

Example 1

If $y = \sqrt{x^2 + 1}$ find $\frac{dy}{dx}$

$\therefore$ Let $u = x^2 + 1$ Then $y = \sqrt{u} = u^{1/2}$

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dy}{du} = \frac{1}{2} u^{-1/2}$$

So via The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2} u^{-1/2} \cdot 2x$$

$$= \frac{x}{\sqrt{u}}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 1}}$$

Example 2

If $y = e^{x^2 - 3}$ find $\frac{dy}{dx}$

Let $u = x^2 - 3$ then $y = e^u$

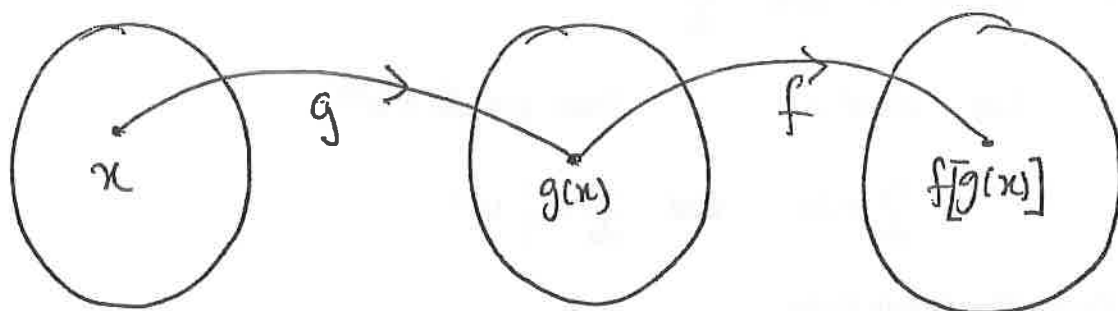
$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dy}{du} = e^u$$

So by the Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \longrightarrow$

$$\frac{dy}{dx} = e^u \cdot 2x$$

$$\frac{dy}{dx} = 2x e^{x^2 - 3}$$

The Chain Rule (function of a function)



Consider the above composite function when:

$$g(x) = x^2 + 1$$

$$f(x) = \sqrt{x}$$

Then $fg(x) = f(g(x))$

$$= f(x^2 + 1)$$

$$fg(x) = \sqrt{x^2 + 1}$$

Thus note the function $x^2 + 1$ lies within the square root function.

We now consider how to differentiate such composite functions.

If $y = f(g(x))$ then

Let $u = g(x)$ then $y = f(u)$

$$\longrightarrow \frac{du}{dx} = g'(x) \quad \text{and} \quad \frac{dy}{du} = f'(u) \quad \text{The Derivatives}$$

$$\text{So} \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\longrightarrow \frac{dy}{dx} = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

7.5**Differentiation of $\ln x$**

- 1** (i) $\frac{3}{x}$ (ii) $-\frac{6}{x}$ (iii) $\frac{1}{5x}$
(iv) $\frac{1}{x}$ (v) $2x + \frac{2}{x}$

- 2** (i) 1.75 (ii) 9 (iii) 19.7
(iv) -3.17 (v) -0.192

3 $\frac{2}{2x+1}$

- 4** When x is large the gradient gets very small. When x is very small the gradient is very large.

- 5** (i) $\frac{1}{x}$ (ii) $\frac{1}{x}$ (iii) $\frac{3}{x}$
(iv) 1 (v) $\frac{3}{x}$ (vi) $\frac{1}{2x}$

7.5

Differentiation of $\ln x$

1 Differentiate:

(i) $3 \ln x$

(ii) $-6 \ln x$

(iii) $\frac{1}{5} \ln x$

(iv) $7 + \ln x$

(v) $x^2 + 2 \ln x$

2 Find the gradient of the following curves at the points indicated:

(i) $y = 7 \ln x + 1$ when $x = 4$

(ii) $y = 5x^2 - \ln x$ when $x = 1$

(iii) $y = \frac{5x^3 - 2 \ln x}{3}$ when $x = 2$

(iv) $y = \frac{\ln x}{8} - 4e^x$ when $x = 0.1$

(v) $y = \frac{3}{2\sqrt{x}} + \frac{1}{x} - \frac{1}{7} \ln x$ when $x = 4$

3 A group of students suggest four possibilities for the derivative of $y = \ln(2x + 1)$. One of them is correct. The suggestions are:

$$\frac{2}{2x+1}, \quad \frac{2x+1}{2}, \quad \frac{1}{2x}, \quad \frac{2}{x}$$

Investigate the gradient of the curve graphically and decide which of the four formulas is correct.

4 Use the derivative of $\ln x$ to describe what happens to the gradient of the curve $y = \ln x$ when x gets very large. What happens to the gradient when x is very small?

5 Use properties of logs to simplify these expressions and hence find the derivatives:

(i) $\ln 10x$

(ii) $\ln \frac{x}{4}$

(iii) $\ln x^3$

(iv) $\ln e^x$

(v) $\ln 5x^3$

(vi) $\ln \sqrt{x}$

Differentiation of $\ln x$ **Example 2**

Find the gradient of the graph of $y = 3 \ln x + 2x^2$ when $x = 3$.

Solution

$$y = 3 \ln x + 2x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{x} + 4x.$$

$$\text{When } x = 3, \frac{dy}{dx} = 1 + 12 = 13.$$

Example 3

Find the gradient of the curve $y = 4x - \frac{1}{3} \ln x$ when $x = 0.2$.

Answer

2.33.

Example 4

Find the derivatives of: (i) $\ln(x^2)$; (ii) $\ln 5x$.

Solution

These, and other expressions involving $\ln x$, can be differentiated after applying the properties of logs.

(i) $\ln(x^2)$ can be written $2 \ln x$.

So, the derivative is $2 \times \frac{1}{x}$ or $\frac{2}{x}$.

(ii) $\ln 5x$ can be written $\ln 5 + \ln x$.

So, the derivative is $\frac{1}{x}$. ($\ln 5$ is a constant, so its derivative is zero.)

7.5

Differentiation of $\ln x$

Prerequisites Lesson 6.1: logarithms; lesson 6.2: logarithmic and exponential functions; lesson 7.2: differentiation; lesson 7.4: differentiation of e^x .

Objectives Students should be able to differentiate $\ln x$.

Lesson**Result**

If $y = \ln x$ then $\frac{dy}{dx} = \frac{1}{x}$.

Proof

If $y = \ln x$ and $x > 0$:

$$x = e^y$$

$$\therefore \frac{dx}{dy} = e^y$$

$$\therefore \frac{dy}{dx} = \frac{1}{e^y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x}.$$

This proof relies on the fact that $\frac{dx}{dy} = \frac{1}{dy/dx}$. This result is not proved as part of the AS-level Core, but it is proved as part of the A-level Core.

Class activity

The above result can be justified in the same way as for other derivatives. Students should plot the graph of $y = \ln x$ and find the gradient of the graph at the points with x coordinates 0.5 and 4. They should agree that when $x = 0.5$ the gradient of the graph is 2 and when $x = 4$ the gradient of the graph is 0.25. Other points can be checked.

Example 1

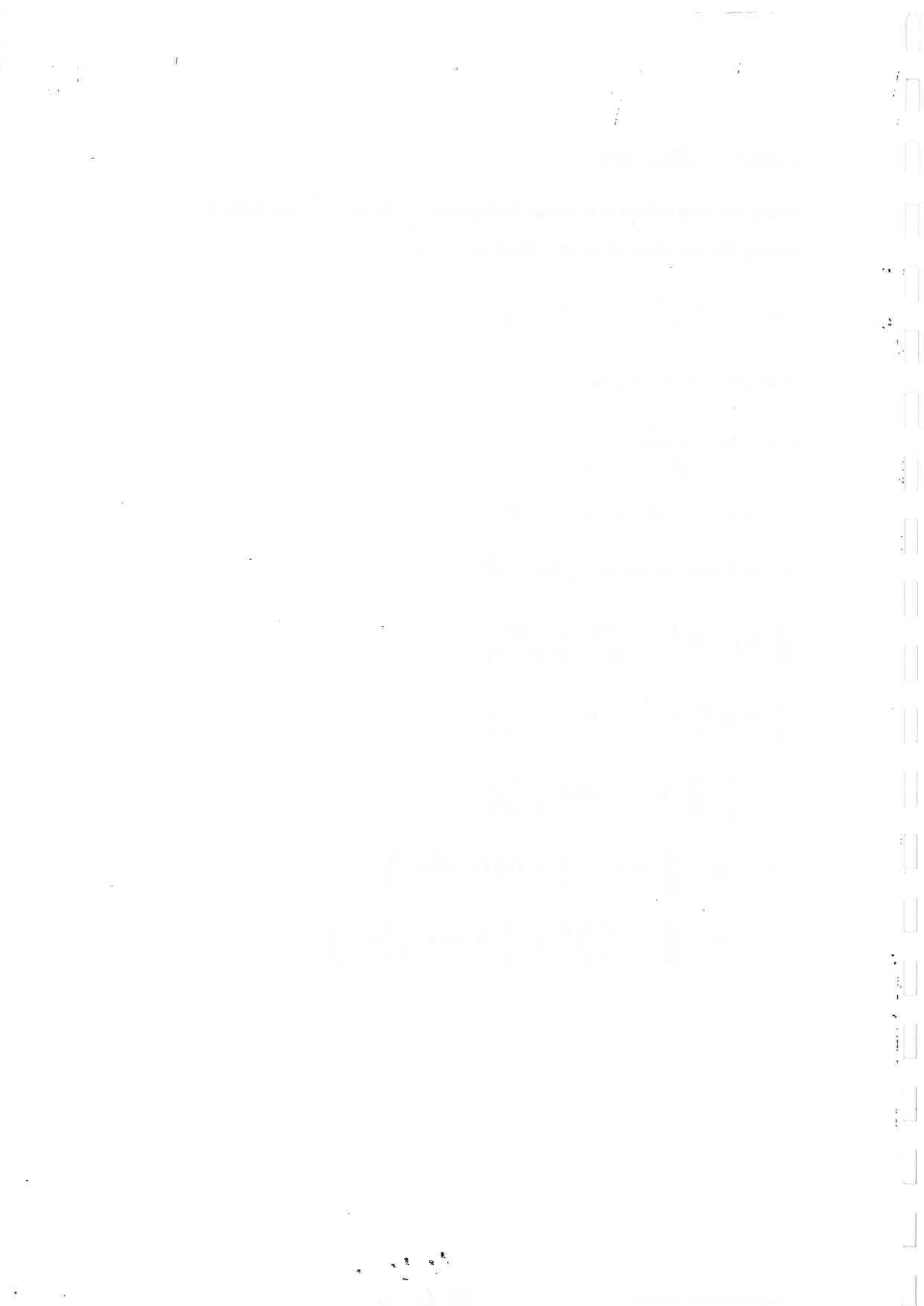
Find the gradient of the graph of $y = 5 \ln x$ at the point with x coordinate 2.

Solution

$$y = 5 \ln x$$

$$\Rightarrow \frac{dy}{dx} = 5 \times \frac{1}{x} = \frac{5}{x}.$$

$$\text{When } x = 2, \frac{dy}{dx} = \frac{5}{2} \text{ or } 2\frac{1}{2}.$$



Logarithmic differentiation

Using the laws of logs and remembering that $\frac{d}{dx} (\ln x) = \frac{1}{x}$ can help in finding the derivative of some difficult functions.

e.g. if $y = \frac{x^3 \sin x}{\sqrt{1+x^2}}$ find $\frac{dy}{dx}$

Taking logs both sides to base 'e' $\longrightarrow$

$$\ln y = \ln \left[\frac{x^3 \cdot \sin x}{\sqrt{1+x^2}} \right]$$

$$= \ln x^3 + \ln (\sin x) - \ln (1+x^2)^{\frac{1}{2}}$$

$$\ln y = 3 \ln x + \ln (\sin x) - \frac{1}{2} \ln (1+x^2)$$

$$\frac{d}{dx} \ln y = 3 \cdot \frac{1}{x} + \frac{\cos x}{\sin x} - \frac{1}{2} \frac{2x}{(1+x^2)}$$

$$\frac{d}{dy} (\ln y) \frac{dy}{dx} = \frac{3}{x} + \cot x - \frac{x}{(1+x^2)}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{x} + \cot x - \frac{x}{(1+x^2)}$$

$$\longrightarrow \frac{dy}{dx} = y \left[\frac{3}{x} + \cot x - \frac{x}{1+x^2} \right]$$

$$\longrightarrow \frac{dy}{dx} = \frac{x^3 \sin x}{\sqrt{1+x^2}} \left[\frac{3}{x} + \cot x - \frac{x}{1+x^2} \right]$$

Exercise 10: Solve the simultaneous equations:-

$$x \cdot y = 80 \text{ ————— I}$$

$$\log x - 2 \log y = 1 \text{ ————— II}$$

from II $\log x - \log y^2 = 1$

$$\log \left(\frac{x}{y^2} \right) = 1$$

$$\longrightarrow \frac{x}{y^2} = 10^1$$

$$\longrightarrow x = 10y^2 \text{ ————— III}$$

Substitute this into I

$$(10y^2) y = 80$$

$$\longrightarrow y^3 = 8$$

$$\therefore \underline{y = 2}$$

as $x = 10y^2$ then in III

$$x = 10(2)^2$$

$$\underline{x = 40}$$

Example 9: Solve $3^x \cdot 4^{2x+1} = 6^{x+2}$

Take logs both sides:-

$$\ln(3^x \cdot 4^{2x+1}) = \ln(6^{x+2})$$

$$\ln 3^x + \ln 4^{2x+1} = \ln 6^{x+2}$$

$$x \ln 3 + (2x + 1) \ln 4 = (x + 2) \ln 6$$

$$x \ln 3 + 2x \ln 4 + \ln 4 = x \ln 6 + 2 \ln 6$$

$$\therefore x \ln 3 + 2x \ln 4 - x \ln 6 = 2 \ln 6 - \ln 4$$

$$x \ln 3 + x \ln 4^2 - x \ln 6 = \ln 6^2 - \ln 4$$

$$x (\ln 3 + \ln 16 - \ln 6) = \ln 36 - \ln 4$$

$$x (\ln (16 \times 3) - \ln 6) = \ln \left(\frac{36}{4} \right)$$

$$x \left(\ln \left(\frac{48}{6} \right) \right) = \ln 9$$

$$x \ln 8 = \ln 9$$

$$\therefore x = \frac{\ln 9}{\ln 8}$$

$$\underline{x = 1.0566 \text{ 4 dp's}}$$

Example 7:- Solve $2^{2x} + 3(2^x) - 4 = 0$

Write as: $(2^x)^2 + 3(2^x) - 4 = 0$

Let $y = 2^x \longrightarrow$

$$y^2 + 3y - 4 = 0$$

$$(y - 1)(y + 4) = 0$$

$\longrightarrow$ either $y = 1$ or $y = -4$

$$\therefore 2^x = 1 \quad 2^x = -4$$

$$\therefore \log_{10} 2^x = \log_{10} 1$$

$$x \log_{10} 2 = 0$$

$$\longrightarrow \underline{x = 0}$$

This has no solution as you can't have logs of -ve numbers.

Example 8: Solve $e^{2x} - 6e^x + 8 = 0$

Write as: $(e^x)^2 - 6(e^x) + 8 = 0$

Let $y = e^x \longrightarrow$

$$y^2 - 6y + 8 = 0$$

$$(y - 2)(y - 4) = 0$$

$$\longrightarrow y = 2 \quad y = 4$$

$$\therefore e^x = 2 \quad \text{and} \quad e^x = 4$$

$$\underline{x = \ln 2} \quad \text{and} \quad \underline{x = \ln 4}$$

Example 5 Solve:-

$$4^{2x+1} = 3$$

$$\log_{10} 4^{2x+1} = \log_{10} 3$$

$$(2x + 1) \log_{10} 4 = \log_{10} 3$$

$$2x + 1 = \frac{\log_{10} 3}{\log_{10} 4}$$

$$2x = \frac{\log_{10} 3}{\log_{10} 4} - 1$$

$$x = \frac{1}{2} \left(\frac{\log_{10} 3}{\log_{10} 4} - 1 \right)$$

$$\underline{x = -0.1038 \quad 4\text{dp's}}$$

Example 6 Solve:-

$$\ln(x + 1) - \ln(x^2 - 1) = 1$$

$$\longrightarrow \ln\left(\frac{x+1}{x^2-1}\right) = 1$$

$$\therefore \frac{x+1}{x^2-1} = e$$

$$x + 1 = e(x^2 - 1)$$

$$ex^2 - x - e - 1 = 0$$

$$e(x^2) - x - (1 + e) = 0$$

via Quadratic formula

$$x = \frac{-1 \pm \sqrt{1 - 4(e)(1+e)}}{2e}$$

$$x = \underline{\hspace{2cm}} \quad (2\text{dp's})$$

Exponential and logarithmic equations

Example 1

$$10^x = 1000$$

Take logs both sides to base 10:

$$\log_{10} 10^x = \log_{10} 1000$$

$$x \log_{10} 10 = \log_{10} 1000$$

$$\underline{x = 3}$$

Example 2

$$\text{solve } 5^x = 10$$

$$\log_{10} 5^x = \log_{10} 10$$

$$x \log_{10} 5 = 1$$

$$x = \frac{1}{\log_{10} 5}$$

$$\underline{x = 1.43 \text{ (3sf)}}$$

Example 3

$$e^x = 7$$

$$\ln e^x = \ln 7$$

$$x \ln e = \ln 7$$

$$x = 1.9459$$

$$\underline{x = 1.95 \text{ 2 dp's}}$$

Example 4

$$e^{2x} = 5$$

$$\ln e^{2x} = \ln 5$$

$$2x = \ln 5$$

$$x = \frac{1}{2} \ln 5$$

$$\underline{x = 0.805 \text{ (3 dp's)}}$$

Laws of logs

If $\log_a b = x$ Then $b = a^x$ and
 $\log_a c = y$ Then $c = a^y$

Then:-

$$b \cdot c = a^x \cdot a^y$$

$$bc = a^{x+y}$$

$$\longrightarrow \log_a(bc) = x + y$$

$$\log_a(b \cdot c) = \log_a b + \log_a c$$

Law I

Similarly: $\frac{b}{c} = \frac{a^x}{a^y}$

$$\frac{b}{c} = a^{x-y}$$

$$\longrightarrow \log_a\left(\frac{b}{c}\right) = x - y$$

$$\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$$

Law II

Now let $\log_a b^n = z$

$$\longrightarrow b^n = a^z$$

$$b = a^{z/n}$$

$$\therefore \log_a b = \frac{z}{n}$$

$$\therefore n \log_a b = z$$

$$\therefore n \log_a b = \log_a b^n$$

Law III

And lastly:

$$\log_a a = 1$$

Law IV

$$\text{i.e. } a^1 = a$$

Frequently used logarithms

If the base of the logarithm of a number 'x' is '10', then this is referred to as a common logarithm.

i.e. $\log_{10} x$ — common logs

very often the base is omitted

i.e. $\log x \longrightarrow \log_{10} x$

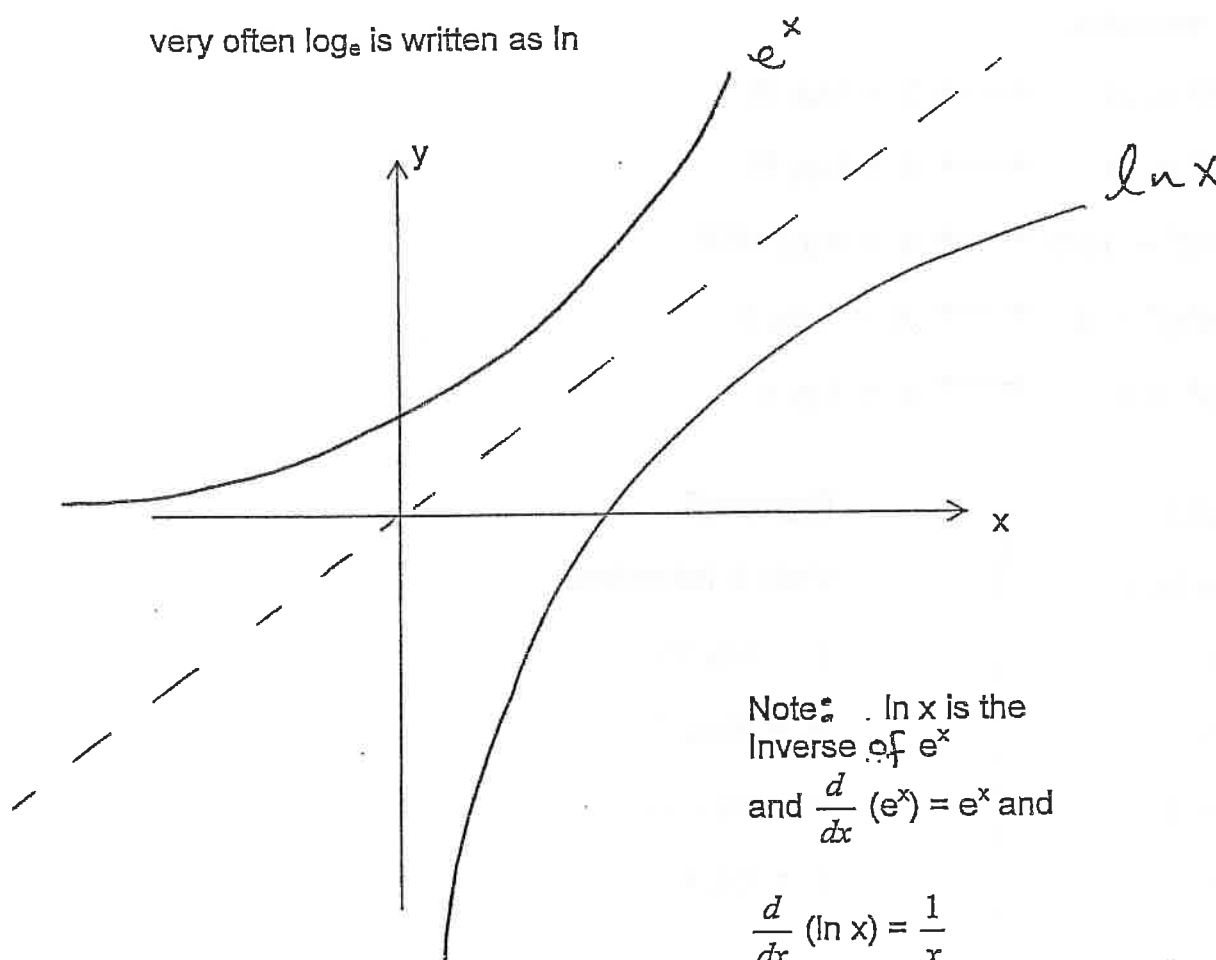
Natural logarithms

This uses 'e' — eulers number

$e = 2.71828\ldots$ then

$\log_e x$ — is referred to as the natural logarithm of x.

very often $\log_e$ is written as $\ln$



Logarithms

A logarithm is another word for an Index or Power.

i.e.

$$\begin{array}{ccc} & 2 & ^3 = 8 \\ & \nearrow & \nwarrow \\ \text{Base No} & & \text{Index} \end{array}$$

That is 3 is the Index/Power to which the base number 2 must be raised to obtain 8.

