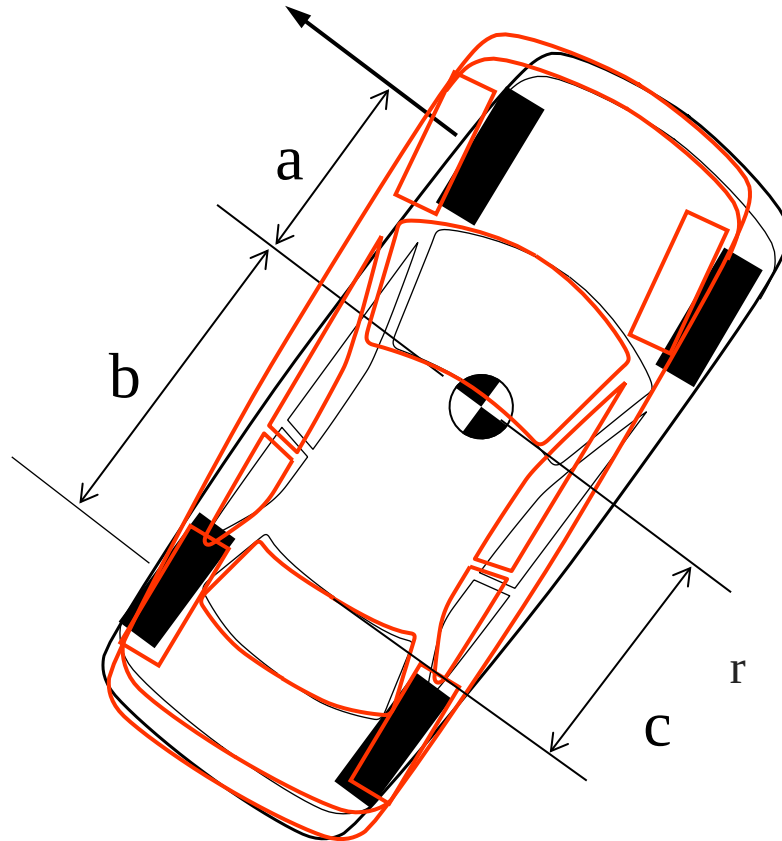
Q 2	Question	Marks
2.a	a) Calculate the rotational inertial (I) for both vehicles $I = M.R^2$ Vehicle A $750.(1.1^2) + 400.(1.3^2) + 200.(0.2^2)$ $= 907.5 + 676 + 8 = 1592 \text{ kg.m}^2$ Vehicle B $840.(1.3^2) + 200.(0.38^2) + 840.(1.5^2)$ $= 1420 + 29 + 1890 = 3338 \text{ kg.m}^2$	2 2
2.b	a) Calculate the Dynamic Index for both vehicles (DI) $DI = \text{Rotational Inertia} / (\text{dist, c of g to front wheels}) . (\text{dist, c of g to rear wheels}). \text{Mass}$ Vehicle A $1592 \text{ kg.m}^2 / (0.95\text{m}) . (1.45\text{m}) . (1350 \text{ kg})$ $= 1592 \text{ kg.m}^2 / 1860$ $= 0.856$ Vehicle B $3338 / (1.3\text{m}) . (1.3\text{m}) . (1880 \text{ kg})$ $= 3338 / 3177$ $= 1.05$	2 2
2.c	a) Discuss the concept centre of percussion (inertial conjugate) and how the position of it relates to the dynamic index.(4 marks) The student should discuss and produce appropriate diagrams, such as: The centre of percussion is the point that an object will rotate about then a force is applied to it. In this case the force applied to the vehicle at vehicle turn is the lateral force generated by the tyres due to the steering angle applied generating a tyre slip angle. A DI of one is when the centre of percussion is on the rear axle A DI > 1 = a very stable vehicle that is reluctant to 'turn in'. A DI < 1 = a stable vehicle that turns in ok If the DI < 0.8 the vehicle is very unstable, turns in too well and is difficult to balance/control.	4
2.d	a) Describe the rear tyre grip behaviour of the two vehicles and how this will affect the vehicle during corner 'turn-in' and the in corner vehicle behaviour.	



The student should explain that when the DI is less than 1; when the vehicle turns in the rear wheels will move outward building a slip angle and generating lateral grip.

Whilst the vehicle is in the corner any additional steering angle application will result in rear wheels moving outward building a slip angle. If the rear tyres are on the limit and the grip is saturated this additional slip angle will reduce lateral grip. A diagram would be helpful.