This can also be written as:-

$$3 = \log_2 8$$

Further examples

$$5^2 = 25 \quad \longleftrightarrow \quad 2 = \log_5 25$$

$$3^4 = 81 \quad \longleftrightarrow \quad 4 = \log_3 81$$

$$10^3 = 1000 \quad \longleftrightarrow \quad 3 = \log_{10} 1000$$

$$(64)^{\frac{1}{2}} = 8 \quad \longleftrightarrow \quad \frac{1}{2} = \log_{64} 8$$

$$a^b = c \quad \longleftrightarrow \quad b = \log_a c$$

Exercise I

Write as logs

$$3^2 = 9$$

$$4^3 = 64$$

$$(25)^{\frac{1}{2}} = 5$$

$$b^x = c$$

Exercise II

Write in index form

$$2 = \log_6 36$$

$$\frac{1}{2} = \log_{49} 7$$

$$3 = \log_{\frac{1}{8}} \left(\frac{1}{8}\right)$$

$$y = \log_x 2$$

Mathematics Exercise



The following text is extremely faint and largely illegible. It appears to be a series of paragraphs or a list of items, possibly related to a mathematics exercise or a report. The text is too light to transcribe accurately.

8.5

Miscellaneous Exercise

1 (i) $\frac{x^4}{4} - \frac{3x^2}{2} + c$ (ii) $\frac{2x^{\frac{3}{2}}}{3} - 2x^{\frac{1}{2}} + c$

2 $y = \frac{x^2}{2} + \frac{4x^{\frac{3}{2}}}{3} - \frac{59}{3}$

3 $\frac{37}{12}$ sq. units

4 $4e^2 - \frac{20}{3}$

5 (i) $(\sqrt{2}, 0), (-\sqrt{2}, 0)$ (ii) $3 + \frac{8}{3}\sqrt{2}$

6 $e - e^{-2} + 6$

7 $-\left(x^{-1} + 4x^{-\frac{3}{2}} + \frac{9}{2}x^{-2}\right) + c$

8 $e^3 - e^2 + \ln\left(\frac{3}{2}\right)$

9 $\frac{x^3}{3} - \frac{2}{5}x^{\frac{5}{2}} - 3x^2 + c$

10 (i) $4\ln 2$ (ii) $P(1, 2); 2\ln 2 - 1$

Miscellaneous Exercise 8.5

- 6** Calculate the area enclosed by the curve $y = e^x + 2$, the x -axis and the ordinates $x = -2$, $x = 1$.

7 Find $\int \frac{(\sqrt{x} + 3)^2}{x^3} dx$

8 Evaluate $\int_2^3 \frac{x e^x + 1}{x} dx$

9 Find $\int x(\sqrt{x} + 2)(\sqrt{x} - 3) dx$

- 10** (i) Find the area enclosed by the curve $y = \frac{2}{x}$, the x -axis and the ordinates $x = \frac{1}{2}$, $x = 2$.
 (ii) The line $y = 2$ intersects the curve $y = \frac{2}{x}$ at point P . Find the coordinates of P .
 Hence, find the area enclosed by the curve $y = \frac{2}{x}$, the line $y = 2$ and the ordinate $x = \frac{1}{2}$.

8.5

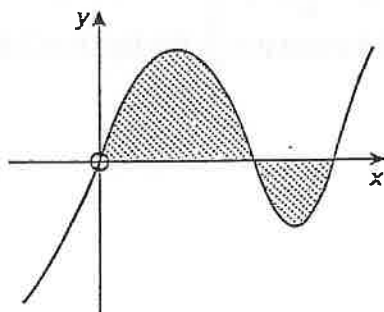
Miscellaneous Exercise

1 Find each of the following:

(i) $\int x(x^2 - 3) dx$ (ii) $\int \frac{1}{\sqrt{x}}(x - 1) dx$

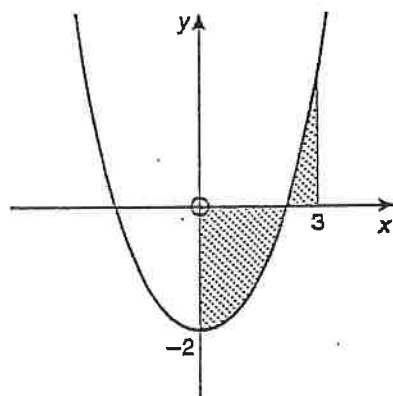
2 A curve has a gradient function given by $\frac{dy}{dx} = x + 2\sqrt{x}$. Given that the curve passes through the point $(4, -1)$, find the equation of the curve.

3 Shown below is the graph of the function $y = x^3 - 5x^2 + 6x$. Find the total shaded area.



4 Evaluate $\int_0^2 4e^x - x^2 dx$

5 (i) Write down the coordinates of the points where the curve $y = x^2 - 2$ intersects the x -axis.
(ii) Calculate the shaded area shown in the diagram below:



8.4**Definite Integrals**

- 1** (i) $\frac{7}{3}$ (ii) 36 (iii) 1.22
(iv) $\frac{124}{3}$ (v) 0.414 (vi) $\frac{2}{5}$

- 2** (i) 7.34 (ii) 2.69 (iii) 2.19
(iv) 2.86 (v) -2.83 (vi) -0.55.

- 3** (i) $\frac{35}{3}$ (ii) $\frac{1}{4}$ (iii) $-\frac{16}{3}$
(iv) 19.95 (v) 1.37 (vi) 1.61

- 4** (i) $\frac{58}{3}$ (ii) $\frac{17}{4}$ (iii) 16.45
(iv) 0.5 (v) $\frac{3}{2}$ (vi) 5.35

8.4

Definite Integrals

1 Evaluate each of the following integrals:

(i) $\int_1^2 x^2 dx$

(ii) $\int_{-1}^3 x^3 + 4 dx$

(iii) $\int_1^2 \sqrt{x} dx$

(iv) $\int_{-2}^2 (x-3)^2 dx$

(v) $\int_1^2 \frac{1}{2\sqrt{x}} dx$

(vi) $\int_0^1 (\sqrt{x})^3 dx$

2 Evaluate each of the following integrals:

(i) $\int_{-3}^2 e^x dx$

(ii) $\int_2^4 \frac{1}{x} + 1 dx$

(iii) $\int_1^2 \frac{x^2 + 1}{x} dx$

(iv) $\int_1^3 \frac{2x+1}{x^2} dx$

(v) $\int_{-2}^{-1} \frac{3x^2 + 2}{x^3} dx$

(vi) $\int_{-1}^1 \frac{e^{2x} - 2e^x}{3e^x} dx$

3 Find the area under each of the following curves between the ordinates given and the x -axis:

(i) $y = x^2; x = -2, x = 3$

(ii) $y = x^3; x = 0, x = 1$

(iii) $y = x^2 - 3; x = -1, x = 1$

(iv) $y = e^x; x = -2, x = 3$

(v) $y = e^x - 2; x = -1, x = 0$

(vi) $y = \frac{1}{x}; x = 1, x = 5$

In each case sketch the graph and indicate with shading the area calculated.

4 Find the area under each of the following curves between the ordinates given and the x -axis:

(i) $y = x^2 - 2x; x = 0, x = 5$

(ii) $y = x^3; x = -1, x = 2$

(iii) $y = e^x - 1; x = -1, x = 3$

(iv) $y = \frac{1}{x} + 2; x = -1, x = -\frac{1}{4}$

(v) $y = x^3 - x^2 - 2x; x = -1, x = 1$

(vi) $y = e^x - 3; x = -1, x = 2$

In each case sketch the graph and indicate with shading the area calculated.

Definite Integrals

Result 2

$$\int_a^b \frac{1}{x} dx = [\ln |x|]_a^b.$$

In other words, we find the modulus of x when the integral is definite.

Back to the example, and we have

$$\begin{aligned} \int_{-3}^{-1} \frac{1}{x} dx &= [\ln |x|]_{-3}^{-1} \\ &= \ln |-1| - \ln |-3| \\ &= \ln 1 - \ln 3 \\ &= -\ln 3, \text{ as expected.} \end{aligned}$$

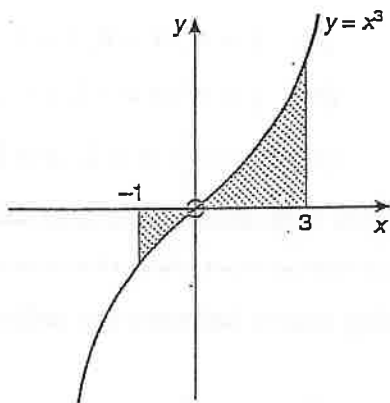
This result tells us that the area is $\ln 3$ and it lies below the x -axis.

Example 6

Find the area under the curve $y = x^3$ between the ordinates $x = -1$, $x = 3$ and the x -axis.

Solution

The required area is shown below:



Calculating $\int_{-1}^3 x^3 dx$ gives

$$\left[\frac{x^4}{4} \right]_{-1}^3 = \frac{3^4}{4} - \frac{(-1)^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

In example 3 we found the shaded area above the x -axis to be $\frac{81}{4}$, which is greater than our answer of 20! There is obviously a problem. Because the required area comprises part below the x -axis and part above, two separate evaluations are required.

$$\int_{-1}^0 x^3 dx = \left[\frac{x^4}{4} \right]_{-1}^0 = -\frac{(-1)^4}{4} = -\frac{1}{4}$$

$$\int_0^3 x^3 dx = \frac{81}{4} \quad (\text{from example 3}).$$

The required area is $\frac{1}{4} + \frac{81}{4} = \frac{82}{4} = 20.5$ sq units.

Definite Integrals

Discussion

Is the area under the curve $y = x^3$ between the ordinates $x = 0$ and $x = -3$ and the x -axis the same?

$$\text{Calculating } \int_{-3}^0 x^3 dx = \left[\frac{x^4}{4} \right]_{-3}^0 = \frac{0^4}{4} - \frac{(-3)^4}{4} = -\frac{81}{4} \text{ sq. units.}$$

Notice that -3 is the lower limit here: $-3 < 0$.

In fact, the answer $-\frac{81}{4}$ tells us that the area is $\frac{81}{4}$ sq. units and the minus sign tells us that the area is below the x -axis.

Example 4

Find the area under the curve $y = e^x - 2$ between the ordinates $x = 0$ and $x = 1$ and the x -axis.

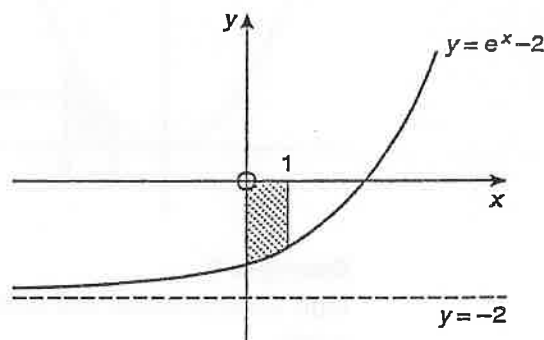
Solution

Calculating $\int_0^1 e^x - 2 dx$ gives

$$[e^x - 2x]_0^1 = [e^1 - 2(1)] - (e^0 - 0) \\ e - 2 - 1 = e - 3 = -0.28.$$

This is a negative value, indicating that the area is below the x -axis.

We write: Area = 0.28 sq. units.



Example 5

Find the area under the curve $y = \frac{1}{x}$ between the ordinates $x = 1$, $x = 3$ and the x -axis.

Solution

Calculating $\int_1^3 \frac{1}{x} dx$ gives

$$[\ln x]_1^3 = \ln 3 - \ln 1 \\ = \ln 3.$$

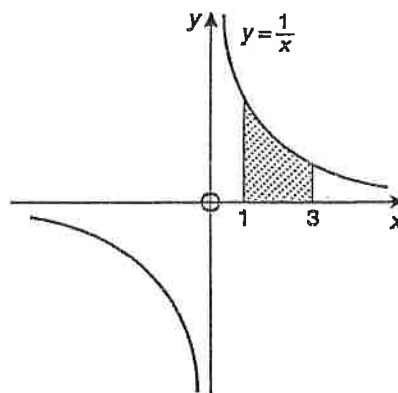
Note

We know that because of the symmetry of the graph of $y = \frac{1}{x}$ the area under the curve between the ordinates $x = -1$, $x = -3$ and the x -axis will also be $\ln 3$.

However, calculating $\int_{-3}^{-1} \frac{1}{x} dx$ gives $[\ln x]_{-3}^{-1} = \ln(-1) - \ln(-3)$.

Now, the function $y = \ln x$ is not defined for negative values of x and $\ln(-1)$ and $\ln(-3)$ cannot be calculated.

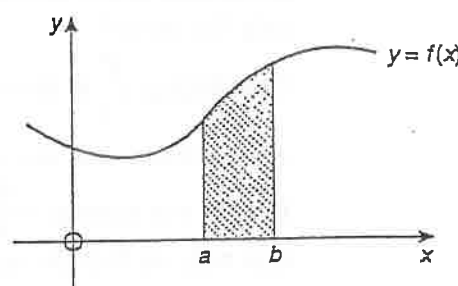
To overcome this problem we have the following result.



Definite Integrals

Result 1

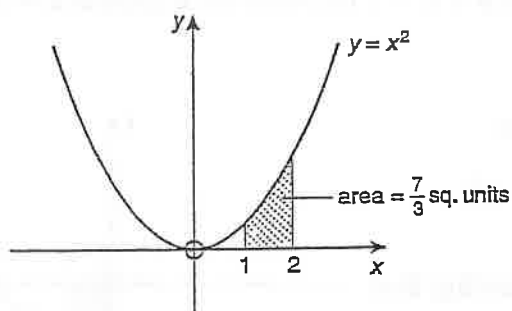
$\int_a^b f(x) dx$ represents the area under the curve $y = f(x)$ between the ordinates $x = a$, $x = b$ and the x -axis.



Example 2

Interpret the result $\int_1^2 x^2 dx = \frac{7}{3}$.

Solution

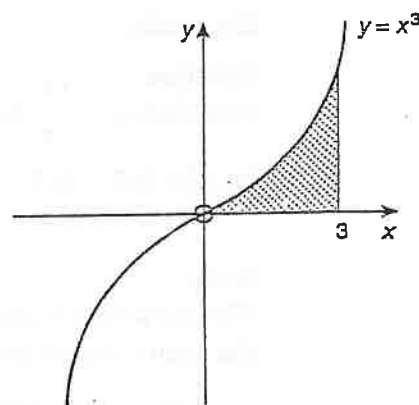


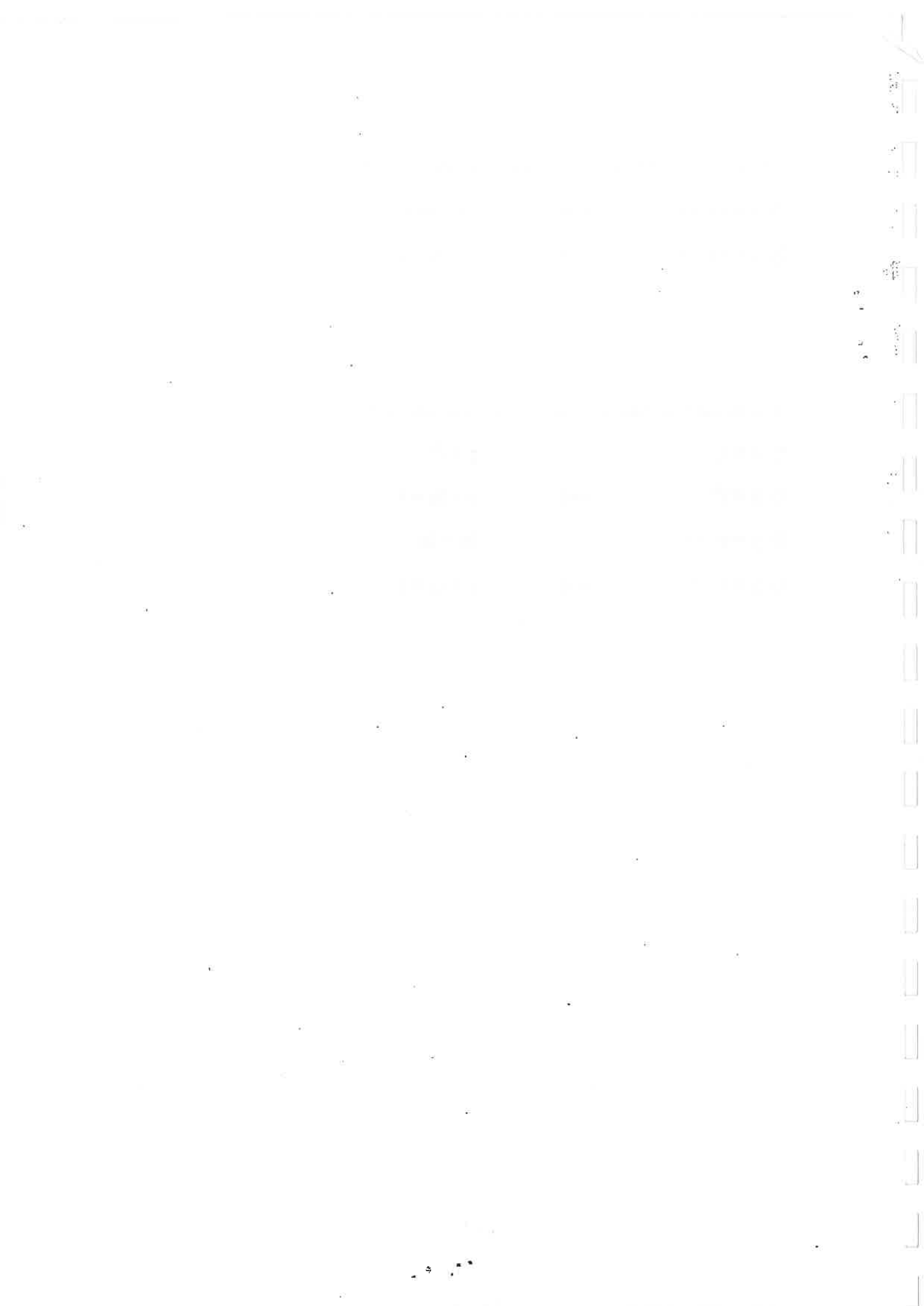
Example 3

Find the area under the curve $y = x^3$ between the ordinates $x = 0$, $x = 3$ and the x -axis.

Solution

$$\text{Area} = \int_0^3 x^3 dx = \left[\frac{x^4}{4} \right]_0^3 = \frac{3^4}{4} - \frac{0^4}{4} = \frac{81}{4} \text{ sq. units.}$$





Find the co-ordinates of the points of intersection of:

1) $x + 2y = 2$ and $3x - 2y = 14$

2) $y + 2x = 5$ and $y = 3x - 5$

3) $y = 2x - 1$ and $y = x + 1$

4) $y = x + 2$ and $2y = 3x + 1$

Find where the following lines and curves intersect:

1) $y = x$, $y = x^2$

2) $y = x^2$ and $y + 3x = 4$

3) $y = x^2 + 1$ $2y = 5x$

4) $y = x - 3$ and $y + xy = 5$

Find where the following lines intersect

1) $x + y = 5$

$$x - y = 2$$

2) $x + y = 7$

$$x - y = 3$$

3) $2x + y = 7$

$$x - y = 2$$

4) $x + 3y = 4$

$$2x - y = 1$$

5) $\alpha y + \beta x = \gamma$

$$\delta y - \epsilon x = \eta$$

Intersection of Lines

If $y = m_1x + c_1$ and $y = m_2x + c_2$ are 2 straight lines then their point of intersection is given by the solution of the two simultaneous equations.

$$\text{i.e. } y = m_1x + c_1 \quad \text{—————} \quad \textcircled{1}$$

$$y = m_2x + c_2 \quad \text{—————} \quad \textcircled{2}$$

$$\text{Sub } 0 = (m_1 - m_2)x + (c_1 - c_2)$$

$$\rightarrow (c_2 - c_1) = (m_1 - m_2)x$$

$$\rightarrow \frac{(c_2 - c_1)}{(m_1 - m_2)} = x$$

$$\therefore y = \frac{m_1(c_2 - c_1)}{(m_1 - m_2)} + c_1$$

$$y = \frac{m_1(c_2 - c_1) + c_1(m_1 - m_2)}{(m_1 - m_2)}$$

$$y = \frac{m_1c_2 - m_1c_1 + m_1c_1 - m_2c_1}{(m_1 - m_2)}$$

$$y = \frac{(m_1c_2 - m_2c_1)}{(m_1 - m_2)}$$

Completing the Square

- 6 (i) $-1 \pm \sqrt{\frac{7}{3}}$ (ii) no real solutions (iii) 1, 2
(iv) no real solutions (v) $-1, \frac{3}{5}$ (vi) no real solutions
(vii) $\frac{5}{4} \pm \frac{\sqrt{33}}{4}$ (viii) $-\frac{1}{4} \pm \frac{\sqrt{13}}{4}$ (ix) $\frac{7}{6} \pm \frac{\sqrt{73}}{6}$

1.6

Completing the Square

1 (i) $x^2 + 8x + 12$ (ii) $x^2 - 10x + 28$ (iii) $2x^2 - 4x - 1$
 (iv) $2x^2 + 2x + 1$ (v) $5x^2 - \frac{15}{2}x + \frac{37}{16}$

2 (i) $-2, -6$ (ii) no real solutions (iii) $1 \pm \sqrt{\frac{3}{2}}$
 (iv) no real solutions (v) $\frac{3}{4} \pm \sqrt{\frac{1}{10}}$

3 (i) $(x+4)^2 - 19$ (ii) $(x-3)^2$ (iii) $(x-5)^2 - 20$
 (iv) $(x+5)^2 - 25$ (v) $\left(x + \frac{5}{2}\right)^2 - \frac{53}{4}$ (vi) $\left(x - \frac{7}{2}\right)^2 - \frac{25}{4}$
 (vii) $\left(x + \frac{11}{2}\right)^2 - \frac{125}{4}$ (viii) $\left(x - \frac{13}{2}\right)^2 - \frac{169}{4}$ (ix) $(x+0)^2 - 3$

4 (i) $-4 \pm \sqrt{19}$ (ii) 3 (iii) $5 \pm 2\sqrt{5}$
 (iv) $-10, 0$ (v) $-\frac{5}{2} \pm \frac{\sqrt{53}}{2}$ (vi) 1, 6
 (vii) $\frac{-11}{2} \pm \frac{5\sqrt{5}}{2}$ (viii) 0, 13 (ix) $\pm\sqrt{3}$

5 (i) $3(x+1)^2 - 7$; $a=3, b=1, c=-7$
 (ii) $2\left(x - \frac{5}{4}\right)^2 + \frac{23}{8}$; $a=2, b=-\frac{5}{4}, c=\frac{23}{8}$
 (iii) $-\left(x - \frac{3}{2}\right)^2 + \frac{1}{4}$; $a=-1, b=-\frac{3}{2}, c=\frac{1}{4}$
 (iv) $-5\left(x - \frac{3}{10}\right)^2 - \frac{291}{20}$; $a=-5, b=-\frac{3}{10}, c=-\frac{291}{20}$
 (v) $5\left(x + \frac{1}{5}\right)^2 - \frac{16}{5}$; $a=5, b=\frac{1}{5}, c=-\frac{16}{5}$
 (vi) $7\left(x - \frac{1}{7}\right)^2 + \frac{6}{7}$; $a=7, b=-\frac{1}{7}, c=\frac{6}{7}$
 (vii) $-2\left(x - \frac{5}{4}\right)^2 + \frac{33}{8}$; $a=-2, b=-\frac{5}{4}, c=\frac{33}{8}$
 (viii) $-4\left(x + \frac{1}{4}\right)^2 + \frac{13}{4}$; $a=-4, b=\frac{1}{4}, c=\frac{13}{4}$
 (ix) $-3\left(x - \frac{7}{6}\right)^2 + \frac{73}{12}$; $a=-3, b=-\frac{7}{6}, c=\frac{73}{12}$

1.6

Completing the Square

- 1 Express each of the following in the form $ax^2 + bx + c$:

(i) $y = (x + 4)^2 - 4$ (ii) $y = (x - 5)^2 + 3$ (iii) $y = 2(x - 1)^2 - 3$

(iv) $y = 2\left(x + \frac{1}{2}\right)^2 + \frac{1}{2}$ (v) $y = 5\left(x - \frac{3}{4}\right)^2 - \frac{1}{2}$

- 2 Using the given form, solve, where possible, each of the equations $y = 0$ in question 1, leaving your answer in surd form.

- 3 Express each of the following quadratic functions in the form $(x + b)^2 + c$:

(i) $y = x^2 + 8x - 3$ (ii) $y = x^2 - 6x + 9$ (iii) $y = x^2 - 10x + 5$

(iv) $y = x^2 + 10x$ (v) $y = x^2 + 5x - 7$ (vi) $y = x^2 - 7x + 6$

(vii) $y = x^2 + 11x - 1$ (viii) $y = x^2 - 13x$ (ix) $y = x^2 - 3$

- 4 Use the form $(x + b)^2 + c$ to solve each of the equations $y = 0$ in question 3, leaving your answer in surd form.

- 5 Express each of the following quadratic functions in the form $a(x + b)^2 + c$, stating the values of a , b and c :

(i) $y = 3x^2 + 6x - 4$ (ii) $y = 2x^2 - 5x + 6$ (iii) $y = -x^2 + 3x - 2$

(iv) $y = -5x^2 + 3x - 15$ (v) $y = 5x^2 + 2x - 3$ (vi) $y = 7x^2 - 2x + 1$

(vii) $y = -2x^2 + 5x + 1$ (viii) $y = 3 - 2x - 4x^2$ (ix) $y = -3x^2 + 7x + 2$

- 6 Use the form $a(x + b)^2 + c$ to solve, where possible, each of the equations $y = 0$ in question 5, leaving your answer in surd form.

Completing the Square

Example 3

Express each of the following quadratic functions in the form $a(x + b)^2 + c$:

- (i) $y = x^2 + 8x - 1$ (ii) $y = x^2 + x - 6$ (iii) $y = 5x^2 + 2x + 10$
 (iv) $y = -3x^2 + 6x + 7$

Answers

- (i) $(x + 4)^2 - 17$ (ii) $\left(x + \frac{1}{2}\right)^2 - \frac{25}{4}$ (iii) $5\left(x + \frac{1}{5}\right)^2 + \frac{49}{5}$
 (iv) $-3(x - 1)^2 + 10$

Discussion

Why is the form $y = x^2 + 2x + 4 = (x + 1)^2 + 3$ useful?

This form gives us a lot of information about the graph of the quadratic function. We will look at this in the Functions section. It is also a useful form for solving the associated quadratic equation.

Example 4

Express $y = x^2 + 6x + 2$ in the form $(x + b)^2 + c$ and hence solve the equation $y = 0$ for x .

Solution

$$y = x^2 + 6x + 2 = (x + 3)^2 - 9 + 2$$

$$\therefore y = (x + 3)^2 - 7$$

The equation $y = 0$ becomes

$$(x + 3)^2 - 7 = 0$$

$$\text{i.e. } (x + 3)^2 = 7$$

$$\therefore x + 3 = \pm\sqrt{7}$$

$$\therefore x = -3 \pm \sqrt{7}$$

Example 5

Using the form $a(x + b)^2 + c$, solve each of the equations $y = 0$ in example 2.

Answers

- (i) $3 \pm \sqrt{12}$ (ii) no real roots (iii) $-2 \pm \sqrt{\frac{5}{2}}$ (iv) $-\frac{2}{3} \pm \sqrt{\frac{31}{9}}$ (v) $3 \pm \sqrt{6}$

1.6

Completing the Square

Prerequisites Lesson 1.5: quadratic equations.

Objectives

Students should be able to:

- (i) express a quadratic function $f(x)$ in the form $a(x + b)^2 + c$;
- (ii) use the form in (i) to solve the equation $f(x) = 0$.

Lesson

We know that 9, 16, 25 are examples of perfect squares since $9 = 3^2$, $16 = 4^2$ and $25 = 5^2$. Similarly, y^6 is a perfect square since $y^6 = (y^3)^2$. In fact, $x^2 + 2x + 1$ is also a perfect square since $x^2 + 2x + 1 = (x + 1)^2$. Not all quadratic expressions are perfect squares. For example, the expression $x^2 + 2x + 4$ cannot be written in the form $(x + k)^2$. However, it can be written in the form $(x + b)^2 + c$, which proves to be a useful form for writing a quadratic expression.

Example 1

Express $y = x^2 + 2x + 4$ in the form $(x + b)^2 + c$.

Solution

The terms $x^2 + 2x$ can be generated from $(x + 1)^2$ since $(x + 1)^2 = x^2 + 2x + 1$.

$$\text{Therefore, } x^2 + 2x = (x + 1)^2 - 1$$

$$\text{and } y = x^2 + 2x + 4 = (x + 1)^2 - 1 + 4$$

$$\therefore y = (x + 1)^2 + 3, \text{ the required form.}$$

This technique is called 'completing the square'.

The following example should be used very carefully. A suggested approach is to focus on types (i) and (ii) first before exploring non-unity a terms.

Example 2

Express each of the following quadratic functions in the form $a(x + b)^2 + c$:

- (i) $y = x^2 - 6x - 3$ (similar to example 1)
- (ii) $y = x^2 + 3x + 4$ (introduces fractional term)
- (iii) $y = 2x^2 + 8x + 3$ (introduces a value of a other than 1)
- (iv) $y = 3x^2 + 4x - 9$ (introduces fractional x terms)
- (v) $y = -x^2 + 6x - 3$ (introduces a value of a which is negative).

Answers

$$(i) (x - 3)^2 - 12 \quad (ii) \left(x + \frac{3}{2}\right)^2 + \frac{7}{4} \quad (iii) 2(x + 2)^2 - 5$$

$$(iv) 3\left(x + \frac{2}{3}\right)^2 - 3\frac{1}{3} \quad (v) -(x - 3)^2 + 6$$

1.5

Quadratic Equations

- 1 (i) $-2, -5$ (ii) $-3, 7$ (iii) $-5, 1$
 (iv) $-2, \frac{1}{2}$ (v) $-5, \frac{1}{2}$ (vi) $-\frac{1}{3}, 6$
 (vii) $-\frac{1}{2}, \frac{2}{5}$ (viii) $0, 7$ (ix) $-5, 5$
 (x) $-\frac{4}{3}, 0$

- 2 (i) $-8.11, 1.11$ (ii) $-1.83, 3.83$ (iii) $0.35, 5.65$
 (iv) $\frac{1}{3}, -2$ (v) $-1, 1.4$

- 3 (i) $-5, 3$ (ii) $-1.14, 2.64$ (iii) $2, 5$
 (iv) $-2.79, 1.79$ (v) $-1.58, 7.58$ (vi) $-0.5, 3$
 (vii) $-3.65, 1.65$ (viii) 1 (ix) $-2, -4$
 (x) $3, \frac{-133}{6}$

- 4 (i) $4, 81$ (ii) $9, 16$ (iii) $\pm\sqrt{3}$
 (iv) $\pm\sqrt{2}, \pm 2\sqrt{2}$ (v) $\pm\frac{3}{2}, \pm 2$ (vi) $-(4)^{\frac{1}{2}}$
 (vii) $-3, 1$ (viii) $0, 25$ (ix) $0, \frac{1}{16}$
 (x) $\left(\frac{4}{9}\right)^{\frac{1}{2}}, \left(\frac{9}{4}\right)^{\frac{1}{2}}$

1.5

Quadratic Equations

1 Solve each of the following using factorisation methods:

- (i) $x^2 + 7x + 10 = 0$ (ii) $x^2 - 4x - 21 = 0$ (iii) $x^2 + 4x - 5 = 0$
 (iv) $2x^2 + 5x + 2 = 0$ (v) $2x^2 + 9x - 5 = 0$ (vi) $3x^2 - 17x - 6 = 0$
 (vii) $10x^2 + x - 2 = 0$ (viii) $x^2 - 7x = 0$ (ix) $x^2 - 25 = 0$
 (x) $6x^2 + 8x = 0$

2 Solve each of the following giving answers to two decimal places where appropriate:

- (i) $x^2 + 7x - 9 = 0$ (ii) $x^2 - 2x - 7 = 0$ (iii) $x^2 - 6x + 2 = 0$
 (iv) $3x^2 + 5x - 2 = 0$ (v) $5x^2 - 2x - 7 = 0$

3 Solve each of the following equations for x :

- (i) $x + 2 = \frac{15}{x}$ (ii) $\frac{2x}{x+2} = \frac{3}{x}$ (iii) $\frac{x-3}{2x-5} = \frac{2}{x}$
 (iv) $\frac{2x+1}{x+2} = \frac{x+4}{x+3}$ (v) $\frac{x+2}{x-2} = \frac{x+3}{2x-9}$ (vi) $\frac{3}{x} + \frac{4}{x+1} = 2$
 (vii) $\frac{1}{x-1} = \frac{2}{x+2} + 1$ (viii) $\frac{2}{x+1} - \frac{3}{x-3} = \frac{5}{3-x}$ (ix) $\frac{2}{x+1} + \frac{1}{2x+5} = -1$
 (x) $\frac{1}{2x+1} - \frac{1}{x+7} = \frac{3}{70}$

4 Solve each of the following equations for x :

- (i) $x + 7\sqrt{x} = 18$ (ii) $x - x^{\frac{1}{2}} = 12$ (iii) $2x^4 + x^2 - 21 = 0$
 (iv) $x^4 = 10x^2 - 16$ (v) $36 = 25x^2 - 4x^4$ (vi) $x^6 + 8x^3 + 16 = 0$
 (vii) $26x^3 = 27 - x^6$ (viii) $x = 5\sqrt{x}$ (ix) $4x - \sqrt{x} = 0$
 (x) $6x^5 - 5x^3 - 6 = 0$

Quadratic Equations

The original equation has simplified to give a standard quadratic equation. Solving now gives

$$x^2 - 4x + 4 = (x - 2)(x + 2) = 0$$

$$\text{i.e. } x - 2 = 0 \quad \text{or} \quad x + 2 = 0$$

$$\therefore x = 2 \quad \text{or} \quad x = -2.$$

It is worth noting that the cross-multiplication technique could have been used on the original equation. This would have given $4(x - 1) = x^2$, missing out a step.

- (ii) In this case cross-multiplication will not work because the fraction on the LHS is not expressed in terms of a single denominator. We first multiply by the LCM, which is $(x + 1)(2x + 1)$. This gives

$$(x + 1)(2x + 1) \times \frac{6}{x + 1} + (x + 1)(2x + 1) \times \frac{5}{2x + 1} = 3 \times (x + 1)(2x + 1)$$

$$\text{i.e. } 6(2x + 1) + 5(x + 1) = 3(x + 1)(2x + 1)$$

$$\therefore 17x + 11 = 3(2x^2 + 3x + 1)$$

$$\therefore 6x^2 - 8x - 8 = 0$$

$$\therefore 2(3x^2 - 4x - 4) = 0.$$

Solving gives

$$3x^2 - 4x - 4 = (3x + 2)(x - 2) = 0$$

$$\therefore 3x + 2 = 0 \quad \text{or} \quad x - 2 = 0$$

$$\therefore x = -\frac{2}{3} \quad \text{or} \quad x = 2.$$

Discussion

How do we solve an equation such as $x - 5\sqrt{x} + 6 = 0$?

On first viewing this does not appear to be a quadratic equation, but if we let $Y = x^{\frac{1}{2}}$ then the equation becomes

$$(x^{\frac{1}{2}})^2 - 5x^{\frac{1}{2}} + 6 = Y^2 - 5Y + 6 = 0$$

We see that it is a quadratic equation in $x^{\frac{1}{2}}$.

Now solving $Y^2 - 5Y + 6 = 0$ i.e. $(Y - 2)(Y - 3) = 0$, gives $Y = 2$ or $Y = 3$.

But $Y = x^{\frac{1}{2}}$, $\therefore x^{\frac{1}{2}} = 2$ or 3

$\therefore x = 4$ or 9 .

Example 5

Solve each of the following equations for x :

$$(i) x^4 - 5x^2 - 14 = 0 \quad (ii) x - 10\sqrt{x} + 25 = 0 \quad (iii) x^3 - 6x^{3/2} = 0$$

Answers

$$(i) \pm\sqrt{7} \quad (ii) 25 \quad (iii) 0, (36)^{\frac{1}{2}}$$

Quadratic Equations

Example 3

Solve each of the following:

(i) $x^2 + 3x - 10 = 0$ (ii) $2x^2 - 5x - 6 = 0$.

Solution

(i) Comparing with $ax^2 + bx + c = 0$ we see that $a = 1$, $b = 3$, $c = -10$.

Using Result 1 now gives

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-10)}}{2(1)} = \frac{-3 \pm \sqrt{49}}{2} = \frac{-3 \pm 7}{2}$$

$$\therefore x = \frac{-3 + 7}{2} = 2 \quad \text{or} \quad x = \frac{-3 - 7}{2} = -5.$$

We see that the two solutions are whole numbers. This tells us that the original quadratic equation could have been solved by factorising the LHS. In fact

$$x^2 + 3x - 10 = (x + 5)(x - 2).$$

(ii) Comparing with $ax^2 + bx + c = 0$ we see that $a = 2$, $b = -5$, $c = -6$.

Using Result 1 now gives

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-6)}}{2(2)} = \frac{5 \pm \sqrt{73}}{4} = \frac{5 \pm 8.54}{4}$$

$$\therefore x = \frac{5 + 8.54}{4} = 3.39 \quad \text{or} \quad x = \frac{5 - 8.54}{4} = -0.89.$$

In this case the solutions have been rounded to two decimal places.

Discussion

Sometimes equations do not look like quadratic equations because they are not expressed in the standard form $ax^2 + bx + c = 0$. However, after algebraic manipulation their identity becomes clear.

Example 4

Solve each of the equations:

(i) $\frac{4}{x} = \frac{x}{x-1}$ (ii) $\frac{6}{x+1} + \frac{5}{2x+1} = 3$

Solution

(i) In this case we must aim to express the equation in a form which does not involve fractions. Multiplying both sides by the LCM of x and $x + 1$ gives

$$x(x-1) \times \frac{4}{x} = x(x-1) \times \frac{x}{x-1}$$

$$\text{i.e.} \quad 4(x-1) = x^2$$

$$\text{i.e.} \quad x^2 - 4x + 4 = 0.$$

1.5

Quadratic Equations

Prerequisites Background knowledge: elementary algebra; lesson 1.4: algebraic fractions.

Objectives Students should:

- (i) know the two standard methods for solving quadratic equations, namely factorisation and the formula method;
- (ii) be able to manipulate equations involving algebraic fractions which give rise to quadratic equations;
- (iii) be able to recognise and solve quadratic equations in functions of x .

Lesson

We know that expansion of the brackets $(x+3)(x+5)$ will give the quadratic expression $x^2 + 8x + 15$. The reverse technique is called factorisation. The technique of factorisation is used to solve quadratic equations.

Example 1

Solve the quadratic equation $x^2 + 8x + 15 = 0$.

Solution

Factorising the LHS gives

$$x^2 + 8x + 15 = 0$$

$$\text{i.e. } (x+3)(x+5) = 0$$

$$\therefore x+3=0 \quad \text{or} \quad x+5=0$$

$$\therefore x=-3 \quad \text{or} \quad x=-5.$$

Example 2

Solve each of the following:

- (i) $x^2 + 5x - 6 = 0$ (ii) $x^2 + 4x = 0$ (iii) $x^2 - 6x + 9 = 0$ (iv) $x^2 - 9 = 0$
 (v) $2x^2 - 5x - 3 = 0$

Answers

- (i) 1, -6 (ii) 0, -4 (iii) 3 (iv) 3, -3 (v) 3, $-\frac{1}{2}$

Discussion

Not all quadratic expressions will factorise in this way. However, this does not mean that the associated quadratic equation cannot be solved. In such cases we use the 'quadratic formula'.

Result 1

If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Although this formula looks complicated, it is very easy to use.

Example 3

Find 'k' such that the equation:

$$x^2 + kx + 1 = 0 \quad \text{has Complex Roots.}$$

$$\begin{aligned}\text{Discriminant} &= b^2 - 4ac \\ &= k^2 - 4(1)(1) \\ &= k^2 - 4\end{aligned}$$

$\therefore$ for Complex Roots $b^2 - 4ac < 0$

$$\longrightarrow k^2 - 4 < 0$$

$$\longrightarrow k > -2 \quad \text{and} \quad k < 2$$

Example 4

Show that the equation:

$(a + 2b)x^2 - (a + b)x - b = 0$ always has 2 Real, Rational and distinct Roots.

$$\begin{aligned}\text{Here: } b^2 - 4ac &= \left[-(a + b) \right]^2 - 4(a + 2b)(-b) \\ &= (a + b)^2 + 4b(a + 2b) \\ &= a^2 + 2ab + b^2 + 4ab + 8b^2 \\ &= a^2 + 6ab + 9b^2 \\ &= (a + 3b)^2\end{aligned}$$

i.e. a Perfect Square, thus, 2 Real, Rational and distinct Roots will be obtained.

Example 1

Find k such that:

$$x^2 + 4x + k = 0 \quad \text{has equal Roots}$$

$$\text{here } a = 1, \quad b = 4, \quad c = k$$

$$\therefore \text{ For equal Roots } b^2 - 4ac = 0$$

$$\longrightarrow 4^2 - 4(1)(k) = 0$$

$$16 - 4k = 0$$

$$\longrightarrow 16 = 4k$$

$$\underline{4 = k}$$

Check

$$x^2 + 4x + 4 = 0$$

$$\longrightarrow (x + 2)^2 = 0$$

$$\longrightarrow x = -2 \text{ (twice)}$$

Example 2

Find 'k' such that the equation:

$$kx^2 + 5x + 6 = 0 \quad \text{has Real Roots.}$$

$$\text{For Real Roots } b^2 - 4ac \geq 0$$

$$\longrightarrow 5^2 - 4(k)(6) \geq 0$$

$$25 - 24k \geq 0$$

$$25 \geq 24k$$

$$\frac{25}{24} \geq k$$

$$\text{or } k \leq \frac{25}{24}$$

Case IV

If $b^2 - 4ac < 0$ then the roots are said to be complex (imaginary) and distinct.

Example

Discuss the nature of the roots of:-

$$4x^2 + 7x + 5 = 0$$

Here $a = 4$, $b = 7$, $c = 5$

$$\text{Discriminant} = b^2 - 4ac$$

$$= 7^2 - 4(4)(5)$$

$$= 49 - 80$$

$$= -31$$

$$< 0$$

∴ Roots are Complex & Distinct.

Case III

If $b^2 - 4ac = 0$ then 2 Real, Rational and Coincident Roots will be obtained.

Example

Discuss the nature of the Roots of

$$4x^2 + 20x + 25 = 0$$

Here $a = 4$, $b = 20$, $c = 25$

Then: $b^2 - 4ac = (20)^2 - 4(4)(25)$

$$= 400 - 400$$

$$= 0$$

∴ 2 Real, Rational and Coincident roots are produced.

Check

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\longrightarrow x = \frac{-20 \pm \sqrt{0}}{2(4)}$$

$$\longrightarrow x = \frac{-20}{8}$$

$$x = \frac{-5}{2}$$

$$\underline{x = -2\frac{1}{2} \text{ (twice)}}$$

Case II $b^2 - 4ac > 0$ but not a Perfect Square.

Then 2 real, unequal and irrational roots will be obtained.

Example

Discuss the nature of the roots of:

$$3x^2 + 5x + 1 = 0$$

Here $a = 3$, $b = 5$, $c = +1$

$$\begin{aligned}\text{So Discriminant} &= b^2 - 4ac \\ &= 5^2 - 4(3)(1) \\ &= 25 - 12 \\ &= 13\end{aligned}$$

> 0 but not a Perfect Square

Thus 2 Real, Irrational but distinct Roots will be obtained.

Check

$$x = \frac{-5 \pm \sqrt{13}}{2(3)}$$

$$x = \frac{-5 \pm \sqrt{13}}{6}$$

either

$$x = \frac{-5 - \sqrt{13}}{6}$$

or

$$x = \frac{-5 + \sqrt{13}}{6}$$

Consider the following cases

Case I

If $b^2 - 4ac > 0$ and is a Perfect Square, then 2 Real, Rational and unequal Roots will be obtained.

Example

Discuss the nature of the roots of the quadratic equation $2x^2 + 3x - 9 = 0$

Here $a = 2$, $b = 3$, $c = -9$

$$\begin{aligned}\therefore \text{Discriminant} &= b^2 - 4ac \\ &= 3^2 - 4(2)(-9) \\ &= 9 + 72 \\ &= 81 \\ &= (9)^2 \text{ (i.e. a Perfect Square)}\end{aligned}$$

Therefore, 2 Real, Rational & unequal Roots will be obtained.

Check

$$\text{Using formula} \rightarrow x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-9)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{81}}{4}$$

$$x = \frac{-3 \pm 9}{4}$$

either

$$x = \frac{-3 - 9}{4}$$

$$x = \frac{-12}{4}$$

$$x = -3$$

or

$$x = \frac{-3 + 9}{4}$$

$$x = \frac{6}{4}$$

$$x = \frac{3}{2}$$

The Nature of the Roots of a Quadratic Equation

The solution of the equation $ax^2 + bx + c = 0$ is given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

upon studying this formula it becomes apparent that the quantity residing within the square root is the entity which determines the roots of a quadratic.

For if $b^2 - 4ac \geq 0$ we have Real Solutions.

But if $b^2 - 4ac < 0$ then no Real Roots are possible as we cannot obtain the square root of negative numbers.

The Quantity $b^2 - 4ac$ is called the 'Discriminant'.

Example 3

Einstein's famous equation $E = mc^2$ is quadratic in nature. It explains how galaxies are formed in our universe and also explains how energy is released from atomic bombs!

If $E = 7.20 \times 10^{18}$ Joules and $m = 80$ kg find 'c'.

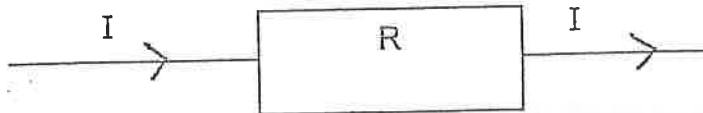
$$\longrightarrow 7.20 \times 10^{18} = 80 \times c^2$$

$$\sqrt{\frac{7.20 \times 10^{18}}{80}} = c$$

$$\longrightarrow c = 3 \times 10^8 \text{ ms}^{-1} \quad (\text{Speed of Light!})$$

Example 2

The Power dissipated through a resistor 'R' when a current 'I' passes through it, is given by the equation $P = I^2 R$.