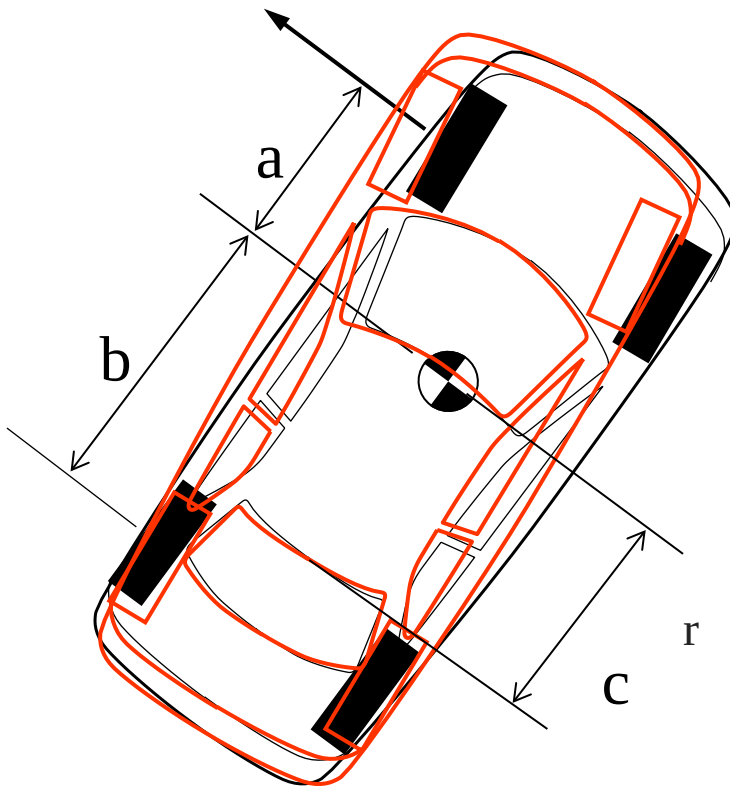
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The student should explain that when the DI is greater than 1; when the vehicle turns in the rear wheels will move inward reducing the slip angle and generating lateral grip. Whilst the vehicle is in the corner any additional steering angle application will result in rear wheels moving inward reducing slip angle. If the rear tyres are on the limit and the grip is saturated this reduction in slip angle will increase lateral grip. A diagram would be helpful.

4





Q3	Question	Marks
3a	a) What tractive effort is required to maintain the constant 5 m/s vehicle speed in the snow? Rolling resistance = $1500\text{kg} \times 9.81 \times 0.1 = 1472\text{ N}$ Aero drag = $0.5 \times c_d \times \text{area} \times \text{density of air} \times \text{velocity}^2$ $= 0.5 \times 0.5 \times 3 \times 1.3 \times 5 \times 5 = 23\text{N}$ Therefore total drive force required = $1472 + 23 = 1495\text{N}$	2
3b	What is the centripetal acceleration? At 5 m/s Centripetal accel = $\text{vel}, \text{vel}/\text{rad} = 5 \times 5 / 15 = 1.67\text{ m/s/s}$	2
3c	What is the vehicle yaw rate in a steady state condition? Circumference = $\pi \times \text{dia} = 3.14 \times 30 = 94.2\text{m}$ $94.2\text{m}/5\text{m/s} = 18.84\text{ sec/lap}$ Yaw rate = $360/18.84 = 19.1\text{ degs/sec}$ or $2 \times \pi / 18.84 = 0.33\text{ rads/sec}$	2
3d	Assuming the spring rate is constant front and rear. What is the weight on each wheel? Static weight on each wheel is $1500\text{kg} \times 9.81 / 4 = 3679\text{N}$ Weight transfer = $\text{CofG height} \times \text{mass}/2 \times \text{centripetal accel} / \text{track}$ $= 0.6\text{m} \times 1500\text{kg}/2 \times 1.67\text{m/s/s} / 1.65$ $= 455\text{N}$ on each axle Therefore: Left side outer wheels $3679\text{N} + 455\text{N} = 4134\text{N}$ each Right side inner wheels = $3679\text{N} - 455\text{N} = 3224\text{ N}$	2
3e	What is the lateral force on each axle ? $1500\text{kg}/2 \times 1.67 = 1253\text{N}$	1
3f	What is the lateral force resistance on the inside and the outside wheels? Each wheel has the same amount of slip therefore the lat grip will be shared out in proportion to the vertical loading. Total vert force on one axle = $750\text{kg} \times 9.81 = 7358\text{N}$ Left side outer wheel loading percentage = $4134 / 7358 = 56.2\%$ Right side inner wheel = $3224 / 7358 = 43.8\%$	3