If the Power $P = 100$ watts and $R = 5$ ohms, find I .

$$\therefore P = I^2 R$$

$$100 = I^2 5$$

$$\frac{100}{5} = I^2$$

$$20 = I^2$$

$$\sqrt{20} = I$$

$$\sqrt{4 \times 5} = I$$

$$2\sqrt{5} \text{ A} = I$$

In Physics

Example 1

The displacement of an object from a fixed origin 'o' is give by the equation:

$$s = 2t^2 - 7t + 3$$

- Find (i) s when $t = 4$
(ii) the time when the displacement $s = 0$

Solution

(i) when $t = 4$, $s = 2(4)^2 - 7(4) + 3$

$$s = 32 - 28 + 3$$

$$\underline{s = 7 \text{ units}}$$

(ii) when $s = 0 \longrightarrow 0 = 2t^2 - 7t + 3$
 $0 = (2t - 1)(t - 3)$

$\longrightarrow$ either

$$2t - 1 = 0$$

$$t = \frac{1}{2}$$

or

$$t - 3 = 0$$

$$\longrightarrow t = 3$$

$$4x^2 = (1-x)(5-x)$$

$$4x^2 = 5 - 5x - x + x^2$$

$$4x^2 = 5 - 6x + x^2$$

$$\longrightarrow 3x^2 + 6x - 5 = 0$$

$$\longrightarrow x = \frac{-6 \pm \sqrt{6^2 - 4(3)(-5)}}{2(3)}$$

$$x = \frac{-6 \pm \sqrt{36 + 60}}{6}$$

$$x = \frac{-6 \pm \sqrt{96}}{6}$$

$$x = \frac{-6 \pm \sqrt{16 \times 6}}{6}$$

$$x = \frac{-6 \pm 4\sqrt{6}}{6}$$

$$\longrightarrow x = \frac{-3 \pm 2\sqrt{6}}{3}$$

either

$$x = \frac{-3 - 2\sqrt{6}}{3}$$

not possible

or

$$x = \frac{-3 + 2\sqrt{6}}{3}$$

$$\underline{x = 0.63 \text{ moles (2 dp)}}$$

In Chemistry:

Example 2

How many moles of Ethanoic acid will be present when 1 mole of Ethyl Ethanoate is added to 5 moles of water and allowed to reach equilibrium. The equilibrium constant $k_c = 4$.



Initially 0 0 1 mole 5 moles

Let the number of moles of ethanoic acid obtained at equilibrium be x then:-

Equilibrium

x x $(1-x)$ $(5-x)$ moles

$\therefore$ Concentration at equilibrium

$$\frac{x}{v} \quad \frac{x}{v} \quad \frac{(1-x)}{v} \quad \frac{(5-x)}{v}$$

$$\therefore k_c = \frac{(\text{CH}_3\text{COO}^- \text{C}_2\text{H}_5^+)(\text{H}_2\text{O})}{(\text{CH}_3\text{COOH})(\text{C}_2\text{H}_5\text{OH})}$$

$$\longrightarrow 4 = \frac{\frac{(1-x)}{v} \cdot \frac{(5-x)}{v}}{\frac{x}{v} \cdot \frac{x}{v}}$$

$$\longrightarrow 4 = \frac{(1-x)(5-x)}{x^2}$$

In Chemistry

Example 1

The number of electrons in each shell is defined by the quadratic function:

$$n = 2s^2 \quad \text{where}$$

n is the number of electrons in each shell

s is the shell number

i.e. when:

$$s = 1, \quad n = 2(1)^2 = 2$$

$$s = 2, \quad n = 2(2)^2 = 8$$

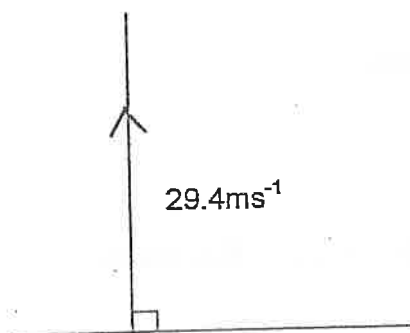
$$s = 3, \quad n = 2(3)^2 = 18$$

$$s = 4, \quad n = 2(4)^2 = 32$$

$$s = 5, \quad n = 2(5)^2 = 50 \quad \text{etc}$$

Example 2

Consider an object thrown vertically upwards with a speed of 29.4ms^{-1} . Find the times at which the object is at a height of 9.8m above the floor.



From the equations of motion

$$\left. \begin{array}{l} v = \\ u = 29.4\text{ms}^{-1} \\ a = -9.8\text{ms}^{-2} \\ s = 9.8\text{m} \\ t = ? \end{array} \right\}$$

Using $s = ut + \frac{1}{2}at^2$.

$$9.8 = (29.4)t + \frac{1}{2}(-9.8)t^2$$

$\therefore$ dividing through by 9.8 $\longrightarrow$

$$1 = 3t - \frac{1}{2}t^2$$

$\therefore$ Multiplying by 2 $\longrightarrow$

$$2 = 6t - t^2$$

$$\therefore t^2 - 6t + 2 = 0$$

$$\longrightarrow t = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(2)}}{2(1)}$$

$$t = \frac{+6 \pm \sqrt{28}}{2}$$

$$t = \frac{6 \pm 2\sqrt{7}}{2}$$

$$\longrightarrow t = 3 \pm \sqrt{7} \text{ seconds}$$

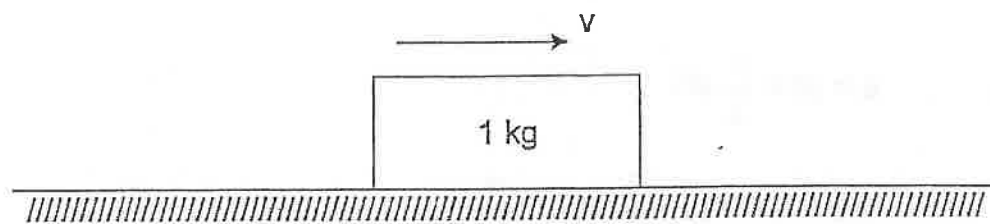
In Mechanics

Example 1

Consider the Kinetic Energy (KE) of an object of mass m kg.

Then $KE = \frac{1}{2} mv^2$, where v is the speed of the object.

Consider an object travelling at a speed $v \text{ ms}^{-1}$ having a KE of 72J. If its mass is 1kg, find its speed.



$$KE = \frac{1}{2} mv^2$$

$$72 = \frac{1}{2} \cdot (1) \cdot v^2$$

$$144 = v^2$$

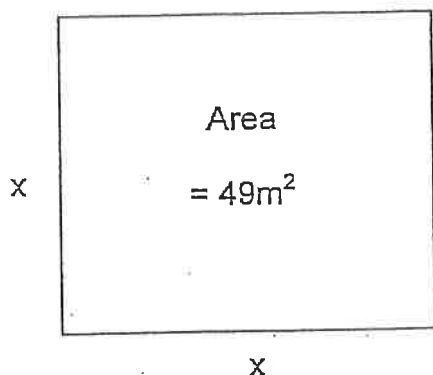
$$\sqrt{144} = v$$

$$\underline{12 \text{ ms}^{-1} = v}$$

Examples of Quadratic Equations in Nature

In Geometry:

If the Area of a Square is 49m^2 , what must be the length of each side?



If square is of side 'x' metres then

$$\text{Area} = x^2$$

$$\therefore 49 = x^2$$

$$\sqrt{49} = x$$

$$7\text{m} = x$$

Likewise the Area of a Circle is a quadratic function as $\text{Area} = \pi r^2$ where 'r' is the radius and also the Surface Area of a sphere it's Surface Area being equal to $4\pi r^2$.

Example 2

In Mathematics and Computing

The solution to the sum $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ a quadratic

function. The answer to this sum is of importance to computing students who often employ 'looping' procedures in their calculations. The time required to perform these often complex procedures can be calculated using this formula.

Example

Solve by the formula $x^2 - x - 1 = 0$

Compare with $ax^2 + bx + c = 0$

→ $a = 1, \quad b = -1, \quad c = -1$

∴ from formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{+1 \pm \sqrt{1+4}}{2}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

either

$$x = \frac{1 - \sqrt{5}}{2}$$

or

$$x = \frac{1 + \sqrt{5}}{2}$$

→ $x = -0.62$ (2 dp)

$x = 1.62$ (2 dp)

The Formula

$$ax^2 + bx + c = 0$$

÷ Throughout by a as $a \neq 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

∴ Completing the square $\longrightarrow$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\longrightarrow x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore x + \frac{b}{2a} = \frac{\pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula will solve any Quadratic equation and is approximately 6,000 years old!!

Example 2

Solve by completing the square:-

$$2x^2 - 4x - 5 = 0$$

$$2(x^2 - 2x) - 5 = 0$$

$$2\left[(x-1)^2 - 1\right] - 5 = 0$$

$$2(x-1)^2 - 2 - 5 = 0$$

$$2(x-1)^2 = 7$$

$$(x-1)^2 = \frac{7}{2}$$

$$x-1 = \pm \sqrt{\frac{7}{2}}$$

$$\therefore x = 1 \pm \sqrt{\frac{7}{2}}$$

Either

$$x = 1 - \sqrt{\frac{7}{2}}$$

$$x = -0.87 \text{ (2 dp's)}$$

or

$$x = 1 + \sqrt{\frac{7}{2}}$$

$$x = 2.87 \text{ (2 dp's)}$$

Completing the Square

Here we attempt to write the quadratic equation:

$$ax^2 + bx + c = 0 \quad \text{in the form}$$

$$\alpha (x \pm \beta)^2 + \gamma = 0$$

Where α , β and γ are to be determined.

Example 1

Solve: $x^2 + 6x - 2 = 0$

$$(x + 3)^2 - 11 = 0$$

$$\therefore (x + 3)^2 = 11$$

$$x + 3 = \pm\sqrt{11}$$

$$x = -3 \pm \sqrt{11}$$

either

or

$$x = -3 - \sqrt{11}$$

$$x = -3 + \sqrt{11}$$

$$x = -6.32 \text{ (2 dp)}$$

$$x = 0.32 \text{ (2 dp)}$$

Case III

The full Quadratic equation of the form $ax^2 + bx + c = 0$ can be solved

- (a) graphically (b) Iteratively (c) by factorising
(d) by completing the square (e) by the FORMULA.

Usually we attempt to factorise first.

Example

Solve $x^2 + 3x + 2 = 0$

We try to write this as a product of 2 brackets.

We think of two numbers which when multiplied give (+2) and when added give (+3)

i.e. $1 \times 2 = 2$ and $1 + 2 = 3$

Thus the equation factorised becomes:-

$$(x + 1) \cdot (x + 2) = 0$$

Thus either or

$$x + 1 = 0$$

$$x + 2 = 0$$

$$\therefore \underline{x = -1}$$

$$\underline{x = -2}$$

Graphical and Iterative methods will be considered at a later date.

Case II

$$c = 0 \rightarrow ax^2 + bx = 0$$

This type can be solved directly by factorising

$$\text{i.e.} \quad ax^2 + bx = 0$$

$$x(ax + b) = 0$$

$\rightarrow$ either or

$$x = 0$$

$$ax + b = 0$$

$$ax = -b$$

$$x = \frac{-b}{a}$$

Example

$$\text{Solve:} \quad 3x^2 + 27x = 0$$

$$3x(x + 9) = 0$$

$\rightarrow$ either or

$$3x = 0$$

$$x + 9 = 0$$

$$\therefore \quad \underline{x = 0}$$

$$\underline{x = -9}$$

Quadratic Equations

These are an important type of equation which occur naturally in science subjects as well as being fundamental in mathematics.

They are of the form:-

$$ax^2 + bx + c = 0 \quad \text{where}$$

a, b and c are constants and $a \neq 0$ and ALWAYS have 2 solutions.

Methods of solution:

Case 1 If $b = 0$ then $ax^2 + c = 0$.

This type can be solved by using the $\sqrt{\quad}$ function.

Example

Solve: $2x^2 - 8 = 0$

$$\longrightarrow 2x^2 = 8$$

$$\therefore x^2 = 4$$

$$\therefore x = \pm\sqrt{4}$$

$$x = \pm 2$$

as $(+2) \times (+2) = +4$

and $(-2) \times (-2) = +4$

1.3

Surds

1 (i) $3\sqrt{5}$ (ii) $2\sqrt{3}$ (iii) $9\sqrt{2}$
(iv) $5\sqrt{6}$ (v) $6\sqrt{6}$

2 (i) $6\sqrt{3}$ (ii) $18\sqrt{2}$ (iii) $6\sqrt{2}$
(iv) $11\sqrt{5}$ (v) $27\sqrt{2} - 4\sqrt{5}$

3 (i) $5 + 3\sqrt{3}$ (ii) $-1 + \sqrt{5}$ (iii) $-9\sqrt{3}$
(iv) $7 + 8\sqrt{2}$ (v) $-1 - 16\sqrt{3}$

4 (i) $2\sqrt{2} + 2$ (ii) $1 - \sqrt{5}$ (iii) $18 - 20\sqrt{2}$
(iv) $-31 - 43\sqrt{2}$ (v) $60 - 31\sqrt{3}$

5 (i) $\frac{\sqrt{7}}{7}$ (ii) $-2 + \sqrt{5}$ (iii) $-\frac{3}{11} - \frac{6}{11}\sqrt{3}$
(iv) $2 + \frac{3}{2}\sqrt{2}$ (v) $-\frac{9}{2} + \frac{5}{2}\sqrt{3}$ (vi) $-\frac{7}{17} - \frac{4}{17}\sqrt{2}$
(vii) $\frac{17}{44} + \frac{7}{44}\sqrt{5}$ (viii) $\frac{45}{671} - \frac{2}{671}\sqrt{3}$ (ix) $23 - 16\sqrt{2}$
(x) $\frac{326}{1271} - \frac{44}{1271}\sqrt{5}$

1.3

Surds

1 Simplify:

(i) $\sqrt{45}$

(ii) $\sqrt{12}$

(iii) $\sqrt{162}$

(iv) $\sqrt{150}$

(v) $\sqrt{216}$

2 Simplify:

(i) $2\sqrt{3} + 2\sqrt{12}$

(ii) $4\sqrt{50} - \sqrt{8}$

(iii) $3\sqrt{2} + \sqrt{18}$

(iv) $5\sqrt{45} - 2\sqrt{20}$

(v) $3\sqrt{32} + 3\sqrt{50} - 2\sqrt{20}$

3 Express each of the following in the form $a + b\sqrt{n}$:

(i) $4\sqrt{3} + (5 - \sqrt{3})$

(ii) $(1 + 3\sqrt{5}) - (2 + 2\sqrt{5})$

(iii) $(5 - 6\sqrt{3}) - (5 + 3\sqrt{3})$

(iv) $(4 - 2\sqrt{2}) + (3 + 5\sqrt{8})$

(v) $(1 + 4\sqrt{12}) - (2 + 6\sqrt{48})$

4 Express each of the following in the form $a + b\sqrt{n}$:

(i) $\sqrt{2}(2 + \sqrt{2})$

(ii) $(2 - \sqrt{5})(3 + \sqrt{5})$

(iii) $(1 - 3\sqrt{2})(6 - \sqrt{8})$

(iv) $(5 + 2\sqrt{2})(1 - 3\sqrt{18})$

(v) $(3 - 2\sqrt{3})(2 - 3\sqrt{27})$

5 Express each of the following in the form $a + b\sqrt{n}$:

(i) $\frac{1}{\sqrt{7}}$

(ii) $\frac{1}{2 + \sqrt{5}}$

(iii) $\frac{3}{1 - 2\sqrt{3}}$

(iv) $\frac{1 + \sqrt{2}}{2 - \sqrt{2}}$

(v) $\frac{3 - 2\sqrt{3}}{1 + \sqrt{3}}$

(vi) $\frac{1 + \sqrt{2}}{1 - 3\sqrt{2}}$

(vii) $\frac{2 - \sqrt{5}}{1 + \sqrt{45}}$

(viii) $\frac{\sqrt{3}}{2 + 3\sqrt{75}}$

(ix) $\frac{5 - \sqrt{8}}{3 + 2\sqrt{2}}$

(x) $\frac{2 + 2\sqrt{20}}{3 - 4\sqrt{80}}$



Example 3

Rationalise: $\frac{a}{c+\sqrt{b}}$

Multiply top and bottom by $c-\sqrt{b}$

$$\frac{a}{c+\sqrt{b}} = \frac{a}{(c+\sqrt{b})} \cdot \frac{(c-\sqrt{b})}{(c-\sqrt{b})}$$

$$= \frac{a(c-\sqrt{b})}{c^2 + c\sqrt{b} - c\sqrt{b} - b}$$

$$= \frac{a(c-\sqrt{b})}{c^2 - b}$$

Example 4

Without using a calculator find the value of:

$$\sin 60^\circ \times \cos 45^\circ \times \tan 30^\circ$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}\sqrt{3}}$$

$$= \frac{1}{2\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{4}$$

Rationalising Surds (Simplifying)

It is customary to remove all Surds from the denominator in an expression.

e.g. to rationalise $\frac{1}{\sqrt{3}}$

Multiply top and bottom by $\sqrt{3}$

$$\begin{aligned}\therefore \frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

Example 2

Rationalise: $\frac{2}{\sqrt{5}-3}$

Multiply top and bottom by $\sqrt{5} + 3$

$$\frac{2}{\sqrt{5}-3} = \frac{2}{(\sqrt{5}-3)} \cdot \frac{(\sqrt{5}+3)}{(\sqrt{5}+3)}$$

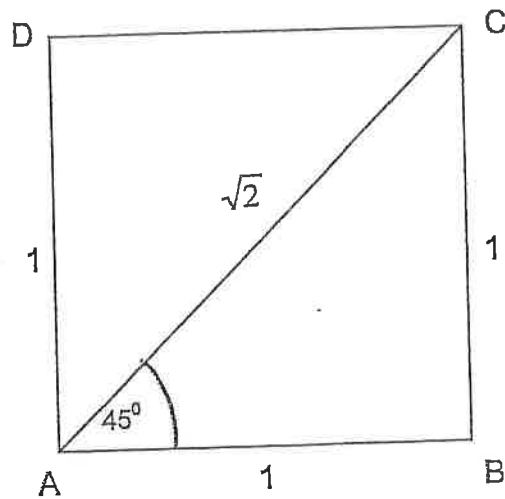
$$= \frac{2(\sqrt{5}+3)}{5-3\sqrt{5}+3\sqrt{5}-9}$$

$$= \frac{2(\sqrt{5}+3)}{-4}$$

$$= \frac{-(\sqrt{5}+3)}{2}$$

Also

Consider the square of side 1 unit.



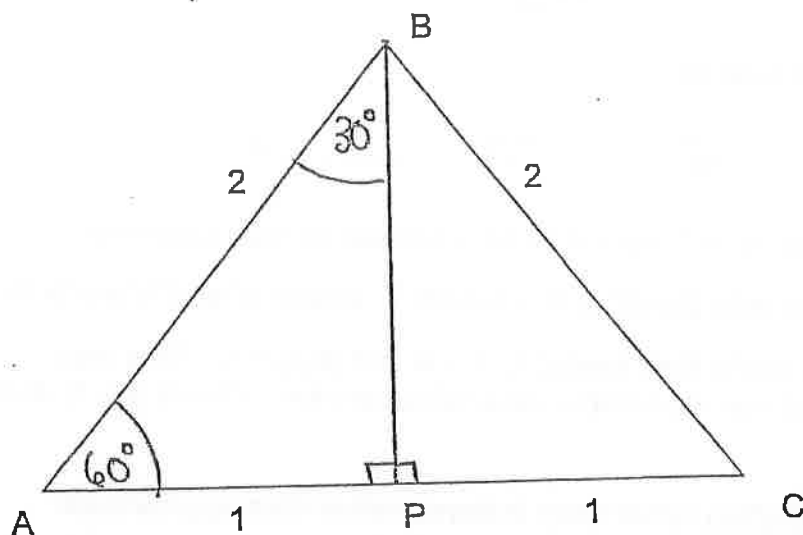
From Pythagoras' Theorem: $AC = \sqrt{2}$

$$\therefore \sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = \frac{1}{1} = 1$$

Then



From Triangle ABP

Using Pythagoras' Theorem

$$BP = \sqrt{3}$$

$$\therefore \sin 60^\circ = \frac{BP}{AB} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{AP}{AB} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{BP}{AP} = \frac{\sqrt{3}}{1}$$

Also

$$\sin 30^\circ = \frac{AP}{AB} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{BP}{AB} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{AP}{BP} = \frac{1}{\sqrt{3}}$$

Surds

Surds are expressions such as:

$$\sqrt{2}, \quad \sqrt{3}, \quad \sqrt{5} \quad \sqrt[3]{7} \quad \dots \text{etc}$$

Things like $\sqrt{2}$ are known as Irrational numbers meaning they cannot be expressed exactly as a ratio (fraction) in the form $\frac{a}{b}$ where 'a' and 'b' are both Integers (whole numbers) in their lowest form; i.e. $\text{hcf}(a,b) = 1$. They also possess an infinite and non repeatable decimal expansion. Others are π and 'e'.

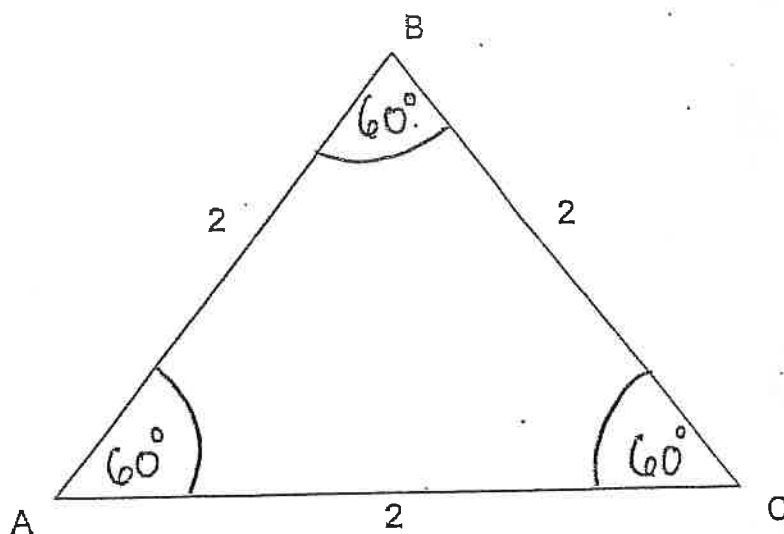
Mathematicians would often rather work in Surds rather than approximate decimal expansions.

i.e. $\sqrt{2}$ is exact 1.414..... is only an approximation.

Uses of Surds

Surds have a particular use in representing certain angles

i.e. Consider the following Equilateral Triangle.



Surds

Example 2

Simplify the following:

$$(i) (3 + \sqrt{2}) + (5 + 2\sqrt{2}) \quad (ii) (2 - \sqrt{5}) - (3 + 5\sqrt{5}) \quad (iii) (1 + \sqrt{3})(5 + 2\sqrt{3})$$

Each answer should be given in the form $a + b\sqrt{n}$.

Answers

$$(i) 8 + 3\sqrt{2} \quad (ii) -1 - 6\sqrt{5} \quad (iii) 11 + 7\sqrt{3}$$

Discussion

How do we express $\frac{1}{1 + \sqrt{5}}$ in the form $a + b\sqrt{5}$?

Multiplying both numerator and denominator by $1 - \sqrt{5}$ is exactly the same as multiplying the fraction by 1 since $\frac{1 - \sqrt{5}}{1 - \sqrt{5}} = 1$.

So we have

$$\begin{aligned} \frac{1}{1 + \sqrt{5}} \times \frac{1 - \sqrt{5}}{1 - \sqrt{5}} &= \frac{1 - \sqrt{5}}{1 - 5} = \frac{1 - \sqrt{5}}{-4} \\ &= -\frac{1}{4} + \frac{\sqrt{5}}{4} \end{aligned}$$

which is in the required form.

Example 3

Express each of the following in the form $a + b\sqrt{n}$:

$$(i) \frac{1}{2 - \sqrt{3}} \quad (ii) \frac{1 + \sqrt{2}}{2 + \sqrt{2}} \quad (iii) \frac{7}{\sqrt{5}} \quad (iv) \frac{2 - \sqrt{3}}{1 + 3\sqrt{3}}$$

Answers

$$(i) 2 + \sqrt{3} \quad (ii) 0 + \frac{1}{2}\sqrt{2} \quad (iii) 0 + \frac{7\sqrt{5}}{5} \quad (iv) -\frac{11}{26} + \frac{7}{26}\sqrt{3}$$

Can $\sqrt{8}$ be simplified?

The answer is yes because 8 can be written as 4×2 , where 4 is a square number. Therefore

$$\begin{aligned} \sqrt{8} &= \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} \\ &= 2\sqrt{2}. \end{aligned}$$

This then enables expressions such as $\sqrt{2} + \sqrt{8}$ to be simplified further. In other words $\sqrt{2} + \sqrt{8} = \sqrt{2} + 2\sqrt{2} = 3\sqrt{2}$.

Example 4

Simplify each of the following:

$$(i) \sqrt{32} \quad (ii) \sqrt{50} \quad (iii) 3\sqrt{3} + \sqrt{48} - 2\sqrt{27}$$

Answers

$$(i) 4\sqrt{2} \quad (ii) 5\sqrt{2} \quad (iii) \sqrt{3}$$

1.3

Surds

Prerequisites None.

Objectives

Students should:

- (i) know what is meant by rational and irrational numbers;
- (ii) be able to manipulate surds via the four rules $+$, $-$, $\times$, $\div$;
- (iii) simplify surds.

Lesson

Using the calculator it is easy to see that $\sqrt{2}$ and π are not exact numbers when written in decimal form.

In other words, neither can be written in the fraction form $\frac{a}{b}$, where a and b are whole numbers.

However, each of the following decimals can be written in fraction form:

$$0.\dot{3} = \frac{1}{3}, \text{ as required.}$$

$$0.52 = \frac{52}{100} = \frac{13}{25}, \text{ as required.}$$

Definition

Numbers which can be written in the form $\frac{a}{b}$, where a and b are whole numbers, are called **rational numbers**. Numbers which cannot be written in this form are called **irrational numbers**.

Example 1

State, with a reason, which of the following are rational:

- (i) $\sqrt{5}$ (ii) $\frac{1}{\sqrt{16}}$ (iii) 0.32 (iv) 0.245

Solution

- (i) Irrational (ii) $\frac{1}{\sqrt{16}} = \frac{1}{4}$, rational (iii) $0.32 = \frac{32}{100}$, rational

- (iv) $0.245 = \frac{245}{1000}$, rational.

Definition

A root, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, which cannot be written in fraction form is called a **surd**.

Result 1

Properties of surds

$$(i) a\sqrt{n} \pm b\sqrt{n} = (a \pm b)\sqrt{n} \quad (ii) \sqrt{m} \times \sqrt{n} = \sqrt{mn} \quad (iii) \sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}$$

Note: $\sqrt{m} + \sqrt{n} \neq \sqrt{m+n}$! This is a common error and many students write simplifications which are not true, such as $\sqrt{x^2 + 4} = x + 2$.

Example 4

$$\text{If } T = 2\pi \sqrt{\frac{L}{g}}$$

Make 'L' the subject.

Square both sides $\rightarrow$

$$T^2 = 4\pi^2 \cdot \frac{L}{g}$$

Now multiply both sides by g

$$T^2 \cdot g = 4\pi^2 L$$

Now divide both sides by $4\pi^2$

$$\frac{T^2 \cdot g}{4\pi^2} = L$$

Re-arranging Formulae

Example 1

If $V = u + at$ make 'a' the subject.

Try to isolate the term with the 'a' in it, by subtracting 'u' from both sides:-

$$\therefore V - u = at$$

Now $\div$ b.s. by t

$$\frac{V - u}{t} = a \quad ('a' \text{ is now the subject of the formula})$$

Example 2

$$\text{If } P = mgh$$

Make 'g' the subject

$$P = m \times g \times h$$

$\div$ both sides by mh

$$\frac{P}{mh} = g$$

Example 3

If $k = \frac{1}{2}mv^2$ make v the subject

Multiply by 2 $\rightarrow$

$$2k = mv^2$$

Now divide b.s. by m

$$\frac{2k}{m} = v^2$$

Now square root both sides

$$\sqrt{\frac{2k}{m}} = v$$

Class Exercise

Solve the following simple linear equations:

1. $2x = 12$

2. $-3x = 6$

3. $\frac{1}{2}x = -4$

4. $x + 1 = 5$

5. $x - 1 = 7$

6. $x + 2 = -3$

7. $2x + 1 = 19$

8. $2x - 2 = 14$

9. $3x + 1 = x + 19$

10. $3x - 1 = 2x + \frac{1}{2}$

Example 4

Solve:

$$\alpha x + \beta = \gamma$$

$$-\beta = -\beta$$

$$\alpha x = \gamma - \beta$$

$\div$ by α

$$\rightarrow x = \frac{\gamma - \beta}{\alpha}$$

Harder linear equations

$$3x + 1 = x - 9$$

The same idea applies, try to get the x's on one side, all the numbers the other, thus subtracting '1' from both sides $\rightarrow$

$$3x + 1 - 1 = x - 9 - 1$$

$$3x = x - 10$$

Now subtract 'x' from both sides

$$3x - x = x - x - 10$$

$$2x = -10$$

Now $\div$ both sides by 2 $\rightarrow$

$$\frac{2x}{2} = \frac{-10}{2}$$

$$x = -5$$

Example 2

$$2x - 1 = 11$$

Here add 1 to both sides:-

$$2x - 1 = 11$$

$$+ 1 \quad + 1$$

$$2x = 12$$

Now ÷ both sides by 2

$$\frac{2x}{2} = \frac{12}{2}$$

$$x = 6$$

Example 3

$$\frac{1}{2}x + 3 = 7$$

$$-3 = -3$$

$$\rightarrow \frac{1}{2}x = 4$$

$$x2 \rightarrow$$

$$x = 4 \times 2$$

$$x = 8$$

But what if the equation is more complex?

We need a method of obtaining the solution if we are asked to find 'x' in :-

$$\pi^2 x + \frac{1}{e} = \sqrt{7}$$

Method of solution of simple linear equations of the form $ax + b = c$

Example

$$x + 1 = 3$$

Try to get the x's on one side, the numbers on the other.

So subtracting '1' from both sides gives:

$$x + 1 = 3$$

Subtracting from both sides

$$\begin{array}{rcl} -1 & -1 & \end{array}$$

$$x = 2$$

Simple Linear Equations

An equation is a statement of equality so that what is on the left hand side (LHS) of an equation is the same (equal) to what is on the right hand side (RHS).

Examples

$$4 = 4$$

$$1 + 3 = 2 + 2$$

WE USUALLY think of equations like BALANCE SCALES.
Usually we have to find a value (x) which satisfies an equation i.e.

$$x + 1 = 2$$

Then $x = 1$ because

$$\text{LHS} = 1 + 1$$

$$= 2$$

$$= \text{RHS}$$

$$x^{m/n} = \sqrt[n]{x^m}$$

$$\begin{aligned}\rightarrow 16^{3/2} &= [(16)^{1/2}]^3 \\ &= 4^3 \\ &= 64\end{aligned}$$

$$25^{-3/2} = \frac{1}{(25)^{3/2}}$$

$$= \frac{1}{[(25)^{1/2}]^3}$$

$$= \frac{1}{(5)^3}$$

$$= \frac{1}{125}$$

Rational Indices

$$x^{1/2} = \sqrt{x} \quad \text{Square Root}$$

$$x^{1/3} = \sqrt[3]{x} \quad \text{Cubed Root}$$

$$x^{1/4} = \sqrt[4]{x} \quad \text{Fourth Root}$$

$$x^{1/n} = \sqrt[n]{x} \quad n^{\text{th}} \text{ Root}$$

Examples

$$25^{1/2} = \sqrt{25} = 5$$

$$64^{1/3} = \sqrt[3]{64} = 4$$

$$81^{1/4} = \sqrt[4]{81} = 3$$

$$1^{1/n} = \sqrt[n]{1} = 1$$

Example 3

Simplify: $\frac{3x^4 \cancel{x} 4x^6}{2\cancel{x} 3x^5} = \frac{12\cancel{x}^{10}}{6x^6}$

$$= 2x^4$$

Example 4

Simplify: $\frac{(2x^3)^3 \cancel{x} (3x^2)^2}{36x^{-2}}$

$$= \frac{8x^9 x 9x^4}{36x^{-2}}$$
$$= \frac{72x^{13}}{36x^{-2}}$$
$$= 2x^{13-(-2)}$$
$$= 2x^{15}$$

Harder Examples

Example 1

Simplify: $\frac{2x^3 \times 6x^4}{3x^5}$

deal with numbers first, then the indicial expressions.

$$\begin{aligned}\text{i.e. } \frac{2x^3 \times 6x^4}{3x^5} &= \frac{(2 \times 6) \times (x^3 \times x^4)}{3x^5} \\ &= \frac{12 \times x^7}{3x^5} \\ &= 4x^{7-5} \\ &= 4x^2\end{aligned}$$

Example 2

$$\begin{aligned}\text{Simplify: } (2x^3)^2 &= (2x^3) \times (2x^3) \\ &= (2 \times 2) \times (x^3 \times x^3) \\ &= 4x^6\end{aligned}$$

Example

1. Simplify $x^{-2} \times x^{-3} = x^{-2-3} = x^{-5} = \frac{1}{x^5}$

2. Simplify $\frac{1}{x^3} \times \frac{1}{x^5} = \frac{1}{x^{3+5}} = \frac{1}{x^8} = x^{-8}$

3. Simplify $\frac{x^{-3} \cancel{x} x^{-4}}{x^{-2}} = \frac{x^{-7}}{x^{-2}} = x^{-7-(-2)} = x^{-5}$

4. Simplify $\frac{x^3 \cancel{x} x^6}{x^{-5}} = \frac{x^9}{x^{-5}} = x^{9-(-5)} = x^{14}$

5. Simplify $\frac{x^2 \cancel{x} x^3 \cancel{x} x^{-4}}{x^{-5} \cancel{x} x^6} = \frac{x^{2+3-4}}{x^{-5+6}} = \frac{x^1}{x^1} = x^0 = 1$
If $x \neq 0$

NEGATIVE INDICES

Consider the following:-

$$x^n$$

$$x^4$$

$$x^3$$

$$x^2$$

$$x$$

$$x^0 = 1 \quad (\text{if } x \neq 0)$$

$$x^{-1} = \frac{1}{x}$$

$$x^{-2} = \frac{1}{x^2}$$

$$x^{-3} = \frac{1}{x^3}$$

$$x^{-4} = \frac{1}{x^4}$$

$$x^{-n} = \frac{1}{x^n}$$

The Power '0'

$$2^0 = 1$$

$$3^0 = 1$$

$$10^0 = 1$$

$$\left(\frac{1}{2}\right)^0 = 1$$

In fact any base number (APART from '0') raised to the power of zero has a value of 1.

$$0^0 = ? \quad (\text{NOT DEFINED})$$

Examples

1. Simplify $2^2 \times 2^4 \times 2^1 = 2^{2+4+1} = 2^7$

2. Simplify $x^2 \times x^3 = x^{2+3} = x^5$

3. Simplify $(x^3)^2 \times x^4 = x^6 \times x^4 = x^{6+4} = x^{10}$

4. Simplify $\frac{x^3 \times x^6}{x^4} = \frac{x^9}{x^4} = x^{9-4} = x^5$

5. Simplify $\frac{x^0 \times x^4 \times x^6}{x^7}$ if $x \neq 0$

$$= \frac{1 \times x^{10}}{x^7}$$

$$= x^{10-7}$$

$$= x^3$$

Division

Consider simplifying $\frac{2^5}{2^3}$ then:

$$\frac{2^5}{2^3} = \frac{2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2 \times 2 = 2^2$$

A quicker way would be to subtract the bottom power from the top one.

i.e. $\frac{2^5}{2^3} = 2^{5-3} = 2^2$

IN GENERAL $x^n \div x^m = x^{n-m}$

Raising by a Power

Consider simplifying $(2^3)^2$

$$(2^3)^2 = (2^3) \times (2^3)$$

$$= 2^{3+3}$$

$$= 2^6$$

In general

$$(x^n)^m = x^{n \times m} = x^{nm}$$

BASIC ALGEBRA

INDICES

Consider $4^2 = 16$

4 – is called the base number

2 – is called the index or power

Manipulation of Indical Expressions

Consider simplifying:-

$$\begin{aligned}2^2 \times 2^3 &= (2 \times 2) \times (2 \times 2 \times 2) \\ &= 2^5\end{aligned}$$

An easier way would be to add the powers

i.e. $2^2 \times 2^3 = 2^{2+3} = 2^5$

Likewise

$$3^4 \times 3^5 = 3^{4+5} = 3^9$$

In general

$$x^n \times x^m = x^{n+m}$$

1.2

Indices II

1 (i) $3^{\frac{1}{2}}$

(ii) $\frac{3}{4}$

(iii) 4

(iv) $\frac{1}{125}$

(v) $3125^{\frac{1}{5}}$

2 (i) $\frac{1}{3}$

(ii) $\frac{3}{2}$

(iii) $\frac{3}{2}$

(iv) $\frac{3}{5}$

(v) $\frac{3}{4}$

3 (i) $-\frac{1}{3}$

(ii) $-\frac{2}{3}$

(iii) $-\frac{1}{4}$

(iv) $-\frac{1}{3}$

(v) $-\frac{1}{3}$

4 (i) $\frac{25}{16}$

(ii) 8

(iii) 8

(iv) $\frac{25}{36}$

(v) -1

(vi) $\frac{1}{4}$

(vii) $\frac{4}{3}$

(viii) $\frac{5}{2}$

(ix) $\frac{2}{13}$

5 (i) -3, 0

(ii) -3, 6

(iii) 0, 1

(iv) -6, 6

(v) 2, 4

1.2

Indices II

1 Solve each of the following for x :

(i) $8x^2 = 24$

(ii) $3x^{-1} = 4$

(iii) $5x^{\frac{1}{2}} = 10$

(iv) $x^{-\frac{1}{2}} = 5$

(v) $2x^{\frac{3}{2}} = 10$

2 Solve each of the following for x :

(i) $8^x = 2$

(ii) $4^x = 8$

(iii) $81^x = 729$

(iv) $32^x = 8$

(v) $625^x = 125$

3 Solve each of the following for x :

(i) $27^x = \frac{1}{3}$

(ii) $8^x = \frac{1}{4}$

(iii) $16^{3x} = 8^{-1}$

(iv) $\frac{125^x}{3} = \frac{1}{15}$

(v) $216^{2x} = \frac{1}{36}$

4 Solve each of the following for x :

(i) $5x^{-\frac{1}{2}} = 4$

(ii) $4x^{-\frac{3}{2}} = 1$

(iii) $\frac{8}{x^{\frac{1}{2}}} = \frac{x}{2}$

(iv) $\frac{6}{x^{-\frac{1}{2}}} = 5$

(v) $3^{2x} = \frac{1}{9}$

(vi) $2401^x = 7$

(vii) $5^{3x} = 25^2$

(viii) $3^{2x+1} = 729$

(ix) $1331^{5x} = 121^{x+1}$

5 Solve each of the following for x :

(i) $2^{x(x+3)} = 1$

(ii) $5^{x^2} = 125^{x+6}$

(iii) $7^{2x^2} = 49^x$

(iv) $5^{x^2} = (125)^{12}$

(v) $3^{x^2} = \frac{729^x}{81^2}$

Indices II

The technique used for solving such an equation is to express both sides of the equation in terms of the same base number.

So, if $2^x = 8$

then $2^x = 2^3$.

Now, comparing indices gives $x = 3$, as expected.

Example 2

Solve: (i) $27^x = 9$ (ii) $625^x = 3125$

Solution

(i) Both sides can be expressed as powers of 3, thus giving

$$(3^3)^x = 3^2$$

$$\text{i.e. } 3^{3x} = 3^2$$

$$\therefore 3x = 2$$

$$\therefore x = \frac{2}{3}$$

(ii) Both sides can be expressed as powers of 5, thus giving

$$(5^4)^x = 5^5$$

$$\text{i.e. } 5^{4x} = 5^5$$

$$\therefore 4x = 5$$

$$\therefore x = \frac{5}{4}$$

Example 3

Solve: (i) $8^x = \frac{1}{16}$ (ii) $81^x = \frac{1}{27}$

Answers

$$(i) -\frac{4}{3} \quad (ii) -\frac{3}{4}$$

Example 4

Solve: (i) $2^{-2} = 8^{2x-3}$ (ii) $5^{x^2} = \frac{25}{5^x}$

Answers

$$(i) 3 \quad (ii) -2, 1$$

1.2

Indices II

Prerequisites Background knowledge: elementary algebra; lesson 1.1: indices I.

Objectives Students should be able to use their knowledge of indices to solve equations in which:

- (i) the x term is raised to a fractional power;
- (ii) the unknown is an index.

Lesson

Suppose we want to solve equations in which the x term is raised to some power. The following example illustrates some of the techniques that are used for solving such equations.

Example 1

Solve: (i) $2x^{\frac{1}{4}} = 3$ (ii) $5x^{\frac{2}{3}} = 45$

Solution

(i) Given $2x^{\frac{1}{4}} = 3$

$$\therefore x^{\frac{1}{4}} = \frac{3}{2}$$

raising both sides to the power 4 gives

$$(x^{\frac{1}{4}})^4 = \left(\frac{3}{2}\right)^4$$

$$\therefore x = \frac{81}{16}$$

(ii) Given $5x^{\frac{2}{3}} = 45$ then $x^{\frac{2}{3}} = 9$. Raising both sides to the power 3 gives

$$(x^{\frac{2}{3}})^3 = 9^3$$

$$\therefore x^2 = 729$$

$$\therefore x = 729^{\frac{1}{2}}$$

$$\therefore x = 27$$

Discussion

If $2^x = 8$ what is the value of x ?

This is a straightforward example and the answer is $x = 3$. However, in many cases the answer is not quite as obvious.

1.1

Indices I

- 1 (i) x^8 (ii) x^7 (iii) $3x^3$
 (iv) $8x^3$ (v) $15x^7$ (vi) $10x^4$
 (vii) $2x^5$ (viii) $12x^6$ (ix) $\frac{5}{4}x^6$

- 2 (i) x^3 (ii) x^2 (iii) $3x$
 (iv) 15 (v) $\frac{1}{2}x^5$ (vi) $4x^5$
 (vii) $2x$ (viii) $\frac{1}{3}x^5$ (ix) $\frac{1}{5}x^3$

- 3 (i) 2 (ii) 2 (iii) 3
 (iv) $\frac{1}{2}$ (v) 4 (vi) 16
 (vii) 125 (viii) $\frac{4}{9}$ (ix) $\frac{32}{243}$

- 4 (i) $\frac{1}{16}$ (ii) 1 (iii) 8
 (iv) $\frac{1}{125}$ (v) 2 (vi) 4
 (vii) 5 (viii) $\frac{1}{9}$ (ix) 4

- 5 (i) $2^{\frac{1}{2}}$ (ii) 1 (iii) 2
 (iv) 3^3 (v) $2^{-\frac{3}{2}}$ (vi) 2^3
 (vii) $3^{\frac{1}{2}} \cdot 2^{\frac{3}{2}}$ (viii) $2^5 \cdot 3^{\frac{1}{2}}$ (ix) $2^{\frac{1}{2}} \cdot 5^{-\frac{1}{2}}$

- 6 (i) $x - x^{\frac{1}{2}} - 2$ (ii) $2x^2 + 8x - 3x^3 - 12x^{\frac{3}{2}}$
 (iii) $2x^3$ (iv) $648x^6$ (v) $\frac{3x^2}{y^4}$

Indices I

5 Without the use of a calculator, simplify each of the following:

(i) $8^{\frac{1}{3}} \times 2^{\frac{1}{3}}$

(ii) $3^2 \times 9^{-1}$

(iii) $4^{-2} \times 2^5$

(iv) $9^3 \times 27^{-1}$

(v) $\frac{8^{\frac{1}{3}} \times 4^{-2}}{2^2}$

(vi) $\frac{27^{-\frac{1}{3}} \times 3^2 \times 4^2}{9^{\frac{1}{2}} \times 2^{-1}}$

(vii) $\frac{64^{\frac{1}{4}} \times 3^3 \times 8^{\frac{1}{2}}}{27^{\frac{1}{3}} \times 6^{\frac{1}{2}}}$

(viii) $\frac{2^5 \times 3^{-2} \times 27^{-\frac{1}{3}}}{8^{\frac{1}{3}} \times 9^{-2} \times 18^{-1}}$

(ix) $\frac{4^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 25^{-\frac{1}{2}}}{10^{\frac{1}{2}} \times 5}$

6 Simplify each of the following:

(i) $(x^{\frac{1}{2}} + 1)(x^{\frac{1}{2}} - 2)$

(ii) $(2x^{\frac{3}{2}} - 3x^{\frac{1}{2}})(x^{\frac{4}{3}} + 4x^{\frac{1}{3}})$

(iii) $\left(\frac{1}{2x^{-3}}\right)^2 \times 8x^{-3}$

(iv) $(36x^2)^{\frac{3}{2}} \times \frac{x^2}{(3x)^{-1}}$

(v) $\sqrt[3]{\frac{27x^6}{y^{12}}}$

1.1

Indices I

1 Simplify each of the following:

(i) $x^7 \times x$

(ii) $x^2 \times x^5$

(iii) $3x \times x^2$

(iv) $4x^2 \times 2x$

(v) $5x^5 \times 3x^2$

(vi) $2x^2 \times 5x^2$

(vii) $\frac{x^3}{4} \times 8x^2$

(viii) $\frac{3x^2}{7} \times 28x^4$

(ix) $\frac{3x^5}{2} \times \frac{5x}{6}$

2 Simplify each of the following:

(i) $x^4 \div x$

(ii) $x^7 \div x^5$

(iii) $3x^2 \div x$

(iv) $15x^3 \div x^3$

(v) $4x^9 \div 8x^4$

(vi) $\frac{12x^7}{3x^2}$

(vii) $\frac{14x^2}{7x}$

(viii) $\frac{x^{10}}{3x^5}$

(ix) $\frac{4x^7}{20x^4}$

3 Without the use of a calculator, simplify each of the following:

(i) $4^{\frac{1}{2}}$

(ii) $8^{\frac{1}{3}}$

(iii) $27^{\frac{1}{3}}$

(iv) $\left(\frac{1}{8}\right)^{\frac{1}{3}}$

(v) $8^{\frac{2}{3}}$

(vi) $64^{\frac{1}{3}}$

(vii) $25^{\frac{1}{2}}$

(viii) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

(ix) $\left(\frac{4}{9}\right)^{\frac{3}{2}}$

4 Without the use of a calculator, simplify each of the following:

(i) 4^{-2}

(ii) 17^0

(iii) $\left(\frac{1}{2}\right)^{-3}$

(iv) 5^{-3}

(v) $\left(\frac{1}{8}\right)^{-\frac{1}{3}}$

(vi) $\left(\frac{1}{4}\right)^{-1}$

(vii) $\left(\frac{1}{25}\right)^{-\frac{1}{2}}$

(viii) $27^{-\frac{2}{3}}$

(ix) $\frac{1}{16^{-\frac{1}{2}}}$

Indices I

Result 2

$$x^{-n} = \frac{1}{x^n} \text{ for all } x (x \neq 0).$$

Example 4

Simplify: (i) 5^{-2} (ii) $\left(\frac{2}{5}\right)^{-3}$ (iii) $8^{-\frac{1}{3}}$ (iv) $\left(\frac{1}{64}\right)^{-\frac{1}{2}}$ (v) $\left(\frac{4}{9}\right)^{-\frac{1}{2}}$

Answers

(i) $\frac{1}{25}$ (ii) $\frac{125}{8}$ (iii) $\frac{1}{2}$ (iv) 8 (v) $\frac{27}{8}$

Example 5

Simplify each of the following:

(i) $(x^2)^{\frac{1}{2}} \times x^{-3}$ (ii) $\left(\frac{9}{x^{-4}}\right)^{\frac{1}{2}}$ (iii) $\frac{2}{x^5} \times \left(\frac{x^4}{y}\right)^6$

Answers

(i) x^{-2} (ii) $3x^2$ (iii) $2x^{-1}y^{-6}$

Indices I

Definition 1

$x^{1/n} = \sqrt[n]{x}$, i.e. the n th root of x .