	Therefore Lat force resistance on outside wheel = $56.2\% \times 1253\text{N} = 704\text{N}$ Lat force resistance on inside wheel = $43.8\% \times 1253 = 549\text{N}$	
3g	a) Draw and label two frictions circles with the longitudinal and lateral forces. One for the rear outer wheel and one for the rear inner wheel. Calculate the additional drive force that could be applied before the inside wheel forces exceed the available grip. (assume the vehicle has an open differential.) Using Pythagoras $(1240\text{N}^2) - (704\text{N}^2) = \text{longitudinal drive force grip}^2$ Therefore available longitudinal drive force available from the outside wheel $= 1022\text{N}$ Using Pythagoras $(967\text{N}^2) - (548\text{N}^2) = \text{longitudinal drive force grip}^2$ Therefore available longitudinal drive force available from the inside wheel $= 797\text{N}$ The vehicle already needs $1495\text{N} / 2 = 748\text{N}$ from each wheel There the additional drive force is $797\text{N} - 748\text{N} = 49\text{N}$ on each wheel.	3
3h	What initial acceleration could the vehicle achieve with a standard open differential front wheel drive arrangement? (assume no longitudinal weight transfer) 46N from each driven wheel = $49\text{N} \times 2 = 98\text{N}$ Acceleration = force / mass $= 98\text{N} / 1500\text{Kg}$ $= 0.065 \text{ m/s/s}$	1
3i	The all wheel drive control system wishes to provide maximum acceleration. What is the maximum acceleration on the curve? Discuss limiting factors. The drive force is no longer symmetrical but can be asymmetrical. Inside wheel longitudinal tractive effort available 49N Outside wheel longitudinal tractive effort available $= 1022\text{N} - 748\text{N} = 274\text{N}$ Total rear tractive effort = $274\text{N} + 49\text{N} = 323\text{N}$ Total front tractive effort = 98N Potential new acceleration = $421\text{N} / 1500\text{kg} = 0.28 \text{ m/s/s}$ The driver is on the limit of lateral grip, which is low in the snow. Potential for more acceleration is limited.	2



	With front wheel drive only, the rear of the car has some spare lateral grip. With AWD all the available longitudinal grip can be used at the rear: leading to potential lateral instability.	
3j	What is the resulting yaw moment on the vehicle? Discuss system compromises and driver reaction requirements. The front wheel drive is symmetrical. The rear drive is asymmetrical. Outside = 274N Inside = 49N Resulting asymmetric drive = 274N – 49N = 225N Yaw moment = Force / half track = 225 / 1.65 / 2 = 68Nm This will turn the vehicle into the corner.	2

Q4	Question	Marks
4a	What is the rate of deceleration? Vertical force on the front of the vehicle is 1100kg x 9.81 = 10791N For each front wheel vertical load = 5396N Vertical force on the rear of the vehicle is 750kg x 9.81 = 7358N For each rear wheel vertical load = 3679N Low μ side grip = (5396 + 3679) x 0.1 = 907.5N High μ side grip = (5396 + 3679) x 0.6 = 5445N Total grip = 5445 + 907.5 = 6352.5 N Deceleration = force / mass = 6352.5N / 1850kg = 3.43 m/sec/sec	3
4b	What is the yaw moment generated ? Difference in braking forces = 5445 – 908 = 4537N Moment = 4537N x (1.9m/2) = 4310 Nm	2
4c	Calculate the resulting rotational acceleration Moment = Vehicle inertia x rotational accel Accel = moment / inertia = 4310Nm / 3000 kgm² = 1.44 rads/sec/sec	2



4d	What lateral force from the front steered wheels is required to counter the asymmetric braking force rotational moment? Yaw moment from asymmetric braking was 4310Nm Distance from front axle to centre of gravity = rear mass/total mass x wheel base = 750kg/1850kg x 2.5m = 1.014m Torque = force x radius. Force = torque / radius = 4310 Nm / 1.014m = 4028N	2
4e	The tyre on the high μ side provides a tyre cornering stiffness of 1000N/degree. What steering handwheel angle would be required to counter the rotational moment ? The student would be expected to ignore the grip on the low μ side as it is so little. 4250N / 1100N/deg = 3.86 degs at the road wheel 3.86 degs x 12 (steering ratio) = 46.4 degrees steering correction required.	2
4f	In ideal conditions what would be the maximum rate of deceleration Vertical force on the front of the vehicle is 1100kg x 9.81 = 10791N For each front wheel vertical load = 5396N Vertical force on the rear of the vehicle is 750kg x 9.81 = 7358N For each rear wheel vertical load = 3679N Low μ side grip = (5396 + 3679) x 0.2 = 1815 N High μ side grip = (5396 + 3679) x 0.8 = 7260 N Total grip = 7260 + 1815 = 9075 N Deceleration = force / mass = 9075N / 1850kg = 4.9 m/sec/sec	3
4g	Describe using graphs and diagrams how the ABS system would operate in order to achieve controllable vehicle deceleration. The student should explain the humans have a limited response rate due to. Sensors monitoring the yaw acceleration (eyes and ear-balance) The rate of assimilating the information and selecting a course of action The delay in the response of the hand and arm action to the request. This response will vary between people and an individual can have different response rates due to a variety of factors eg mood, concentration, alertness to dangers etc.	6



	Consequently the ABS system will. 1. Control the rate of increase of braking force on the high grip side so that the driver has time to react and apply a suitable steering handwheel compensatory angle. 2. Utilise a rear axle 'select low' control system so that the rear wheel have some lateral grip and hence will resist the asymmetric braking moment. Both of the above will reduce the deceleration rate and hence increase stopping distances. However vehicle stability will be improved. It is considered more important to maintain path following and body slip angles than to reduce stopping distances.	
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