Example 2

Simplify: (i) $25^{\frac{1}{2}}$ (ii) $\left(\frac{1}{4}\right)^{\frac{1}{2}}$

Answers

(i) 5 (ii) $\frac{1}{2}$

Definition 1 can now be extended to show that

$$x^{m/n} = (x^m)^{1/n} = \sqrt[n]{x^m}.$$

Alternatively, we have $x^{m/n} = (x^{1/n})^m = (\sqrt[n]{x})^m$.

Example 3

Simplify: (i) $9^{\frac{3}{2}}$ (ii) $\left(\frac{1}{4}\right)^{\frac{3}{2}}$ (iii) $\left(\frac{27}{8}\right)^{\frac{2}{3}}$

Answers

(i) 27 (ii) $\frac{1}{32}$ (iii) $\frac{9}{4}$

Discussion

What is the meaning of 3^0 ?

Start with the fact that $3 \div 3 = 1$, but we also know that $3^1 \div 3^1 = 3^{1-1} = 3^0$.
Therefore, we take 3^0 to be equal to 1.

Result 1

$x^0 = 1$ for all x except $x = 0$.

Discussion

What is the meaning of 3^{-1} ?

We know that $3^0 \div 3^1 = 3^{0-1} = 3^{-1}$, but we also know that $3^0 = 1$.

Therefore $3^0 \div 3^1 = 1 \div 3 = \frac{1}{3}$ and we take $3^{-1} = \frac{1}{3}$.

What is the meaning of 3^{-2} ?

We know that $3^0 \div 3^2 = 3^{0-2} = 3^{-2}$.

But $3^0 = 1$; therefore, $3^0 \div 3^2 = 1 \div 3^2 = \frac{1}{3^2} = \frac{1}{9}$ and we take $3^{-2} = \frac{1}{3^2}$.

1.1

Indices I

Prerequisites Background knowledge: the laws of indices for positive integer exponents.

Objectives Students should understand the meaning of:

- (i) fractional indices and be able to evaluate expressions involving fractional indices;
- (ii) negative indices and be able to evaluate expressions involving negative indices.

Lesson Start the lesson with a review of the laws of indices.

Example 1

Simplify each of the following:

(i) $x^3 \times x^5$ (ii) $3x^5 \times 2x^3$ (iii) $\frac{x^5}{x^2}$ (iv) $\frac{10x^7}{2x^3}$ (v) $(x^5)^4$

Answers

(i) x^8 (ii) $6x^8$ (iii) x^3 (iv) $5x^4$ (v) x^{20}

Discussion

Can we express $2^3 \times 5^2$ in the form x^n ?

A common error is to write $2^3 \times 5^2 = 10^5$. Evaluation of both the LHS and the RHS shows that this is not true. It is important to stress that because the 2 and the 5 are different then the law $x^n \times x^m = x^{n+m}$ cannot be used.

What is the meaning of $9^{\frac{1}{2}}$?

Start with a discussion around the fact that

$$9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9^{\frac{1}{2} + \frac{1}{2}} = 9^1.$$

Therefore, $9^{\frac{1}{2}}$ is the square root of 9.

What is the meaning of $9^{\frac{1}{3}}$?

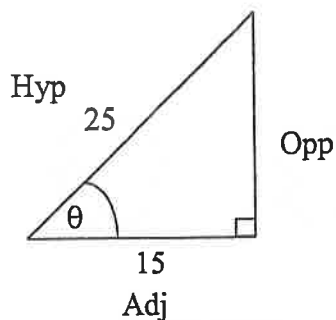
$$\text{In this case } 9^{\frac{1}{3}} \times 9^{\frac{1}{3}} \times 9^{\frac{1}{3}} = 9^1.$$

Therefore, $9^{\frac{1}{3}}$ must be the cube root of 9.

In this next example we show how to work out an unknown angle when two lengths are known.

Example 2

Find the angle of θ in this triangle:



1. Label the sides Hyp, Adj and Opp.
2. We know the hypotenuse and the adjacent. Which trigonometric ratio links these two? The answer is **Cos**.
3. Now do the calculation:

$$\begin{aligned}\cos \theta &= \frac{\text{adj}}{\text{hyp}} \\ &= \frac{15}{25} \\ &= 0.6\end{aligned}$$

To work out the value of θ we need to work out the **INVERSE COS** of 0.6. To do this we use the **INV** key on the calculator.

$$\begin{aligned}\theta &= \text{INV COS}(0.6) \\ &= 53.1\end{aligned}$$

To do this calculation on your calculator you will need to do one of the following:

PRESS	0.6	INV	COS	=
PRESS	INV	COS	0.6	=
PRESS	2 nd F	COS	0.6	=

Example

Make y the subject of the formula: $z = \sqrt{2x - \frac{x^2}{2y+1}}$

Firstly we must get rid of the square root, to do this we square both sides:

$$z^2 = 2x - \frac{x^2}{2y+1}$$

To get rid of the fraction we must multiply both sides by $(2y+1)$

$$z^2 \times (2y+1) = \left(2x - \frac{x^2}{2y+1}\right) \times (2y+1)$$

$$z^2(2y+1) = 2x \times (2y+1) - \frac{x^2}{2y+1} \times (2y+1)$$

$$z^2(2y+1) = 2x(2y+1) - x^2$$

Now we need to expand the brackets

$$2yz^2 + z^2 = 4xy + 2x - x^2$$

Let's get all terms involving y on the LHS and the rest on the RHS. To get rid of $+4xy$ we use $-4xy$

$$2yz^2 + z^2 - 4xy = 4xy + 2x - x^2 - 4xy$$

$$2yz^2 - 4xy + z^2 = 2x - x^2$$

To get rid of $+z^2$ we use $-z^2$

$$2yz^2 - 4xy + z^2 - z^2 = 2x - x^2 - z^2$$

$$2yz^2 - 4xy = 2x - x^2 - z^2$$

Now let's factorise out the y term on the LHS:

$$y(2z^2 - 4x) = 2x - x^2 - z^2$$

To undo the $\times (2z^2 - 4x)$ we use $\div (2z^2 - 4x)$, this gives

$$\frac{y(2z^2 - 4x)}{(2z^2 - 4x)} = \frac{2x - x^2 - z^2}{(2z^2 - 4x)}$$

$$y = \frac{2x - x^2 - z^2}{(2z^2 - 4x)}$$

Note: The question in your GCSE examination will probably be easier than the above example.

22. Rearranging Formulae

Consider the formula

$$3z - x = 5y + 2p$$

We could re-arrange this formula so that we had

$$x = \text{"something"}$$

where there is no x on the right hand side. We say we have made x the **subject** of the formula.

Alternatively we could rearrange to obtain

$$y = \text{"something"}$$

where there is no y on the right hand side. This time we've made y the **subject** of the formula.

You could also make x or p the subject of the formula.

To re-arrange formulae we use the same methods that we used when we learnt to solve equations.

Example

Make p the subject of the formula: $\frac{p}{p+y} = 4$

Firstly we multiply both sides by $(p+y)$ to get rid of the fraction:

$$\begin{aligned}\frac{p}{p+y} \times (p+y) &= 4 \times (p+y) \\ p &= 4p + 4y\end{aligned}$$

To get rid of $+4p$ we use $-4p$

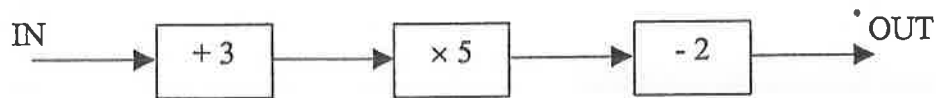
$$\begin{aligned}p - 4p &= 4p + 4y - 4p \\ -3p &= 4y\end{aligned}$$

To undo $\times -3$ we use $\div -3$ which gives

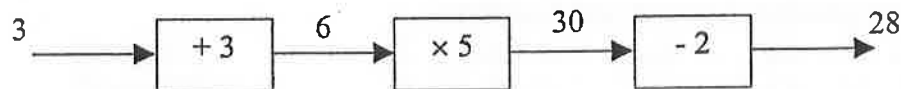
$$\begin{aligned}\frac{-3p}{-3} &= \frac{4y}{-3} \\ p &= -\frac{4}{3}y\end{aligned}$$

21. Function Machines

An example of a function machine is



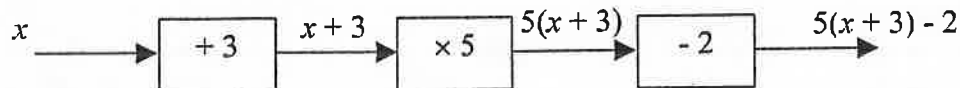
The secret of a function machine is to write down what is coming out from each box.
If we put 3 in the above machine we'd have



Put 3 in the 1 st box	$3 + 3 = 6$	6 comes out.
Put 6 in the 2 nd box	$6 \times 5 = 30$	30 comes out.
Put 30 in the 3 rd box	$30 - 2 = 28$	28 comes out.

Example

What comes out if you put x in?



Put x in the 1 st box	$x + 3$	$x + 3$ comes out.
Put $x + 3$ in the 2 nd box	$5 \times (x + 3)$	$5(x + 3)$ comes out.
Put $5(x + 3)$ in the 3 rd box	$5(x + 3) - 2$	$5(x + 3) - 2$ comes out.

What comes out can be simplified as follows:

$$\begin{aligned}
 5(x + 3) - 2 &= 5x + 15 - 2 \\
 &= 5x + 13
 \end{aligned}$$

Sometimes you may be told what comes out and asked to find what was originally put in the machine. To do this you simply use the machine from right to left and do the opposite operations.

Example

For the above machine what would you need to put in to get 48 out.

Working from right to left and doing the opposite operations gives.

48 goes in.	$48 + 2 = 50$	50 comes out.
50 goes in.	$50 \div 5 = 10$	10 comes out.
10 goes in.	$10 - 3 = 7$	7 comes out.

So you need to put 7 in.

Example 3

Solve the equation

$$x^2 + x - 12 = 0$$

Ans

Firstly we must factorise the quadratic:

$$(x \quad)(x \quad)$$

We want to find two numbers that multiply to give -12 and add to give 1. Since -12 is negative the numbers must have different signs and since they add to give 1 the bigger number must be positive. Let's try and find them:

$-1 \times 12 = -12$	and	$-1 + 12 = 11$	[No good]
$-2 \times 6 = -12$	and	$-2 + 6 = 4$	[No good]
$-3 \times 4 = -12$	and	$-3 + 4 = 1$	[Success!!!]

Hence

$$x^2 + x - 12 = (x - 3)(x + 4)$$

Thus

$$(x - 3)(x + 4) = 0$$

So we have

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

OR

$$\begin{aligned} x + 4 &= 0 \\ x &= -4 \end{aligned}$$

Example 4

Solve the equation

$$x^2 - 4x - 12 = 0$$

Ans

Firstly we must factorise the quadratic:

$$(x \quad)(x \quad)$$

We want to find two numbers that multiply to give -12 and add to give -4. Since -12 is negative the numbers must have different signs and since they add to give -4 the bigger number must be negative. Let's try and find them:

$1 \times -12 = -12$	and	$1 - 12 = -11$	[No good]
$2 \times -6 = -12$	and	$2 - 6 = -4$	[Success!!!]

Hence

$$x^2 - 4x - 12 = (x + 2)(x - 6)$$

Thus

$$(x + 2)(x - 6) = 0$$

So we have

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

OR

$$\begin{aligned} x - 6 &= 0 \\ x &= 6 \end{aligned}$$

Example 1

Solve the equation

$$x^2 + 7x + 10 = 0$$

Ans

Firstly we must factorise the quadratic:

$$(x \quad)(x \quad)$$

We want to find two numbers that multiply to give 10 and add to give 7. Since 10 is positive the numbers must have the same sign and since they add to give 7 they must both be positive. Let's try and find them:

$1 \times 10 = 10$	but	$1 + 10 = 11$	[No good]
$2 \times 5 = 10$	and	$2 + 5 = 7$	[Success!!!]

Hence

$$x^2 + 7x + 10 = (x + 5)(x + 2)$$

Thus

$$(x + 5)(x + 2) = 0$$

So we have

$$\begin{aligned} x + 5 &= 0 \\ x &= -5 \end{aligned}$$

OR

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

Example 2

Solve the equation

$$x^2 - 5x + 6 = 0$$

Ans

Firstly we must factorise the quadratic:

$$(x \quad)(x \quad)$$

We want to find two numbers that multiply to give 6 and add to give -5. Since 6 is positive the numbers must have the same sign and since they add to give -5 they must both be negative. Let's try and find them:

$-1 \times -6 = 6$	but	$-1 + -6 = -7$	[No good]
$-2 \times -3 = 6$	and	$-2 + -3 = -5$	[Success!!!]

Hence

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

Thus

$$(x - 2)(x - 3) = 0$$

So we have

$$\begin{aligned} x - 2 &= 0 \\ x &= 2 \end{aligned}$$

OR

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

20. Solving Quadratic Equations

20.1. What if $a \times b = 0$?

An important fact that you should know is: "If two terms multiply together to give zero then at least one of them must be zero", i.e.

If $a \times b = 0$ then either $a = 0$ OR $b = 0$

This can be useful in solving some types of equations.

OR $a = 0$ and $b = 0$

Example

Solve the equation

$$(x + 3)(x - 7) = 0$$

From above we know that either

$$x + 3 = 0$$

OR

$$x - 7 = 0$$

$$x = -3$$

$$x = 7$$

Example

Solve the equation

$$(x - 5)(x + 11) = 0$$

From above we know that either

$$x - 5 = 0$$

OR

$$x + 11 = 0$$

$$x = 5$$

$$x = -11$$

20.2. Solving Quadratics

A quadratic equation is one that looks like

$$ax^2 + bx + c = 0$$

In the quadratic equations you will consider we will always have $a = 1$, namely

$$x^2 + bx + c = 0$$

Examples of such equations include:

$$x^2 - 2x + 1$$

$$x^2 + 5x - 3$$

$$x^2 - 11x + 22$$

To solve a quadratic equation you need to factorise it using the method below:

- First write the equation in the form $x^2 + bx + c = 0$
- Our aim is to write the equation in the form $(x \quad)(x \quad) = 0$
- To achieve this we need to find two numbers that multiply to give c but also add to give b .
- If c is positive then both numbers must have the same sign, but if c is negative the numbers will have different signs.
- Once the equation has been factorised it can then be easily solved.

19.4. Equations with Brackets

If an equation involves brackets **always** multiply out the brackets first before you attempt to solve the equation.

Example

Solve $2(x + 5) = 5(x - 7)$

Ans

Firstly we expand the brackets, this gives

$$2x + 10 = 5x - 35$$

Subtract $2x$ from both sides:

$$\begin{aligned} 2x + 10 - 2x &= 5x - 35 - 2x \\ 10 &= 3x - 35 \end{aligned}$$

Add 35 to both sides:

$$\begin{aligned} 10 + 35 &= 3x - 35 + 35 \\ 45 &= 3x \end{aligned}$$

Divide both sides by 3:

$$\begin{aligned} \frac{45}{3} &= \frac{3x}{3} \\ 15 &= x \end{aligned}$$

Check:

$$\begin{array}{llll} \text{LHS: } 2(x + 5) & = & 2(15 + 5) & \text{RHS: } 5(x - 7) \\ & = & 2 \times 20 & = 5(15 - 7) \\ & = & 40 & = 5 \times 8 \\ & & & = 40 \end{array}$$

Since LHS = RHS, our answer is correct.

19.5. Equations with Fractions

If an equation involves a fraction it is easier to get rid of the fraction first before attempting to solve the equation.

Example

Solve $\frac{2x-1}{3} = 7$

Ans

Multiply each side by the denominator 3:

$$3 \times \frac{(2x-1)}{3} = 3 \times 7$$

$$2x - 1 = 21$$

Add 1 to each side

$$\begin{aligned} 2x - 1 + 1 &= 21 + 1 \\ 2x &= 22 \end{aligned}$$

Divide both sides by 2

$$\begin{aligned} \frac{2x}{2} &= \frac{22}{2} \\ x &= 11 \end{aligned}$$

As always you should check this answer, we leave this as an exercise.

19.3. More Complicated Examples

Example

Solve $2x + 5 = 11$

Ans

To get rid of $+5$ we use -5 , so we subtract 5 from both sides:

$$\begin{array}{rcl} 2x + 5 - 5 & = & 11 - 5 \\ 2x & = & 6 \end{array}$$

To undo the $\times 2$ we use $\div 2$, so we divide both sides by 2:

$$\begin{array}{rcl} \frac{2x}{2} & = & \frac{6}{2} \\ x & = & 3 \end{array}$$

Check:

$$\begin{array}{lcl} \text{LHS: } 2x + 5 & = & 2 \times 3 + 5 \\ & = & 6 + 5 \\ & = & 11 \\ \text{RHS: } & & 11 \end{array}$$

Since LHS = RHS our answer is correct.

Example

Solve $3x + 4 = 7 - 2x$

Ans

To get rid of $-2x$ we use $+2x$, so we add $2x$ to **both sides**

$$\begin{array}{rcl} 3x + 4 + 2x & = & 7 - 2x + 2x \\ 5x + 4 & = & 7 \end{array}$$

To get rid of $+4$ we use -4 , so we subtract 4 from **both sides**

$$\begin{array}{rcl} 5x + 4 - 4 & = & 7 - 4 \\ 5x & = & 3 \end{array}$$

To undo the $\times 5$ we use $\div 5$, so we divide **both sides** by 5

$$\begin{array}{rcl} \frac{5x}{5} & = & \frac{3}{5} \\ x & = & \frac{3}{5} = 0.6 \end{array}$$

Check:

$$\begin{array}{lcl} \text{LHS: } 3x + 4 & = & 3 \times 0.6 + 4 = 5.8 \\ \text{RHS: } 7 - 2x & = & 7 - 2 \times 0.6 = 5.8 \end{array}$$

Since LHS = RHS our answer is correct.

Example

Solve $4x = 8$

Ans

To undo the $\times 4$, we use $\div 4$. So we must divide **both sides** by 4.

$$\frac{4x}{4} = \frac{8}{4}$$

Now since $4 \div 4 = 1$ we have

$$x = 2$$

Check:

$$\begin{array}{lcl} \text{LHS: } 4x & = & 4 \times 2 \\ & = & 8 \end{array}$$

$$\text{RHS: } 8$$

Since $\text{LHS} = \text{RHS}$ our answer is correct.

Example

Solve $\frac{x}{7} = 3$

Ans

To undo $\div 7$, we use $\times 7$. So we must multiply **both sides** by 7.

$$\frac{x}{7} \times 7 = 3 \times 7$$

Now since $\frac{1}{7} \times 7 = 1$ we have

$$x = 21$$

Check:

$$\begin{array}{lcl} \text{LHS: } \frac{x}{7} & = & \frac{21}{7} \\ & = & 3 \end{array}$$

$$\text{RHS: } 3$$

Since $\text{LHS} = \text{RHS}$ our answer is correct.

Example

Solve $x + 3 = 7$

Ans

To get rid of $+ 3$ we use $- 3$. So we subtract 3 from **both sides** of the equation:

$$\begin{array}{rcl} x + 3 - 3 & = & 7 - 3 \\ x & = & 4 \end{array}$$

Let's check that this answer is correct by using substitution:

$$\begin{array}{rcl} \text{LHS: } x + 3 & = & 4 + 3 \\ & = & 7 \end{array} \qquad \text{RHS: } 7$$

Since LHS = RHS our answer is correct.

Example

Solve $x - 4 = 10$

Ans

To get rid of $- 4$ we use $+ 4$. So we add 4 to **both sides** of the equation:

$$\begin{array}{rcl} x - 4 + 4 & = & 10 + 4 \\ x & = & 14 \end{array}$$

Let's check that this answer is correct by using substitution:

$$\begin{array}{rcl} \text{LHS: } x - 4 & = & 14 - 4 \\ & = & 10 \end{array} \qquad \text{RHS: } 10$$

Since LHS = RHS our answer is correct.

19.2. Inverses of $\times$ and $\div$

Recall that x means $1x$. Sometimes after rearranging your equation the letter term may have a coefficient that is not 1. For example,

$3x$ has a coefficient 3; it means $3 \times x$

$\frac{x}{2}$ has a coefficient $\frac{1}{2}$; it means $\frac{x}{2}$

We will want to make the coefficient equal to 1. The inverses of $\times$ and $\div$ will help you do this:

The inverse of $\times$ is $\div$. So to undo a multiplication use a matching division. For example:

If you have a coefficient 3 then you need to divide by 3 since $3 \div 3 = 1$

The inverse of $\div$ is $\times$. So to undo a division use a matching multiplication. for example:

If you have a coefficient $\frac{1}{2}$ then you need to multiply by 2 since $\frac{1}{2} \times 2 = 1$

19. Solving Linear Equations

An **equation** is a statement containing an “=” sign, such as

$$x + 4 = 7 \qquad 12x + 3 = 2x$$

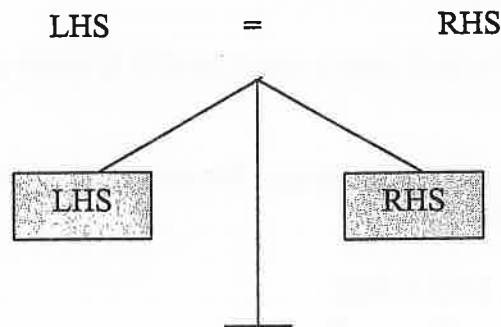
In the above equations x is the unknown. A **linear equation** is one that has no squares or higher power of the unknown.

A **solution** of an equation is a number which, when put instead of the letter, makes the equation true. For example,

$$x + 4 = 7 \quad \text{has the solution} \quad x = 3 \quad \text{because } 3 + 4 = 7$$

19.1. Keeping the Balance

You should think of an equation like a set of scales in balance.



The equation, like scales, will still balance if
whatever you do to one side you also do to the other

The balance will be unchanged if you

- I. add the same quantity to each side
- II. subtract the same quantity from each side
- III. multiply each side by the same quantity
- IV. divide each side by the same quantity

In order to solve an equation the idea is to get all the “letter terms” on one side and all the “number terms” on the other. To obtain this you may need to get rid of unwanted terms from each side of the equation.

The inverse of $+$ is $-$. So to get rid of a ‘+ term’ use the matching ‘- term’. For example,

$$\text{to get rid of '+2' use '-2' because } +2 - 2 = 0$$

The inverse of $-$ is $+$. So to get rid of a ‘- term’ use the matching ‘+ term’. For example,

$$\text{to get rid of '-3' use '+3' because } -3 + 3 = 0$$

Once you have solved the equation always **check** your answer by substitution.

18. Factorising

This is the exact opposite of multiplying out brackets. Now instead of getting rid of brackets we are interested in putting them back in. The idea is to find common factors amongst the terms and bring them out. This is best learnt by example:

Example

Factorise $9x + 12x^2y$

- What is the highest common factor of 9 and 12?
3 can be brought out as a common factor.
- What is the biggest power of x that can be brought out from each term?
We can bring an x out of each term but we can't bring x^2 out of each term. So x can be brought out as a common factor.
- Since y doesn't occur in each term it won't be able to come out as a common factor.

Hence we can bring out $3x$ as a common factor. We then need to work out what will be left once we've taken out the $3x$:

$$9x + 12x^2y = 3x(3 + 4xy)$$

We need 3
to get $9x$

We need $4xy$ to
get $12x^2y$

Example

Factorise $15x^4y + 20x^2y^3z - 35x^3yz^2$

- What is the highest common factor of 15, 20 and 35?
5 can be brought out as a common factor.
- What is the biggest power of x that can be brought out of each term?
 x^2 can be brought out as a common factor.
- What is the biggest power of y that can be brought out of each term?
 y can be brought out as a common factor.
- Since z doesn't occur in each term it won't be able to come out as a common factor.

Hence we can bring out $5x^2y$ as a common factor. Then work out what is left:

$$15x^4y + 20x^2y^3z - 35x^3yz^2 = 5x^2y(3x^2 + 4y^2z - 7xz^2)$$

We need $3x^2$
to get $15x^4y$

We need $4y^2z$
to get $20x^2y^3z$

We need $7xz^2$
to get $35x^3yz^2$

17.2. Brackets Multiplied by Brackets

To expand an expression like $(2x + 1)(x + 2)$ you need to multiply each term in the first bracket by each term in the second:

$$\begin{aligned}(2x + 1)(x + 2) &= 2x \times x + 2x \times 2 + 1 \times x + 1 \times 2 \\ &= 2x^2 + 4x + x + 2 \\ &= 2x^2 + 5x + 2\end{aligned}$$

Example

Expand and simplify $(3x + 7)(5x - 2)$

$$\begin{aligned}(3x + 7)(5x - 2) &= 3x \times 5x + 3x \times -2 + 7 \times 5x + 7 \times -2 \\ &= 15x^2 - 6x + 35x - 14 \\ &= 15x^2 + 29x - 14\end{aligned}$$

17.3. Squared Brackets

To expand something like $(5x + 3)^2$ you need to remember that

$$(5x + 3)^2 = (5x + 3)(5x + 3)$$

Then we can use the same method as above.

$$\begin{aligned}(5x + 3)^2 &= (5x + 3)(5x + 3) \\ &= 5x \times 5x + 5x \times 3 + 3 \times 5x + 3 \times 3 \\ &= 25x^2 + 15x + 15x + 9 \\ &= 25x^2 + 30x + 9\end{aligned}$$

Example

Expand and simplify $(x - 4)^2$

Ans

$$\begin{aligned}(x - 4)^2 &= (x - 4)(x - 4) \\ &= x \times x + x \times -4 + -4 \times x + -4 \times -4 \\ &= x^2 - 4x - 4x + 16 \\ &= x^2 - 8x + 16\end{aligned}$$

17. Expanding Brackets

17.1. Multiplying Out Brackets

Recall that when letters are multiplied together then we just write them next to each other:

$$x \times y = xy$$

$$x \times y \times z = xyz$$

Also remember from our work on indices that:

$$x \times x = x^2$$

$$x \times x \times x = x^3$$

Finally don't forget the important effect of brackets:

$$xy^2 = x \times y \times y$$

$$(xy)^2 = x \times x \times y \times y$$

When expanding brackets the term outside the bracket multiplies each separate term inside the bracket:

$$3(5x + 2) = 3 \times 5x + 3 \times 2 = 15x + 6$$

$$-4(4x^2 + y) = -4 \times 4x^2 + -4 \times y = -16x^2 - 4y$$

$$4x(3y + 2x) = 4x \times 3y + 4x \times 2x = 12xy + 8x^2$$

To expand a bracket that just has a minus sign outside the bracket you simply change the signs of each of the terms inside the bracket:

$$-(x + y) = -x - y$$

$$-(-x - y) = x + y$$

$$-(x - y) = -x + y$$

$$-(-x + y) = x - y$$

Example

Expand and simplify $7x + 8y + 3(2x + 4y)$

Ans

Firstly we expand the bracket:

$$\begin{aligned} 3(2x + 4y) &= 3 \times 2x + 3 \times 4y \\ &= 6x + 12y \end{aligned}$$

Hence we have

$$\begin{aligned} 7x + 8y + 3(2x + 4y) &= 7x + 8y + 6x + 12y \\ &= 7x + 6x + 8y + 12y && \text{[Group together like terms]} \\ &= 13x + 20y && \text{[Combine like terms]} \end{aligned}$$

16. Simplifying Expressions

Consider the expression

$$5x - 7y - y^2 - x + 3y^2$$

We say

$5x$	is an x term
$-7y$	is a y term
$-y^2$	is a y^2 term
$-x$	is an x term
$3y^2$	is a y^2 term

The idea here is to first **collect like terms together**, so we would write:

$$5x - 7y - y^2 - x + 3y^2 = (5x - x) - 7y - y^2 + 3y^2$$

$(5x - x)$

x terms

$-7y$

y terms

$-y^2 + 3y^2$

y^2 terms

Next we **combine like terms**:

$$5x - x = 4x$$

$$-7y = -7y$$

$$-y^2 + 3y^2 = 2y^2$$

So we have

$$5x - 7y^2 - x + 3y^2 = 4x - 7y + 2y^2$$

WARNING: Don't be confused into thinking that y and y^2 are both y terms!!! They are different and hence cannot be combined.

Example

Simplify $7xy + 9yz - 3xy - 3yz + 7xy - 2yz$

Firstly we **collect the like terms together**:

$$\begin{aligned}
 7xy + 9yz - 3xy - 3yz + 7xy - 2yz &= \boxed{7xy - 3xy + 7xy} + \boxed{9yz - 3yz - 2yz} \\
 &\quad \text{(Collect like terms together)} \\
 &= 11xy + 4yz \quad \text{(Combine like terms)}
 \end{aligned}$$

15. Substitution

Prior to doing substitution we need to understand some of the notation used in algebra:

15.1. Notation

- $4x$ means $4 \times x$ [The multiplication sign is often left out]
- $5y$ means $5 \times y$
- xyz means $x \times y \times z$ [The multiplication sign is often left out]
- xy^2 means $x \times y \times y$ [It is only the y that is squared]
- $(xy)^2$ means $x \times y \times x \times y$ [Both the x and y are squared]
- $x(y-z)^2$ means $x \times (y-z) \times (y-z)$ [Only the brackets get squared]

15.2. Substitution

Imagine you were given the formula

$$X = 3A + 2B$$

and asked to find the value of X when $A = 5$ and $B = 7$. All you do is write down the formula again but this time you substitute numbers for letters on the right hand side. So we would have

$$\begin{aligned} X &= (3 \times 5) + (2 \times 7) \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

Make sure you do it in stages as we have done above. **DO NOT TRY TO DO IT ALL IN ONE GO!!!**

Example

If $C = \frac{5}{9}(F - 32)$ find the value of C when $F = 59$.

$$\begin{aligned} \text{Ans} \quad C &= \frac{5}{9}(59 - 32) \\ &= \frac{5}{9}(27) && \text{[Remember we do brackets first]} \\ &= 15 \end{aligned}$$

14. Indices

We use indices as a shorthand method

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^7$$

3 is called the **BASE** and 7 is called the **INDEX**. We say "3 to the power 7".

Examples

$$\begin{array}{lll} 6 \times 6 & = & 6^2 \quad (6 \text{ squared}) \\ 7 \times 7 \times 7 & = & 7^3 \quad (7 \text{ cubed}) \\ 8 \times 8 \times 8 \times 8 & = & 8^4 \quad (8 \text{ to the power 4}) \\ 9 \times 9 \times 9 \times 9 \times 9 & = & 9^5 \quad (9 \text{ to the power 5}) \end{array}$$

You can work out indices on your calculator using the X^Y or Y^X button. To work out 8^4 you would press:

$$8 \quad X^Y \quad 4 \quad =$$

14.1. Rules for Working with Indices

- When multiplying numbers with the same base you add the indices:
 $3^2 \times 3^3 = 3^{2+3} = 3^5$ $5^2 \times 5^7 = 5^9$

- When dividing numbers with the same base you subtract the indices:
 $3^3 \div 3^2 = 3^{3-2} = 3^1$ $5^8 \div 5^{11} = 5^{8-11} = 5^{-3}$

- When raising one power to another you multiply the powers:
 $(3^2)^4 = 3^{2 \times 4} = 3^8$ $(6^3)^9 = 6^{3 \times 9} = 6^{27}$

- Any number to the power 1 is just itself:
 $33^1 = 33$ $99^1 = 99$

- Any number to the power 0 is 1:
 $100^0 = 0$ $9561^0 = 0$

(Apart from 0 itself!)

$$0^0 = ?$$

- Fractional Powers:
The power $\frac{1}{2}$ means square root: $25^{1/2} = \sqrt{25} = 5$
The power $\frac{1}{3}$ means cube root: $64^{1/3} = \sqrt[3]{64} = 4$

- Negative Powers:
 $5^{-6} = \frac{1}{5^6}$ $6^{-2} = \frac{1}{6^2}$ $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

Example

Write 2^{-2} as a decimal.

Answer: $2^{-2} = \frac{1}{2^2} = \frac{1}{4} = 0.25$

Example

Simplify $7^3 \times 3^2 \times 7^5$

Answer: $7^{3+5} \times 3^2 = 7^8 \times 3^2$

13. Square Roots and Cube Roots

13.1. Square Roots

When we **square** a number we multiply it by itself. So

$$\begin{aligned}3^2 &= 3 \times 3 = 9 \\4^2 &= 4 \times 4 = 16 \\x^2 &= x \times x\end{aligned}$$

To find the **square root** of 9 you want to know what number times itself gives 9, the answer is 3 because $3 \times 3 = 9$. We write

$$\sqrt{9} = 3$$

Example

What is $\sqrt{49}$? Since $7 \times 7 = 49$, we have $\sqrt{49} = 7$.

You can also work out square roots using the $\sqrt{\quad}$ button on your calculator. Depending on your calculator you may have to press the number before or after the number you want the square root of.

Example

Using a calculator find $\sqrt{36}$

13.2. Cube Roots

When we **cube** a number we multiply it by itself 3 times. So

$$\begin{aligned}2^3 &= 2 \times 2 \times 2 = 8 \\3^3 &= 3 \times 3 \times 3 = 27 \\x^3 &= x \times x \times x\end{aligned}$$

To find the **cube root** of 27 you want to know what number times itself three times gives 27, the answer is 3 because $3 \times 3 \times 3 = 27$. We write

$$\sqrt[3]{27} = 3$$

Example

What is $\sqrt[3]{64}$? Since $4 \times 4 \times 4 = 64$ we have $\sqrt[3]{64} = 4$

You can find out cube roots on your calculator by using the $\sqrt[3]{\quad}$ button.

Exercise

Using your calculator work out $\sqrt[3]{125}$

Example

If I divide £100 in the ratio 2:3:5 how much is each part worth?

Answer

Total number of parts: $2 + 3 + 5 = 10$
Value of each part: $100 \div 10 = £10$

Hence

2 parts are worth $\rightarrow 2 \times 10 = £20$
3 parts are worth $\rightarrow 3 \times 10 = £30$
5 parts are worth $\rightarrow 5 \times 10 = £50$

Example

A sum of money is split between three people A, B, and C in the ratio 2:3:4.

Assuming that A gets £8 work out how much B and C receive and the total amount of money.

Answer

This is slightly different because we don't know the total amount of money. First we work out the value of each part:

Person A has received 2 parts so each part is worth $8 \div 2 = £4$

Person B receives: $3 \times 4 = £12$

Person C receives: $4 \times 4 = £16$

Hence total amount is $8 + 12 + 16 = £36$

12. Ratio

12.1. Introduction

Ratios are basically just fractions on the side and can be treated in much the same way. If the ratio of boys to girls in a class is 3 : 4 then this means that for every 3 boys there are 4 girls.

Alternatively by dividing both sides of the ratio by 3 we obtain:

$$1 : \frac{4}{3}$$

This means that for every boy then will be $\frac{4}{3}$ of a girl.

So if there were 12 boys there would be

$$12 \times \frac{4}{3} = 16$$

girls.

If there were 20 girls there would be

$$\begin{aligned} 20 \div \frac{4}{3} &= 20 \times \frac{3}{4} \\ &= 15 \end{aligned}$$

boys.

Ratios are usually easier to deal with when they are in the form 1 : n and they can always be got into this form by dividing both sides by the smallest side.

Exercise

1. A drawing of a house is drawn to a scale 1:100.
 - a. If the height of a wall on the drawing is 2.8 cm what will it be in the house?
 - b. If the length of a room in the house is 450 cm what will it be on the drawing?

12.2. Splitting into Ratios

Imagine you were asked to divide £9100 into the ratio 2:4:7. How would you go about doing this? The method is as follows:

- Work out the total number of parts: $2 + 4 + 7 = 13$ parts
- Work out the value of one part: $9100 \div 13 = £700$
- Now find the values of 2, 4 and 7 parts respectively:
 $2 \times 700 = £1400$ $4 \times 700 = £2800$ $7 \times 700 = £4900$

As a check we can add up the parts to check we have the right amount:

$$1400 + 2800 + 4900 = £9100$$

11. Simple and Compound Interest

11.1. Simple Interest

In a simple interest account the interest each year is calculated on the initial amount invested not on your current balance (i.e. unlike a bank account).

Example

Say we invest £50 in a simple interest account that pays 2% interest per annum:

$$\begin{array}{ll} \text{Interest given at end of 1}^{\text{st}} \text{ year:} & 2\% \text{ of } £50 = \frac{2}{100} \times 50 = 0.02 \times 50 = £1 \\ \text{Balance after 1}^{\text{st}} \text{ year:} & 50 + 1 = £51 \end{array}$$

$$\begin{array}{ll} \text{Interest given at end of 2}^{\text{nd}} \text{ year:} & 2\% \text{ of } £50 = \frac{2}{100} \times 50 = 0.02 \times 50 = £1 \\ \text{Balance after 2}^{\text{nd}} \text{ year:} & 51 + 1 = £52 \end{array}$$

$$\begin{array}{ll} \text{Interest given at end of 3}^{\text{rd}} \text{ year:} & 2\% \text{ of } £50 = \frac{2}{100} \times 50 = 0.02 \times 50 = £1 \\ \text{Balance after 3}^{\text{rd}} \text{ year:} & 52 + 1 = £53 \end{array}$$

Because the interest paid is the same each year it is easy to work out the amount of interest paid over a period of time. For example over 61 years we would have received $71 \times 1 = £71$ in interest.

Exercise

I invest £3000 in a simple interest account that pays 5% per annum. How much interest will I have received after 10 years? What will be my balance after 10 years?

11.2. Compound Interest

Compound interest works in the same way as most bank accounts. The interest at the end of each year is calculated on your current balance and **not** the initial amount invested.

Example

I invest £250 in a compound interest account at 5% p.a.:

$$\begin{array}{ll} \text{Interest given at end of 1}^{\text{st}} \text{ year:} & 5\% \text{ of } £250 = \frac{5}{100} \times 250 = £12.50 \\ \text{Balance after 1}^{\text{st}} \text{ year:} & 250 + 12.50 = £262.50 \end{array}$$

$$\begin{array}{ll} \text{Interest given at end of 2}^{\text{nd}} \text{ year:} & 5\% \text{ of } £262.50 = \frac{5}{100} \times 262.50 = £13.13 \\ \text{Balance after 2}^{\text{nd}} \text{ year:} & £262.50 + £13.13 = £275.63 \end{array}$$

Example

Place the following in order from smallest to largest:

$\frac{1}{4}$ 0.2 23%

Firstly we convert them all to decimals:

$$\frac{1}{4} \rightarrow 1 \div 4 = 0.25$$

$$0.2 \rightarrow 0.2$$

$$23\% \rightarrow 23 \div 100 = 0.23$$

It is now easy to put them in order:

0.2 23% $\frac{1}{4}$

Exercise

Place the following in ascending order:

1. $\frac{3}{8}$ 0.41 36%

2. $\frac{7}{8}$ 0.8 87%

10.3. Finding Percentages 2

Imagine in an examination you had 23 out of 50. What percentage have you had?

To work this out we calculate

- $\frac{23}{50} = 0.46$
- $0.46 \times 100 = 46\%$

Exercise

What percentage is

1. 24 out of 50

2. 56 out of 90

3. 43 out of 76

10.4. Percentage Increase or Decrease

Imagine you own a property whose value has increased by 20% to £72,000. How do we find what it was originally worth?

Let's think of the original price as being 100% so if it has increased by 20% we now have 120%:

$$\text{£72,000} = 120\%$$

Divide by 120 to find out what 1% is:

$$\text{£600} = 1\%$$

Multiply by 100 to discover the original value:

$$\text{£60,000} = 100\%$$

Exercise

After an increase of 10% my house is worth £55,000. What was it originally worth?

10. Percentages

10.1. Finding Percentages 1

Percentage means **per hundred**.

Example

$$30\% \text{ means } \frac{30}{100}$$

$$55\% \text{ means } \frac{55}{100}$$

$$79\% \text{ means } \frac{79}{100}$$

To work out 30 % of 200 you do the following:

$$30\% \text{ of } £200 \text{ is } \frac{30}{100} \times 200 = 60$$

Exercise

Calculate

1. 37% of £600

2. 40% of £1.50

3. 93% of 1200

Example

My car is worth £7000 when I purchase it. It's value falls by 10% each year. What will it be worth after one year?

$$\begin{aligned} \text{Ans: } 10\% \text{ of } £7000 &= \frac{10}{100} \times 7000 = £700 \\ \text{Value after 1 year} &= 7000 - 700 \\ &= £6300 \end{aligned}$$

Exercise

If my house is worth £70,000 and over a period of time its value increase by 20%. What will it now be worth?

10.2. Comparing Fractions, Percentages and Decimals

We know how to change fractions into decimals, but it is just as easy to change decimals into percentages.

To change decimals into percentages you MULTIPLY by 100

To change percentages into decimals you DIVIDE by 100.

Example

1. What is 0.36 as a percentage?
 $0.36 \times 100 = 36\%$

2. What is 59% as a decimal?
 $59 \div 100 = 0.59$

Exercise

1. Convert 45%, 73% and 22% into decimals
2. Convert 0.55, 0.88 and 0.01 into percentages

Hence we can now change fractions into decimals and then into percentages. When you are asked to compare fractions, percentages and decimals you must first convert them all into the same type.

Exercise

1. $\frac{2}{3} \times \frac{5}{8}$

2. $\frac{4}{13} \div \frac{2}{5}$

3. $\frac{1}{3} \div \frac{2}{3}$

4. $\frac{2}{3} \times \frac{4}{3} \div \frac{1}{5}$

9.5. Changing Fractions into Decimals

To change a fraction into a decimal you simply **divide the top by the bottom**.

Example

To change $\frac{3}{5}$ into a decimal we work out $3 \div 5 = 0.6$

So $\frac{3}{5} = 0.6$

Example

To change $4\frac{1}{5}$ into a decimal we do the following:

- $\frac{1}{5}$ as a decimal is given by $1 \div 5 = 0.2$
- Now add the 4 wholes to it $4 + 0.2 = 4.2$

So $4\frac{1}{5} = 4.2$

Exercise

Write the following fractions as decimals:

1. $\frac{3}{5}$

2. $\frac{4}{10}$

3. $\frac{10}{20}$

9.6. Calculating a Fraction of a Quantity

Imagine you wanted to work out $\frac{3}{4}$ of £6 to do this we calculate $\frac{3}{4} \times 6$

$$\frac{3}{4} \times \frac{6}{1} = \frac{18}{4} = 4.5 = \text{£}4.50$$

Exercise

1. Calculate $\frac{5}{8}$ of £24

2. Calculate $\frac{3}{5}$ of 10

What if you wanted to calculate $3\frac{3}{4}$ of £6? Do it in stages:

- Calculate $\frac{3}{4}$ of £6 $\frac{3}{4} \times 6 = 4.5$
- Calculate 3 lots of £6 $3 \times 6 = 18$
- So $3\frac{3}{4}$ of £6 will be $18 + 4.5 = \text{£}22.50$

Exercise

1. Calculate $2\frac{3}{8}$ of £24

2. Calculate $7\frac{4}{10}$ of £10

9.3. Adding Fractions

It is very easy to add fractions if they have the **SAME** denominator:

Example

$$\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$$

But how would you work out $\frac{1}{5} + \frac{2}{3}$?

The idea is to replace the fractions by equivalent fractions that have the same denominator. We use the following method:

LCM of 5 and 3 is 15 --- so we look for equivalent fractions with 15 in the denominator.

$$\frac{1}{5} = \frac{3}{15} \quad (\text{Multiplying top and bottom by 3})$$

$$\frac{2}{3} = \frac{10}{15} \quad (\text{Multiplying top and bottom by 5})$$

$$\text{Hence } \frac{1}{5} + \frac{2}{3} = \frac{3}{15} + \frac{10}{15} = \frac{13}{15}$$

Exercise

Calculate the following without the use of a calculator:

1. $\frac{1}{2} + \frac{4}{2}$

2. $\frac{1}{3} + \frac{2}{6}$

3. $\frac{1}{7} - \frac{1}{8}$

4. $\frac{2}{10} + \frac{3}{4}$

9.4. Multiplying and Dividing Fractions

Multiplying fractions is easier still. You simply multiply the numerators and then multiply the denominators:

Example

$$\frac{4}{2} \times \frac{2}{5} = \frac{8}{10}$$

People think that dividing fractions is more difficult, but all you do is turn the fraction you are dividing by **upside down** and then multiply instead.

Example

$$\text{To calculate } \frac{5}{8} \div \frac{3}{6} \quad \rightarrow \quad \text{We work out } \frac{5}{8} \times \frac{6}{3} = \frac{30}{24} = 1\frac{6}{24}$$

CAUTION: MAKE SURE IT IS THE FRACTION YOU ARE DIVIDING BY THAT YOU TURN UPSIDE DOWN!!!!!!!!!!

9. Fractions

9.1. Kinds of Fractions

If a cake is cut into 8 pieces and you eat 2, then you've eaten $\frac{2}{8}$ of the cake.

The bottom number in a fraction is called the **denominator**.

The top number in a fraction is called the **numerator**.

The **denominator** tells us how many equal pieces the whole is divided into whilst the **numerator** tells us how many pieces we are considering. If the **numerator** is bigger than the **denominator** we say the fraction is top heavy.

Example

$\frac{11}{10}$ is top heavy. The whole is split into 10 equal pieces and we have taken 11 of them. This means we have **1 whole** and $\frac{1}{10}$. Hence we can write $\frac{11}{10} = 1\frac{1}{10}$.

Example

Write $6\frac{3}{4}$ as a top heavy fraction.

Here we have 6 wholes and 3 quarters. How many quarters is this in total?

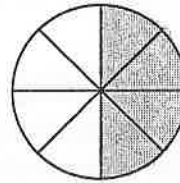
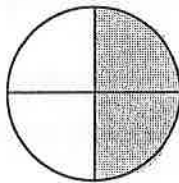
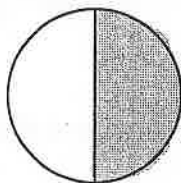
Each whole is 4 quarters and we have 6 wholes $\rightarrow 6 \times 4 = 24$ quarters.

Hence in total we have $24 + 3 = 27$ quarters.

So $6\frac{3}{4} = \frac{27}{4}$

9.2. Equivalent Fractions

Consider if I gave you $\frac{1}{2}$ a cake. You may say you've been given $\frac{2}{4}$ or even $\frac{4}{8}$ of the cake.



We say $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$ are **equivalent fractions** because they represent the same amount. To find equivalent fractions we multiply the top and bottom of a fraction by the same number.

Example

Find two equivalent fractions of $\frac{1}{4}$

Multiplying top and bottom by 2 yields: $\frac{2}{8}$

Multiplying top and bottom by 3 yields: $\frac{3}{12}$

Exercises

1. Write the following as a product of their prime factors:

a. 990

b. 160

8.4. Lowest Common Multiple (LCM)

The lowest common multiple of a group of numbers is the smallest number that they all divide. To find the lowest common multiple following the steps below:

- Write down the multiples of each number.
- The smallest number that occurs in all lists is the lowest common multiple.

Example

1. Find the lowest common multiple of 2 and 3:

Multiples of 2 are:	2	4	6	8	10	12	14	...
Multiples of 3 are:	3	6	9	12	15	18	21	...

The LCM of 2 and 3 is 6

2. Find the lowest common multiple of 2 and 6:

Multiples of 2 are:	2	4	6	8	10	12	14	...
Multiples of 6 are:	6	12	18	24	30	36	42	...

The LCM of 2 and 6 is 6.

Exercise

1. Find the LCM of 8 and 9.

2. Find the LCM of 7 and 9.

8.5. Highest Common Factor (HCF)

The highest common factor of a group of numbers is the biggest number that divides all of them exactly. To find the highest common factor just follow the steps below:

- Write down the factors of each of the numbers..
- The biggest number that occurs in all lists is the highest common factor.

Example

Find the HCF of 36, 54 and 72:

Factors of 36 are:	1	2	3	4	6	9	12	18	36
Factors of 54 are:	1	2	3	6	9	18	27	54	
Factors of 72 are:	1	2	3	4	6	8	9	12	18
	24	36	72						

So the HCF of 36, 54 and 72 is 18.

Exercises

Find the HCF of

1. 10 and 12

2. 15 and 30

3. 10 and 1

8. Factors and Prime Factors

8.1. Multiples

The **multiples** of a number are just its multiplication table. For example the multiples of 15 are 15 30 45 60 75 90 105 etc...

Exercise

1. Write down six multiples of 13.
2. Which of the following are multiples of 3: 12 13 14 15

8.2. Factors

The **factors** of a number are all those numbers that divide the number exactly, i.e. leave no remainder.

Example

The factors of 12 are 1 2 3 4 6 12

To find the factors of 12 consider which numbers multiply together to make 12:

$$1 \times 12 = 12 \qquad 2 \times 6 = 12 \qquad 3 \times 4 = 12$$

Note that for any even number 2 is **always** a factor.

8.3. Prime Factors

You should notice that for any number

- 1 is a factor of the number
- The number itself is a factor

Some numbers such as 7 only have these factors. That is 7 only has the factors 1 and 7. We call such number **prime numbers**. Note that 0 and 1 are **NOT** prime numbers!!!

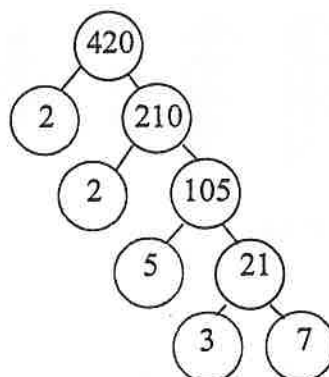
Examples

3, 5, 7, 11 are all prime numbers.

Any number can be written as a product of its prime factors by a factor tree:

Example

Write 420 as a product of its prime factors:



So we can write $420 = 2 \times 2 \times 5 \times 3 \times 7$

7) Estimating

Imagine you wanted to get an estimate for

$$\frac{56.5 \times 3}{17.3 + 42.5}$$

First write each of the numbers to 1 significant figure:

17.3	→	20
42.5	→	40
56.5	→	60
3	→	3

Now

$$\frac{56.5 \times 3}{17.3 + 42.5} \approx \frac{60 \times 3}{20 + 40} = \frac{180}{60} = \frac{18}{6} = 3$$

So we estimate the answer to be 3.

The symbol $\approx$ means **approximately equal to**.

Example

Estimate $11.52 + 13.2 + 37.35$

Round each number to 1 significant figure

11.52	→	10
13.2	→	10
37.35	→	40

So $11.52 + 13.2 + 37.35 \approx 10 + 10 + 40 = 60$

Example

Estimate $135.23 \times 25.36 \times 37.52$

Round each number to 1 significant figure:

135.23	→	100
25.36	→	30
37.52	→	40

So $135.23 \times 25.36 \times 37.52 \approx 100 \times 30 \times 40 = 3000 \times 40 = 120000$

It is also possible to work out such problems by using brackets. We use brackets to tell the calculator that we want to divide all of the top by all of the bottom. For the above example we would enter:

$$(1 2 + 3) \div (5 - 2) = 5$$

If you didn't enter brackets then because your calculator knows the BODMAS rule it would do division first and work out:

$$12 + \frac{3}{5} - 2 = 12 + 0.6 - 2 = 10.6$$

But this is not what we wanted so you **must use brackets**.

Example

1. Use the calculator to calculate the following:

a) $\frac{54}{11+7}$

b) $\frac{5 \times 28}{22+18}$

c) $\frac{13.6 - 5.1}{7 - 2}$

d) $(3.2 - 1.5) \times (4.1 - 1.6)$

2. Write down the keys you would press to calculate $\frac{5+7}{6-2}$

6. BODMAS

Consider the calculation $4 + 2 \times 5$

By using brackets we can change what the answer is:

$$(4 + 2) \times 5 = 6 \times 5 = 30 \quad \text{OR} \quad 4 + (2 \times 5) = 4 + 10 = 14$$

But which one is $4 + 2 \times 5$? Is it $(4 + 2) \times 5$ OR $4 + (2 \times 5)$?

To answer this question we need to know the BODMAS rule:

Brackets Or Division and Multiplication then Addition and Subtraction

So in $4 + 2 \times 5$ we do the multiplication before the addition:

$$4 + 2 \times 5 = 4 + 10 = 14$$

Example

1) Calculate

a) $4 + (8 \div 2)$

b) $(4 + 8) \div 2$

c) $4 + 8 \div 2$

Answer:

a. Brackets first: $4 + (8 \div 2) = 4 + 4 = 8$

b. Brackets first: $(4 + 8) \div 2 = 12 \div 2 = 6$

c. Division first: $4 + 8 \div 2 = 4 + 4 = 8$

Now consider $\frac{10}{2+3}$

When you need to work out fractions you should

- **work out the top**
- **work out the bottom**
- **divide your answers**

So for the above:

Top	=	10	Bottom	=	$2 + 3$
				=	5
Answer	=	$10 \div 5$			
	=	2			

Example

$$\frac{12+3}{5-2}$$

$$\text{Top} = 12 + 3 = 15$$

$$\text{Bottom} = 5 - 2 = 3$$

$$\text{Answer} = 15 \div 3 = 5$$

- 2) Consider the number 45.319461
- To 5 decimal places 45.31946 (5 d.p.)
 - To 4 decimal places 45.3195
 - To 3 decimal places 45.319
 - To 2 decimal places 45.32
- 3) Write the following numbers to 4 decimal places.
- 4.3014643 **Ans:** 4.3015 (4 d.p.)
 - 10.210081 **Ans:** 10.2101 (4 d.p.)
 - 11.234956

The number in the 4th decimal place is 9.
 In the 5th decimal place we have 5, which is **5 or more**.
 We round the 9 up to 10.
 Write down 0 carry the 1. $1 + 4 = 5$
 Hence we have 11.2350 (4 d.p.)

5.3. Significant Figures

The 1st significant figure of any number is the first digit which isn't zero.

Example

The 1st significant figure of 3.562 is 3
 The 1st significant figure of 0.023 is 2

The 2nd, 3rd, 4th, etc. significant figures follow immediately from the 1st **regardless of whether they are zero or not.**

Example

Consider 0.054032

The 1st significant figure is 5
 The 2nd significant figure is 4
 The 3rd significant figure is 0
 The 4th significant figure is 3
 The 5th significant figure is 2

To write numbers to a certain number significant figures you follow the following 3 steps:

Imagine you wanted to write a number to 4 significant figures.

- 1) Locate the 4th significant figure.
- 2) Look at the next digit to the RIGHT.
- 3) If it's 5 or more, then we **ROUND UP** the 4th significant figure. If it's 4 or less then we leave the 4th significant figure as it is.

Example

	4 SF	3 SF	2 SF	1 SF
1) 54.7651	54.77	54.8	55	50
2) 17.0067	17.01	17.0	17	20
3) 0.0045902	0.004590	0.00459	0.0046	0.005
4) 30895.4	30900	30900	31000	30000

5. Rounding Off

5.1. Nearest 10, 100, 1000, etc.

Examples

37 to the nearest 10 is 40	--- 37 is closer to 40 than 30.
537 to the nearest 100 is 500	--- 537 is closer to 500 than 600.
537 to the nearest 10 is 540	--- 537 is closer to 540 than 530.

If we are **exactly** half way between two numbers the convention is to **round up!!**

3350 to the nearest 100 is 3400	--- 3350 is half way between 3300 and 3400
7500 to the nearest 1000 is 8000	--- 7500 is half way between 7000 and 8000

Exercise

- 1) Write these numbers to the nearest 10:
a) 4624 b) 624 c) 58
- 2) Write these numbers to the nearest 100:
a) 4624 b) 624 c) 58
- 3) Write these number to the nearest 1000:
a) 4624 b) 4500 c) 850

5.2. Decimal Places

Writing numbers to a certain number of decimals is easy as long as you remember the following 3 steps:

Imagine you wanted to write a number to 3 decimal places.

- 1) Locate the 3rd decimal place.
- 2) Look at the next digit to the RIGHT.
- 3) If it's 5 or more, then we **ROUND UP** the 3rd decimal place. If it's 4 or less then we leave the 3rd decimal place as it is.

Example

- 1) Write the following to 3 decimal places:

- a) 7.432543

The number in the 3rd decimal place is 2.

In the 4th decimal place we have 5, which of course is **5 or more**.

We round up the 2 to 3

Hence we have 7.433 (3 d.p.)

- b) 0.001232

The number in the 3rd decimal place is 1.

In the 4th decimal place we have 2, which is **4 or less**.

We leave the 1 as it is.

Hence we have 0.001 (3 d.p.)

Examples

1) Calculate 5×-3

We know $5 \times 3 = 15$

From the table our answer should be negative.

So $5 \times -3 = -15$

2) Calculate -7×3

We know $7 \times 3 = 21$

From the table our answer should be negative.

So $-7 \times 3 = -21$

3) Calculate -5×-6

We know $5 \times 6 = 30$

From the table our answer should be positive.

So $-5 \times -6 = 30$

Note: When the signs are different the answer is negative, but when they are the same the answer is positive.

4.3. Dividing Negative Numbers

To do these calculations first ignore the signs and then use the following table to work out what sign your answer should take:

+	÷	+	=	+ve answer
+	÷	-	=	-ve answer
-	÷	+	=	-ve answer
-	÷	-	=	+ve answer

Example

1) Work out $-9 \div 3$

We know $9 \div 3 = 3$

From the table the answer should be negative.

So $-9 \div 3 = -3$

2) Work out $-10 \div -5$

We know $10 \div 5 = 2$

From the table the answer should be positive.

So $-10 \div -5 = 2$

4.4. Using a Calculator with Negative Numbers

To enter negative numbers on your calculator you need to press the +/- key on your calculator. Whether you need to press this button before or after you enter the number will depend on your calculator.

Exercise

Redo all the exercises and examples in this section using a calculator.

4. Negative Numbers

4.1. Introduction

You will probably have seen negative numbers when looking at thermometers or perhaps even your bank balance! If the temperature falls below 0 then we have negative temperatures. If your bank balance falls below 0 then you have a negative balance.



Exercise

- 1) Put these temperatures in order (lowest first)
 - a) 2°C , -2°C , -5°C
 - b) -7°C , -4°C , -10°C
- 2) What is the difference in temperature between
 - a) 3°C and -4°C
 - b) 12°C and -5°C
 - c) -1°C and -13°C
 - d) 1°C and -7°C

Note that if you are taking away a minus number you are actually adding the number. This is because **TWO NEGATIVES MAKE A POSITIVE!!!**

- 3) Calculate
 - a) $3 - (-4)$
 - b) $3 + (-4)$
 - c) $-5 - 7$
 - d) $-5 - (-7)$
 - e) $-5 + (-7)$

4.2. Multiplying Negative Numbers

To work out multiplication problems involving negative numbers is no harder than with positive numbers. You simply ignore the signs and do the calculation as normal. Then use the following table to work out what sign your answer should be:

+	$\times$	+	=	+ve answer
+	$\times$	-	=	-ve answer
-	$\times$	+	=	-ve answer
-	$\times$	-	=	+ve answer

3.3. Multiplication

Examples

$$\begin{array}{r} 50.2 \\ \times \quad 3 \\ \hline 150.6 \end{array}$$

$$\begin{array}{r} 3.64 \\ \times \quad 5 \\ \hline 18.20 \\ \quad 3 \quad 2 \end{array}$$

3.4. Division

When we do divisions with decimals we keep on going until there is no remainder left. If necessary we can add extra zeros onto the end of the number as long as we are to the right of the decimal point.

Example

3.96 could be written as 3.960
or 3.9600
or 3.96000 etc.

Examples

1) Work out $5.36 \div 4$

Ans:
$$4 \overline{) 5.36}$$

2) Work out $96.2 \div 5$

Ans:
$$5 \overline{) 96.20}$$

Method: 5 goes into 9 once with 4 left over.
5 goes into 46 nine times with 1 left over.
5 goes into 12 twice with 2 left over.
We now need to add an extra 0!!!!
5 goes into 20 four times exactly.
Since there is nothing left over we are finished.

3) Work out $1.0 \div 4$

Ans:
$$4 \overline{) 1.00}$$

Method: 4 goes into 1 zero times with 1 left over.
4 goes into 10 twice with 2 left over.
We now need to add an extra 0!!!
4 goes into 20 five times exactly.
Since there is nothing left over we are finished.

3. Using Decimals

Doing calculations using numbers that involve decimal points is no harder than earlier as long as you remember to keep the decimal point in the same place.

3.1. Simple Calculations

10×3.653	$=$	36.53	[Everything moves 1 place to the left]
100×3.653	$=$	365.3	[Everything moves 2 places to the left]
1000×3.653	$=$	3653	[Everything moves 3 places to the left]
$10,000 \times 3.653$	$=$	36,530	[Everything moves 4 places to the left]
$370 \div 10$	$=$	37	[Everything moves 1 place to the right]
$370 \div 100$	$=$	3.7	[Everything moves 2 places to the right]
$370 \div 1000$	$=$	0.37	[Everything moves 3 places to the right]
$370 \div 10000$	$=$	0.037	[Everything moves 4 places to the right]

Multiplying by 0.1 moves everything 1 place to the right.

Multiplying by 0.01 moves everything 2 places to the right.

Multiplying by 0.001 moves everything 3 places to the right.

Examples

0.1×360	$=$	36	0.001×360	$=$	0.36
0.01×360	$=$	3.6	0.0001×360	$=$	0.036

Slightly harder examples include:

Examples

Calculate	0.2×5		Calculate	0.003×7	
We know	2×5	$= 10$	We know	3×7	$= 21$
So	0.2×5	$= 1$	So	0.003×7	$= 0.021$

3.2. Addition

Examples

1. Add $2.7 + 3.9 + 14.65$

$$\begin{array}{r}
 2 \quad . \quad 7 \quad 0 \\
 3 \quad . \quad 9 \quad 0 \\
 14 \quad . \quad 6 \quad 5 \\
 \hline
 21 \quad . \quad 2 \quad 5 \\
 \hline
 1 \quad 2
 \end{array}$$

2. Add $2.95 + 3.7 + 13$

[Note that 13 can be written 13.00 and 3.7 can be written 3.70]

$$\begin{array}{r}
 2 \quad . \quad 9 \quad 5 \\
 3 \quad . \quad 7 \quad 0 \\
 13 \quad . \quad 0 \quad 0 \\
 \hline
 19 \quad . \quad 6 \quad 5 \\
 \hline
 1
 \end{array}$$

2.5. Short Division

Example

Calculate $861 \div 3$

$$\begin{array}{r} 287 \\ 3 \overline{) 861} \end{array}$$

Method

- 1) 3 into 8 goes twice with 2 left over.
- 2) 3 into 26 goes 8 times with 2 left over.
- 3) 3 into 21 goes 7 times with 0 left over.

Example

Calculate $123 \div 7$

$$\begin{array}{r} 17 \text{ r}4 \\ 7 \overline{) 123} \end{array}$$

Method

1. 7 doesn't go into 1.
2. 9 into 12 goes once with 5 left over.
3. 7 into 53 goes 7 times with 4 left over.

2.6. Long Division

Example

Calculate $325 \div 21$ using long division.

$$\begin{array}{r} 15 \text{ r}10 \\ 21 \overline{) 325} \\ \underline{21} \\ 115 \\ \underline{105} \\ 10 \end{array}$$

Method:

- 1) 21 doesn't go into 3.
- 2) 21 goes into 32 once.
- 3) $1 \times 21 = 21$.
- 4) $32 - 21 = 11$.
- 5) Bring down the 5.
- 6) 21 goes into 115 five times
- 7) $5 \times 21 = 105$
- 8) $115 - 105 = 10$

Example

Calculate $702 \div 33$ using long division.

$$\begin{array}{r} 21 \text{ r}9 \\ 33 \overline{) 702} \\ \underline{66} \\ 42 \\ \underline{33} \\ 9 \end{array}$$

Method:

- 1) 33 doesn't go into 7.
- 2) 33 goes into 70 twice.
- 3) $2 \times 33 = 66$.
- 4) $70 - 66 = 4$.
- 5) Bring down the 2.
- 6) 33 goes into 42 once.
- 7) $1 \times 33 = 33$
- 8) $42 - 33 = 9$

We can also handle slightly harder problems:

Examples

Calculate 300×4

Ans: We know $3 \times 4 = 12$
So $300 \times 4 = 1200$

Calculate 5000×40

Ans: We know $5 \times 40 = 200$
So $5000 \times 40 = 200,000$

2.4. Long Multiplication

Example 1

$$\begin{array}{r} 23 \\ \times 3 \\ \hline 69 \end{array}$$

Method:

- 1) $3 \times 3 = 9$
- 2) $3 \times 2 = 6$

Example 2

$$\begin{array}{r} 23 \\ \times 40 \\ \hline 920 \\ \hline \end{array}$$

Method:

- 1) Write down 0
- 2) $4 \times 3 = 12$
- 3) Write down 2 and carry 1
- 4) $4 \times 2 = 8$
- 5) $8 + 1 = 9$

Example 3

$$\begin{array}{r} 52 \\ \times 43 \\ \hline 156 \\ 2080 \\ \hline 22136 \end{array}$$

Method:

- 1) $3 \times 2 = 6$
- 2) $3 \times 5 = 15$
- 3) Write down 0
- 4) $4 \times 2 = 8$
- 5) $4 \times 5 = 20$
- 6) Calculate $156 + 2080$

Example 4

$$\begin{array}{r} 78 \\ \times 53 \\ \hline 234 \\ 3900 \\ \hline 4134 \end{array}$$

Method:

- 1) $3 \times 8 = 24$
- 2) Write down 4 and carry the 2
- 3) $3 \times 7 = 21$
- 4) $21 + 2 = 23$
- 5) Write down 0
- 6) $5 \times 8 = 40$
- 7) Write down 0 carry the 4
- 8) $5 \times 7 = 35$
- 9) $35 + 4 = 39$
- 10) $234 + 3900 = 4134$

2. Mental Arithmetic

2.1. Addition

You should already know how to add numbers. If you can't remember how then here are some examples that will remind you:

Example

$$\begin{array}{r} 1 \ 3 \\ + \ 2 \ 1 \\ \hline 3 \ 4 \end{array}$$

Method:

- 1) $3 + 1 = 4$
- 2) $1 + 2 = 3$

Example

$$\begin{array}{r} 2 \ 6 \\ + \ 9 \ 6 \\ \hline 1 \ 2 \ 2 \\ \hline 1 \end{array}$$

Method:

- 1) $6 + 6 = 12$
- 2) Write down 2 and carry 1
- 3) $2 + 9 + 1 = 12$

2.2. Subtraction

You should also be familiar with the method used for subtraction. Here are 2 examples that illustrate it:

Example

$$\begin{array}{r} 4 \ 7 \\ - \ 2 \ 5 \\ \hline 2 \ 2 \end{array}$$

Method:

- 1) $7 - 5 = 2$
- 2) $4 - 2 = 2$

Example

$$\begin{array}{r} \cancel{10}^5 \ 13 \\ - \ 4 \ 5 \\ \hline 1 \ 8 \end{array}$$

Method:

- 1) 3 is smaller than 5
- 2) $13 - 5 = 8$
- 3) $5 - 4 = 1$

2.3. Multiplication

Multiplying a number by 10 is easy because all the digits move one place to the left:

Examples

$$10 \times 36 = 360$$

$$10 \times 520 = 5,200$$

$$10 \times 1,100 = 11,000$$

- Multiplying by 100 moves all the digits two places to the left.
 - Multiplying by 1000 moves all the digits three places to the left.
- Etc.

Examples

$$100 \times 52 = 5200$$

$$1000 \times 62 = 62,000$$

$$10,000 \times 6 = 60,000$$

7. Write the number **seven hundred thousand six hundred and fifty four** in numerical form:

Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units
7	0	0	6	5	4

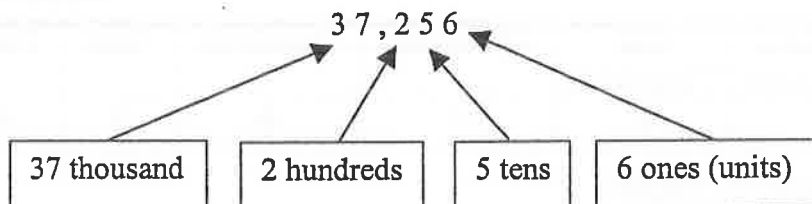
So the number is **700,654**.

Exercise

- 1) Write these numbers as words:
 - a) 8000
 - b) 33 200
 - c) 503 000
 - d) 503 100
 - e) 503 010
 - f) 503 001
 - g) 800 500 300
 - h) 909 606 303
- 2) Write these numbers in numerical form:
 - a) Five thousand
 - b) Five thousand, two hundred and thirty three
 - c) Seventy seven thousand, six hundred and five
 - d) Three hundred thousand and one
 - e) Ninety two million and three thousand

1. Number System

Consider the number



This can also be written by using column headings:

Ten Thousands	Thousands	Hundreds	Tens	Units
3	7	2	5	6

You will need to be able to read and write numbers in both numerical and word form. An important point to remember is that a million has 6 zeros.

Examples

1. 835 **Reads:** Eight hundred and thirty five
2. 57,325 **Reads:** Fifty Seven thousand, three hundred and twenty five.
3. Five thousand, three hundred and twenty **Numerical Form:** 5,020
4. Seventy five thousand and fifty two **Numerical Form:** 70,052

Using Columns Headings can be very helpful when writing numbers as words or vice versa:

Examples

5.

Thousands	Hundreds	Tens	Units
2	6	2	1

We have Two thousands
 Six Hundreds
 Two Tens i.e. Twenty
 One Unit

So the number we have is **Two thousand, six hundred and twenty one.**

6.

Ten Thousands	Thousands	Hundreds	Tens	Units
4	2	6	0	3

We have Four ten thousands, i.e. Forty thousand
 Two thousand
 Six Hundred
 Zero Tens
 Three Units

So the number we have is **Forty two thousand, six hundred and three.**

MATHEMATICS MODULE

ACDA 4008

OME/EES/HNC